

Correlations in totally symmetric self-complementary plane partitions

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ABSTRACT

Totally symmetric self-complementary plane partitions (TSSCPPs) are boxed plane partitions with the maximum possible symmetry. We use the well-known representation of TSSCPPs as a dimer model on a honeycomb graph enclosed in $1/12$ of a hexagon with free boundary to express them as perfect matchings of a family of non-bipartite planar graphs. Our main result is that the edges of the TSSCPPs form a Pfaffian point process, for which we give explicit formulas for the inverse Kasteleyn matrix. Preliminary analysis of these correlations are then used to give a precise conjecture for the limit shape of TSSCPPs in the scaling limit.

1. Introduction

Totally symmetric self-complementary plane partitions (TSSCPPs) of order n are the subset of plane partitions in a $(2n) \times (2n) \times (2n)$ box with the maximum possible symmetry. They have been intensely studied since the initial analysis by Mills, Robbins and Rumsey conjecturing that the number of TSSCPPs of order n are the same as the number of alternating sign matrices (ASMs) of size n [34] (see the formula for A_n in (2.1)). The fact that the number of TSSCPPs of order n is given by A_n was established by Andrews in a difficult paper [1]. It was this paper by Mills, Robbins and Rumsey that led Stanley to initiate the study of symmetry classes of plane partitions [43]. The fact that ASMs are also enumerated by the same sequence of numbers is known as the Alternating Sign Matrix theorem and was proven first by Zeilberger [46] by directly comparing them with TSSCPPs and later by Kuperberg [33] using a connection with the six-vertex model in statistical mechanics. See the book by Bressoud [11] for more about the history of this and related fascinating problems.

Boundary correlations in ASMs have been studied almost from the very beginning. A formula for the enumeration of ASMs according to the position of the unique 1 in the first row is known as the Refined Alternating Sign Matrix theorem and this was proven by Zeilberger [47]. Stroganov [44] gave a formula for the number of ASMs according to the position of the unique 1 in the first and last rows (the *top-bottom* formula), and according to the position of the unique 1 in the first row and left column (the *top-left* formula). The latter formula was subsequently improved in [5]. Other refined enumeration formulas include the *top two* formula [20, 29], the *top two and bottom* formula [19], the *top-left-bottom* formula [2, 19] and the *top-left-bottom-right* formula [2, 4]. The problem of computing bulk correlations seems

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like a difficult and interesting open problem. On the TSSCPP side, no formulas are known for any correlation functions.

On the other hand, correlations for some plane partitions have been established in recent years [3, 38, 40]. The typical perspective here is to view the plane partition as a rhombus or lozenge tiling. Randomness is introduced by picking each configuration at random from the set of all possible configurations in some prescribed manner, the simplest being picking each configuration uniformly at random which is the case considered here for TSSCPPs. For a specific class of tiling models, interesting probabilistic features are observed when the system size becomes large, such as a macroscopic limit shape, which is a type of law of large numbers result. Around this limit shape, there are still microscopic fluctuations which are believed to be governed by universal probability distributions originating in both statistical mechanics and random matrix theory. This assertion has been proved primarily for domino and lozenge tiling models; see [22] and references therein for details.

To study these fluctuations, one of the more successful approaches has been to study the correlations of an associated particle system to the random tiling model using methods originating from random matrix theory. For many types of tiling models, these correlations are governed by the determinant of some matrix, called the correlation kernel. Probability measures of this form are known as determinantal point processes; see, for example, [42]. Finding the correlation kernel can be computationally tricky, but there are now some relatively standard approaches such as using the Eynard–Mehta Theorem [8, 24] which has been particularly useful for those in the Schur process class [7, 16, 18, 25, 40] as well as those that are not Schur processes [6, 12, 17]. Put bluntly, this theorem gives the correlation kernel when the model is expressed in terms of nonintersecting lattice paths with fixed endpoints. There are other approaches for computing correlation kernels, such as vertex operators [9, 10, 38] and also the Harish-Chandra/Itzykson–Zuber integral [36].

The Eynard–Mehta theorem has a Pfaffian analog where the final positions of the nonintersecting lattice paths are free. In this case, the correlations of the associated particle system to the tiling model are given by a Pfaffian point process; see [3] for an example where the authors give a formula for the correlation kernel for both symmetric plane partitions and plane overpartitions. The TSSCPP is another example of this and so the Eynard–Mehta theorem immediately shows that the particle system defined through the nonintersecting lattice paths for TSSCPP is a Pfaffian point process with some correlation kernel[†]. Unfortunately, the formula for this correlation kernel is not known, due to computational difficulties in inverting an arbitrary sized matrix that is found in the Eynard–Mehta theorem. In this paper, we use dimer model techniques to settle this problem and find a formula for the inverse Kasteleyn matrix for TSSCPPs, where the inverse Kasteleyn matrix can be heuristically thought of as the dimer model equivalent to the correlation kernel of a particle system. Since lozenge tilings and its associated particle system are in bijection, this implies a formula for the correlation kernel of the associated particle system.

The rest of the paper is organized as follows. In Section 2, we convert plane partitions to perfect matchings of a class of nonbipartite graphs bijectively to be able to explain our main results. We then summarize the main results of this article in Section 3, giving a formula for the inverse Kasteleyn matrix and stating a sum rule. Sections 4 and 5 are devoted to computations of this formula at special locations. Section 6 completes the proof of the formula for the inverse Kasteleyn matrix. The proof of the main result use combinatorial identities whose proof is deferred to Section 7. We present boundary recurrences for the inverse Kasteleyn matrix of independent interest, which we use to prove the sum rule, in Section 8. Finally, we end with heuristics for the limit shape and a precise conjecture in Section 9.

[†]As far as we are aware, this observation is due to Dan Romik, who gave a talk at the Clay Mathematics Institute, Oxford in May 2015 pointing out the difficulty in finding the specific form of the correlation kernel.

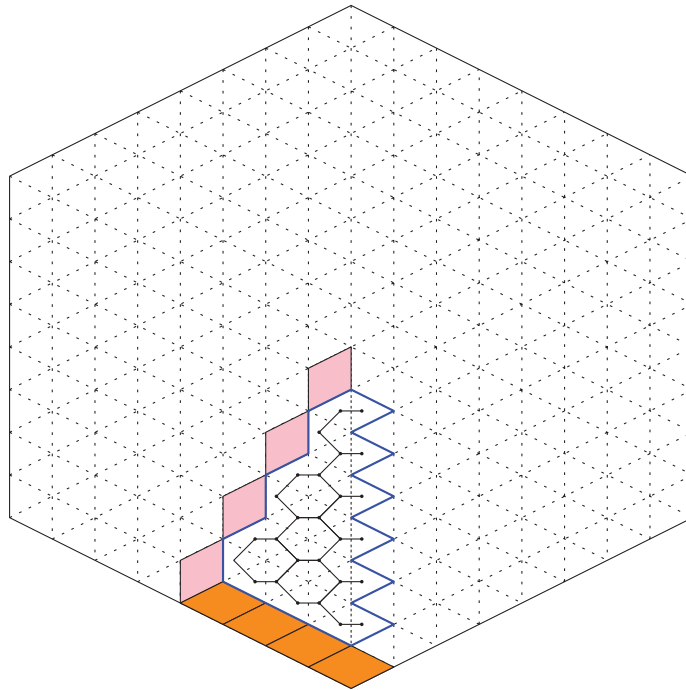


FIGURE 1 (colour online). *The region of the regular hexagon bounded by the blue lines whose tiling is sufficient to determine a totally symmetric self-complementary plane partition. The pink and orange lozenges are forced.*

2. From TSSCPPs to hexagonal graphs

As mentioned earlier, a plane partition inside an $a \times b \times c$ box can be equivalently viewed as a lozenge (or rhombus) tiling of a hexagonal region of side lengths a, b, c, a, b, c of a triangular lattice [32]. A totally symmetric self-complementary plane partition of order n is then a rhombus tiling of a regular hexagon with side length $2n$ with the maximum possible symmetry. In this case, all the information about the tiling is contained in $(1/12)$ 'th of the hexagon [34, Section 8].

This is illustrated in Figure 1, where the region enclosed by the blue lines is to be tiled with lozenges in a maximal way. This means that among the $2(n-1)$ pendant edges, only $n-1$ will be matched. This is known as a *free boundary condition*. Equivalently, we have to find maximum matchings of the dual graph drawn in black. (Recall that a *maximum matching* of a graph is one which has the largest number of matched edges.)

Let T_{n-1} denote this dual graph for TSSCPPs of size n . Figure 2 shows the dual graphs for TSSCPPs of sizes 3 and 4. Define, for $n \geq 1$,

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \quad (2.1)$$

THEOREM 2.1 (Conjectured in [34], proved in [1]). *The number of maximum matchings of T_n is given by A_{n+1} .*

Recall that T_n has $2n$ pendant vertices on the right. We now define a related family of graphs G_n starting from T_n as follows. We add $2n+1$ (respectively, $2n+2$) vertices if n is even (respectively, odd) in a column on the right of the pendant vertices and connect them in

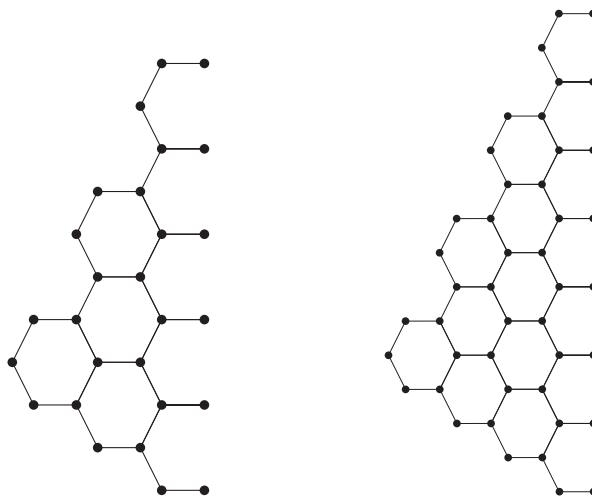


FIGURE 2. The left figure shows the graph T_3 while the right figure shows T_4 .

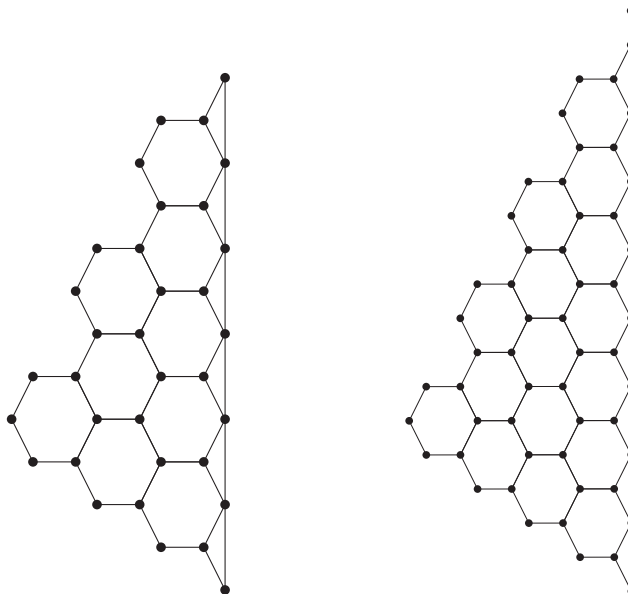


FIGURE 3. The left figure shows the graph G_3 while the right figure shows G_4 .

a triangular fashion as illustrated in Figure 3. Note that if n is odd, the topmost vertex is a leaf, that is, a pendant vertex.

PROPOSITION 2.2. *The number of perfect matchings of G_n is A_{n+1} .*

Proof. We will construct a bijection between maximum matchings of T_n and perfect matchings of G_n . Starting with a perfect matching of G_n and removing the vertices in the rightmost column, we obtain a maximum matching of T_n . To go the other way, we will show that there is a unique way to complete a maximum matching of T_n .

Consider a maximum matching as shown in Figure 4. We will now match the remaining vertices in G_n . We start from the leftmost vertex v_1 and match it to the leftmost available



FIGURE 4 (colour online). An illustration of the pendant vertices of the graph T_n embedded in G_n . The vertices in the top row are labeled u_1, \dots and those in the bottom row are labeled v_1, \dots . The maximum matching of T_n is shown in red. The rightmost vertex exists only if n is odd.

vertex. We then find the leftmost unmatched vertex and match it to the second unmatched vertex on the left. We continue this way until all vertices are matched.

To see that we do not run into a contradiction, consider the leftmost available vertex at any stage. Suppose it is v_i . If u_i is matched, we match v_i to v_{i+1} , and if not, we match it to u_i . Suppose it is u_i . Then we match it to v_{i+1} . This is always possible, since v_{i+1} cannot be matched to v_i as u_i is unmatched. The rightmost vertex has to be matched since there are an even number of unmatched vertices initially. This completes the proof. \square

3. Summary of results

To state our main result, we will need to introduce some notation. We write $[x]_2$ to denote $x \bmod 2$. For $a, m \in \mathbb{Z}$ with $m > 0$, we denote by $\Gamma_{a, a+1, \dots, a+m}$ a positively oriented contour containing the integers a through to $a+m$ and no other integers. In particular, Γ_0 is a positively oriented circle around the origin with radius less than 1. We denote $i = \sqrt{-1}$. We denote Z_G to be the number of dimer configurations of the graph G . For a subset of vertices U , we let $Z_{G \setminus U}$ denote the number of dimer configurations on the subgraph of G induced by removing U . When G_n is the TSSCPP graph of size n defined above, we write $Z_n \equiv Z_{G_n}$. We will also use the notation $Z_n^{\{v_1, \dots, v_m\}} \equiv Z_{G_n \setminus \{v_1, \dots, v_m\}}$, that is the partition function of the dimer model on the induced graph of G_n on the vertex set $V_n \setminus \{v_1, \dots, v_m\}$. Finally, we will use the convention that $\binom{m}{-1} = 0$ for all $m \in \mathbb{N} \cup \{0\}$ throughout the paper.

For the graph $G_n = (V_n, E_n)$, the coordinates of the vertices are given by

$$V_n = \{(x_1, x_2) \mid 0 \leq x_1 \leq 2n, x_1 \leq x_2 \leq 2n+1\} \setminus \{(2n, 2n+1) \mathbb{1}_{n \in 2\mathbb{Z}+1}\} \quad (3.1)$$

and the edges are given by

$$\begin{aligned} E_n = & \{((x_1, x_2), (x_1+1, x_2)) \mid 0 \leq x_1 \leq 2n-1, x_1 \leq x_2 \leq 2n+1, [x_1+x_2]_2 = 1\} \\ & \cup \{((x_1, x_2), (x_1, x_2+1)) \mid 0 \leq x_1 \leq 2n-1, x_1 \leq x_2 \leq 2n\} \\ & \cup \{((x_1, x_1), (x_1+1, x_1+1)) \mid 0 \leq x_1 \leq 2n-1\}; \end{aligned} \quad (3.2)$$

see Figure 5.

We let \mathbf{b} denote the vertex $(2n, 2n+1 - [n]_2)$. The entries of the skew-symmetric Kasteleyn matrix, $K_n = K_n(x, y)_{x, y \in V_n}$ are given by

$$K_n((x_1, x_2), (y_1, y_2)) = k_n((x_1, x_2), (y_1, y_2)) - k_n((y_1, y_2), (x_1, x_2)) \quad (3.3)$$

where

$$k_n((x_1, x_2), (y_1, y_2)) = \begin{cases} 1 & \text{if } [x_1+x_2]_2 = 0, x_2 = y_2, x_1 - y_1 = 1 \\ 1 & \text{if } [x_1+x_2]_2 = 0, |y_2 - x_2| = 1, x_1 = y_1 \\ 1 & \text{if } x_1 = x_2, y_1 = y_2 = x_1 - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

Here the Kasteleyn orientation is chosen so that the number of counter-clockwise arrows around each face is an odd number. From [27, 28], $|\text{Pf}K_n|$ is equal to Z_n .

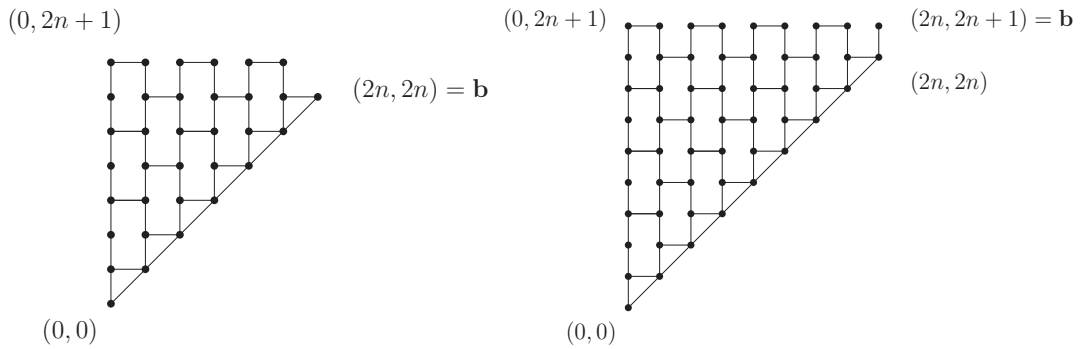


FIGURE 5. The left figure shows G_3 , while the right figures shows G_4 . The coordinates are for the top left, rightmost and bottom left vertices. The Kasteleyn orientation is chosen so that vertices with odd parity are sinks, those with even parity not on the diagonal are sources, and the edges along the diagonal point toward the origin.

Define

$$p(n, k, \ell) = \frac{(n + k - 2\ell + 1)!(2n - k - \ell + 1)!}{(k - \ell)!(3n - k + 2 - 2\ell)!} (-1)^k (3n - 3k + 2) C_{n-k}, \tag{3.5}$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n 'th Catalan number. We now introduce notation helpful for stating the main theorem. Recall that Γ_0 is a positively oriented circle around the origin with radius less than 1. Introduce the following formulas for $0 \leq i \leq 2n - 1$:

$$h_n^{0,\mathbf{b}}(i) = -[i + 1]_2 + \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{(1-r)r^{i-2k}}, \tag{3.6}$$

and

$$h_n^{1,\mathbf{b}}(i) = \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{r^{i-2k+1}}. \tag{3.7}$$

We will first state the formula for K_n^{-1} in the special case when the second vertex is \mathbf{b} . This will prove useful for the general result.

THEOREM 3.1. *If $(x_1, x_2) = (i_1, i_1 + 2i_2 + 1)$ with $0 \leq i_2 \leq n - \lceil i_1/2 \rceil$, then*

$$K_n^{-1}((x_1, x_2), \mathbf{b}) = \sum_{l=0}^{i_1} (-1)^{i_2+l} \binom{i_2 - 1 + l}{l} h_n^{1,\mathbf{b}}(i_1 - l), \tag{3.8}$$

and if $(x_1, x_2) = (i_1, i_1 + 2i_2)$ with $0 \leq i_2 \leq n - \lfloor i_1/2 \rfloor$, then

$$K_n^{-1}((x_1, x_2), \mathbf{b}) = \sum_{l=0}^{i_2} (-1)^{i_2} \binom{i_2}{l} h_n^{0,\mathbf{b}}(i_1 + l). \tag{3.9}$$

Finally, when n is even, $K_n^{-1}((2n, 2n), \mathbf{b}) = -1$.

Introduce the following formulas for $0 \leq i, j \leq 2n - 1$:

$$\begin{aligned}
t_n^{0,0}(i, j) &= \mathbb{1}_{[i < j]} [i + 1]_2 [j]_2 - \mathbb{1}_{[i > j]} [i]_2 [j + 1]_2 + [j + 1]_2 h_n^{0,\mathbf{b}}(i) - [i + 1]_2 h_n^{0,\mathbf{b}}(j) \\
&+ \sum_{k_1=0}^n \sum_{\ell_1=0}^{k_1} \sum_{k_2=0}^n \sum_{\ell_2=0}^{k_2} p(n, k_1, \ell_1) p(n, k_2, \ell_2) \frac{1}{(2\pi i)^4} \int_{\Gamma_0} dr_1 \int_{\Gamma_0} dr_2 \int_{\Gamma_0} ds_1 \int_{\Gamma_0} ds_2 \\
&\times \frac{(1+r_1)^{n-k_1} (1+r_2)^{n-k_2}}{(1-r_1) r_1^{i-2k_1} (1-r_2) r_2^{j-2k_2}} \frac{s_1 - s_2}{(s_1 s_2 - 1) s_1^{\ell_1+1} s_2^{\ell_2+1}},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
t_n^{1,0}(i, j) &= [j + 1]_2 h_n^{1,\mathbf{b}}(i) \\
&+ \sum_{k_1=0}^n \sum_{\ell_1=0}^{k_1} \sum_{k_2=0}^n \sum_{\ell_2=0}^{k_2} p(n, k_1, \ell_1) p(n, k_2, \ell_2) \frac{1}{(2\pi i)^4} \int_{\Gamma_0} dr_1 \int_{\Gamma_0} dr_2 \int_{\Gamma_0} ds_1 \int_{\Gamma_0} ds_2 \\
&\times \frac{(1+r_1)^{n-k_1} (1+r_2)^{n-k_2}}{r_1^{i-2k_1+1} (1-r_2) r_2^{j-2k_2}} \frac{s_1 - s_2}{(s_1 s_2 - 1) s_1^{\ell_1+1} s_2^{\ell_2+1}},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
t_n^{0,1}(i, j) &= -[i + 1]_2 h_n^{1,\mathbf{b}}(j) \\
&+ \sum_{k_1=0}^n \sum_{\ell_1=0}^{k_1} \sum_{k_2=0}^n \sum_{\ell_2=0}^{k_2} p(n, k_1, \ell_1) p(n, k_2, \ell_2) \frac{1}{(2\pi i)^4} \int_{\Gamma_0} dr_1 \int_{\Gamma_0} dr_2 \int_{\Gamma_0} ds_1 \int_{\Gamma_0} ds_2 \\
&\times \frac{(1+r_1)^{n-k_1} (1+r_2)^{n-k_2}}{(1-r_1) r_1^{i-2k_1} r_2^{j-2k_2+1}} \frac{s_1 - s_2}{(s_1 s_2 - 1) s_1^{\ell_1+1} s_2^{\ell_2+1}},
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
t_n^{1,1}(i, j) &= \sum_{k_1=0}^n \sum_{\ell_1=0}^{k_1} \sum_{k_2=0}^n \sum_{\ell_2=0}^{k_2} p(n, k_1, \ell_1) p(n, k_2, \ell_2) \frac{1}{(2\pi i)^4} \int_{\Gamma_0} dr_1 \int_{\Gamma_0} dr_2 \int_{\Gamma_0} ds_1 \int_{\Gamma_0} ds_2 \\
&\times \frac{(1+r_1)^{n-k_1} (1+r_2)^{n-k_2}}{r_1^{i-2k_1+1} r_2^{j-2k_2+1}} \frac{s_1 - s_2}{(s_1 s_2 - 1) s_1^{\ell_1+1} s_2^{\ell_2+1}}.
\end{aligned} \tag{3.13}$$

We are now ready to state the formula for K_n^{-1} . Below, we make repeated use of $K_n^{-1}(x, y) = -K_n^{-1}(y, x)$ for $x, y \in \mathbf{V}_n$ since K_n is antisymmetric.

THEOREM 3.2. *Suppose that $(x_1, x_2) = (i_1, i_1 + 2i_2 + \epsilon_i)$, $(y_1, y_2) = (j_1, j_1 + 2j_2 + \epsilon_j)$ with $0 \leq i_1, j_1 \leq 2n - 1$ and $\epsilon_i, \epsilon_j \in \{0, 1\}$, where $0 \leq i_2 \leq n - \lfloor (i_1 + \epsilon_i)/2 \rfloor$, and $0 \leq j_2 \leq n - \lfloor (j_1 + \epsilon_j)/2 \rfloor$.*

- If $\epsilon_i = \epsilon_j = 1$, then

$$\begin{aligned}
K_n^{-1}((x_1, x_2), (y_1, y_2)) &= \sum_{\ell_1=0}^{i_1} \sum_{\ell_2=0}^{j_1} (-1)^{i_2+j_2} (-1)^{\ell_1+\ell_2} \\
&\times \binom{i_2 - 1 + \ell_1}{\ell_1} \binom{j_2 - 1 + \ell_2}{\ell_2} t_n^{1,1}(i_1 - \ell_1, j_1 - \ell_2).
\end{aligned} \tag{3.14}$$

- If $\epsilon_i = \epsilon_j = 0$, then

$$K_n^{-1}((x_1, x_2), (y_1, y_2)) = \sum_{\ell_1=0}^{i_2} \sum_{\ell_2=0}^{j_2} (-1)^{i_2+j_2} \binom{i_2}{\ell_1} \binom{j_2}{\ell_2} t_n^{0,0}(i_1 + \ell_1, j_1 + \ell_2). \tag{3.15}$$

- If $\epsilon_i = 1$ and $\epsilon_j = 0$, then

$$K_n^{-1}((x_1, x_2), (y_1, y_2)) = \sum_{\ell_1=0}^{i_1} \sum_{\ell_2=0}^{j_2} (-1)^{i_2+j_2} (-1)^{\ell_1} \binom{i_2-1+\ell_1}{\ell_1} \binom{j_2}{\ell_2} \times t_n^{1,0}(i_1 - \ell_1, j_1 + \ell_2) - (-1)^{i_2+j_2} \mathbb{1}_{x_1 \geq y_1} \mathbb{1}_{x_1+x_2 < y_1+y_2} \binom{j_2-i_2-1}{i_1-j_1}. \tag{3.16}$$

Note that the case of $\epsilon_i = 0$ and $\epsilon_j = 1$ in Theorem 3.2 is settled by the antisymmetry of K_n . Theorems 3.1 and 3.2 could be proven in principle by verifying the matrix equation $K_n \cdot K_n^{-1} = \mathbb{I}$ directly. However, this approach seems computationally very difficult.

Our proof strategy is as follows. We first establish Theorem 3.1 in Section 4. Once this is established, we use graphical condensation [31] to establish $K_n^{-1}(x, y)$ for x and y on the top boundary of the TSSCPP. We can then recover the other entries using recursions obtained from the matrix equations $K_n^{-1} \cdot K_n = K_n \cdot K_n^{-1} = \mathbb{I}$ viewed entrywise. The proof is given in Section 6, while postponed proofs of results used in proving Theorem 3.2 are given in Section 7.

This leads to the question whether there is a set of recurrences that establishes K_n^{-1} uniquely, without relying on a priori expressions for $K_n^{-1}(\cdot, \mathbf{b})$. Such recurrences are known for domino tilings of the Aztec diamond [13] and are expected to hold for lozenge tilings [37]. In fact, there is an additional recurrence, along with the recurrence from graphical condensation given in Lemma 5.3 which parametrizes K_n^{-1} on the boundary. These two recurrences give a concrete and explicit example of the additional recurrences postulated in [37] for lozenge tilings. These recurrences combined with the recurrences from the matrix equations $K_n \cdot K_n^{-1} = K_n^{-1} \cdot K_n = \mathbb{I}$ give a unique way to determine K_n^{-1} . This additional recurrence is given in Section 8.

One consequence of finding K_n^{-1} is a formula to compute local statistics [30, 35]. We restate this here in a form that applies to our situation.

THEOREM 3.3 (Montroll–Potts–Ward [35], Kenyon [30]). *The probability of edges $e_1 = (v_1, v_2), \dots, e_m = (v_{2m-1}, v_{2m})$ on \mathbf{V}_n is given by*

$$\mathbb{P}[e_1, \dots, e_m] = \prod_{k=1}^m K_n(v_{2k-1}, v_{2k}) \text{Pf}(K_n^{-1}(v_i, v_j))_{1 \leq i, j \leq 2m}^T. \tag{3.17}$$

The term ‘sum rule’ refers to certain sums of correlation functions that are of importance. In many cases, these sum rules have simple formulas. For the so-called quantum Knizhnik–Zamolodchikov equation, sum rules have been conjectured relating them to the q -enumeration of TSSCPPs by Di Francesco [14], and Di Francesco and Zinn-Justin [15, 49] and proved by Zeilberger [48].

We now state a formula for a sum rule for TSSCPPs. For $0 \leq j \leq 2n - 1$, let

$$g_n^{\mathbf{b}}(j) = |K_n^{-1}((j, j), \mathbf{b})| = (-1)^{j+1} K_n^{-1}((j, j), \mathbf{b}) = \frac{Z_n^{\{(j, j), \mathbf{b}\}}}{Z_n}. \tag{3.18}$$

THEOREM 3.4.

$$\sum_{j=0}^{2n} g_n^{\mathbf{b}}(j) = \begin{cases} n + 1 & n \text{ even,} \\ n + \frac{1}{2} & n \text{ odd.} \end{cases}$$

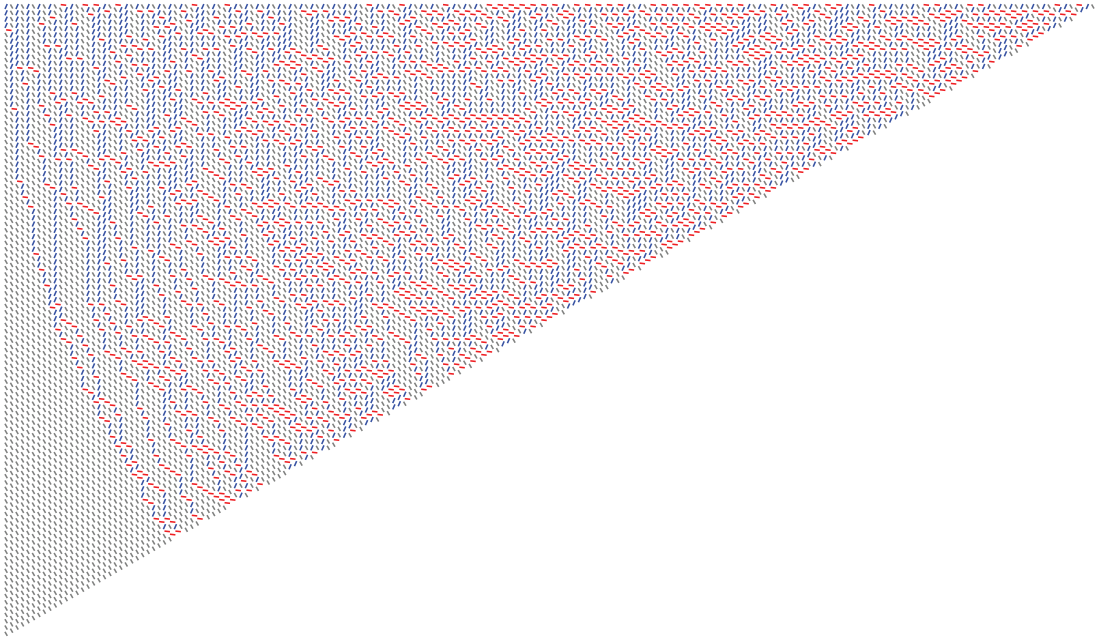


FIGURE 6 (colour online). A simulation, using Glauber dynamics, of a uniformly random TSSCPP of size 100. Here, we have rotated the hexagonal graph in Figure 3 by $\pi/6$. The simulations only show the dimers on this graph, drawn in different three different colors (see online version for colors).

We remark that the entries $g_n^b(j)$ themselves are not so simple. For example,

$$(g_2^b(j))_{j=0}^4 = \left(1, \frac{2}{7}, \frac{1}{7}, \frac{4}{7}, 1\right),$$

and

$$(g_3^b(j))_{j=0}^6 = \left(1, \frac{1}{6}, \frac{1}{3}, \frac{11}{14}, \frac{17}{21}, \frac{17}{42}, 0\right).$$

The formula given in Theorem 3.2 is not in the most convenient form for asymptotic analysis. Nonrigorous computations show that the terms in the formula can be approximated by double contour integral formulas, that share some similarities with lozenge tiling models in the Schur class, for example, see [23] and references therein. Furthermore, these nonrigorous computations show a limit shape and Airy kernel statistics at the edge, which also appears in simulations; see Figure 6. We will not attempt to pursue these here as the computations appear to be long and involved. However, we will give a conjecture of the limit shape, based on a short computation in Section 9.

CONJECTURE 3.5. Rescale the TSSCPP of size n so that the three corners are given by $(-2, 0)$, $(0, 0)$ and $(-2, -2/\sqrt{3})$ as $n \rightarrow \infty$, that is, for $(x_1, x_2) \in \mathbb{V}_n$, rescale so that $x_1 = [(X + 2)n]$ and $x_2 = [(\sqrt{3}Y + 2)n]$. Then, the limit shape curve is given by

$$X^2 + Y^2 = 4, \tag{3.19}$$

so that the region $X^2 + Y^2 \geq 4$ in the rescaled TSSCPP is frozen.

4. Proof of Theorem 3.1

Before giving the proof of Theorem 3.1, we give two combinatorial identities which will be used below. Their proofs are given in Section 7.

THEOREM 4.1. Let $n \in \mathbb{N}$. Let

$$f_n(k) = p(n, k, 0)2^{n-k} = \frac{(n+k+1)!(2n-k+1)!}{k!(3n-k+2)!}(-1)^k 2^{n-k} (3n-3k+2)C_{n-k}.$$

Then

$$\sum_{k=0}^n f_n(k) = \frac{1}{2}(1 + (-1)^n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Krattenthaler has kindly shown us how to derive Theorem 4.1 starting from an identity in the book by Gasper and Rahman [21, Equation (3.8.12)]. However, we use Zeilberger's algorithm to prove it in Section 7.

THEOREM 4.2. Let $n \in \mathbb{N}$ and $0 \leq i \leq n$. Define

$$g_{n,i}(k, j) = \frac{1}{2^{n-k}} f_n(k) \binom{2n+1-i-k}{j}.$$

Then, we have

$$G_{n,i} = \sum_{k=0}^n \sum_{j=0}^{n+i-2k} g_{n,i}(k, j) = 2^{n-i}, \quad (4.1)$$

and

$$G'_{n,i} = \sum_{k=0}^n \sum_{j=0}^{n+k-2i} g_{n,i}(k, j) = (-1)^n 2^{n-i}. \quad (4.2)$$

The proofs of Theorem 4.1, and 4.2 are given in Section 7. First, we need the following two results.

PROPOSITION 4.3. When n is even, we have $K^{-1}((2n, 2n), \mathbf{b}) = -1$.

Proof. Note that for n even, $K_n((2n, 2n), \mathbf{b}) = 1$. By Theorem 3.3, the probability of observing the edge $((2n, 2n), \mathbf{b})$ is equal to

$$K_n((2n, 2n), \mathbf{b})K_n^{-1}(\mathbf{b}, (2n, 2n)) = -K_n^{-1}((2n, 2n), \mathbf{b}).$$

Since the edge $((2n, 2n), \mathbf{b})$ is always observed when n is even, the result follows. \square

PROPOSITION 4.4. From the formulas for $K_n^{-1}(\cdot, \mathbf{b})$ given in (3.8) and (3.9), we have that

$$\begin{aligned} & K_n^{-1}((2n-2, 2n-2), \mathbf{b}) + K_n^{-1}((2n-2, 2n-1), \mathbf{b}) + K_n^{-1}((2n-1, 2n), \mathbf{b}) \\ &= \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \quad (4.3)$$

Proof. We expand out the left side using the definitions of $K_n^{-1}(\cdot, \mathbf{b})$ as given in (3.8) and (3.9) which involves the formulas (3.7) and (3.6). This gives

$$\begin{aligned} & \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \left(\frac{(1+r)^{n-k}}{(1-r)r^{2n-2-2k}} + \frac{(1+r)^{n-k}}{r^{(2n-2)+1-2k}} + \frac{(1-r)^{n-k}}{r^{2n-1+1-2k}} \right) \\ &= \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{(1-r)r^{2n-2k}} = \sum_{k=0}^n p(n, k, 0) 2^{n-k}, \end{aligned} \quad (4.4)$$

where the first equality follows from simplifying the integrand and the last equality follows from pushing the contour through infinity and computing the residue at $r = 1$. The claim follows from Theorem 4.1. \square

We will also repeatedly use the standard integrals,

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{(1+r)^b}{r^a} = \binom{b}{a}, \quad (4.5)$$

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{(1+r)^b}{(1-r)r^a} = \sum_{j=0}^a \binom{b}{j}, \quad (4.6)$$

$$\int_0^\infty dr r^a e^{-r} = a!, \quad (4.7)$$

and

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{e^r}{r^a} = \frac{1}{a!} \quad (4.8)$$

for positive integers a and b .

Proof of Theorem 3.1. We will prove the following, assuming (3.8) and (3.9).

$$K_n \cdot K_n^{-1}(x, \mathbf{b}) = \mathbb{I}_{x=\mathbf{b}}. \quad (4.9)$$

The determinant of K_n is nonzero because it is the square of the partition function. Therefore the equations from (4.9) are linearly independent and they guarantee the uniqueness of $K_n^{-1}(\cdot, \mathbf{b})$. For the purposes of the proof, we introduce

$$E_n = \{(x_1, x_2) \in \mathbf{V}_n \setminus \{\mathbf{b}\} : [x_1 + x_2]_2 = 0\}, \quad (4.10)$$

$$O_n = \{(x_1, x_2) \in \mathbf{V}_n \setminus \{\mathbf{b}\} : [x_1 + x_2]_2 = 1\}, \quad (4.11)$$

and

$$D_n = \{(x_1, x_2) \in E_n : x_1 = x_2\}. \quad (4.12)$$

We will verify (4.9) in each of the four cases: $x \in E_n \setminus D_n$, $x \in D_n$, $x = \mathbf{b}$ and finally $x \in O_n$.

Case (i): We first consider $x = (x_1, x_2) \in E_n \setminus D_n$. Expanding out the left side of (4.9) entrywise gives

$$K_n^{-1}(x + (-1, 0), \mathbf{b}) \mathbb{1}_{x_1 > 0} + K_n^{-1}(x + (0, 1), \mathbf{b}) \mathbb{1}_{x_2 < 2n+1} + K_n^{-1}(x + (0, -1), \mathbf{b}). \quad (4.13)$$

Write $x = (i_1, i_1 + 2i_2)$ and suppose that $x_1 \neq 0$ and $x_2 \neq 2n + 1$. Then (4.13) becomes by (3.8)

$$\begin{aligned}
& (-1)^{i_2} \left(\sum_{\ell=0}^{i_1-1} (-1)^\ell \binom{i_2-1+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1-\ell-1) \right. \\
& \quad \left. + \sum_{\ell=0}^{i_1} (-1)^\ell \binom{i_2-1+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1-\ell) - \sum_{\ell=0}^{i_1} (-1)^\ell \binom{i_2-2+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1-\ell) \right) \\
& = (-1)^{i_2} \left(\sum_{\ell=0}^{i_1-1} (-1)^\ell \binom{i_2-1+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1-\ell-1) \right. \\
& \quad \left. + \sum_{\ell=0}^{i_1} (-1)^\ell \binom{i_2-2+\ell}{\ell-1} h_n^{1,\mathbf{b}}(i_1-\ell) \right) = 0,
\end{aligned} \tag{4.14}$$

where we have used $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ in the last step. This verifies (4.9) when $x = (x_1, x_2) \in E_n \setminus D_n$ and $x_1 \neq 0$ and $x_2 \neq 2n + 1$. We now consider (4.13) when $x_1 = 0$. Note that $K_n^{-1}((0, 2i_2 + 1), \mathbf{b}) = h_n^{1,\mathbf{b}}(0)(-1)^{i_2}$ and so by (3.8) we have also verified (4.9) for $(0, x_2) \in E_n \setminus D_n$. Next we consider when $x_2 = 2n + 1$, which means that (4.13) becomes

$$K_n^{-1}(x + (-1, 0), \mathbf{b}) \mathbb{1}_{x_1 > 0} + K_n^{-1}(x + (0, -1), \mathbf{b}). \tag{4.15}$$

Take $x + (-1, 0) = (x_1 - 1, 2n + 1) = (i_1, 2n + 1)$ so that $i_1 = x_1 - 1$. This means that we have

$$x + (-1, 0) = \left(i_1, i_1 + 2 \frac{2n - i_1}{2} + 1 \right)$$

and

$$x + (0, -1) = \left(i_1 + 1, i_1 + 1 + 2 \frac{2n - i_1 - 2}{2} + 1 \right).$$

Using (3.8), we have that (4.15) becomes

$$\begin{aligned}
& \sum_{\ell=0}^{i_1} (-1)^{n-\frac{i_1}{2}+\ell} \binom{n-\frac{i_1}{2}-1+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1-\ell) \\
& \quad + \sum_{\ell=0}^{i_1+1} (-1)^{n-\frac{i_1}{2}-1+\ell} \binom{n-\frac{i_1}{2}-2+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1+1-\ell) \\
& = - \sum_{\ell=0}^{i_1+1} (-1)^{n-i_1/2+\ell} \binom{n-\frac{i_1}{2}-1+\ell}{\ell} h_n^{1,\mathbf{b}}(i_1+1-\ell),
\end{aligned} \tag{4.16}$$

where the last equation is found from rearranging the relation in (4.14). We need to show that the right side of the above equation is equal to 0. To do so, we expand out the above term using (3.7) which gives

$$(-1)^{n-i_1/2} \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{(1+r)^{n-k} r^{2k}}{r^{i_1+1}} \sum_{\ell=0}^{i_1+1} (-r)^\ell \binom{n-i_1/2+\ell-1}{\ell}. \tag{4.17}$$

The inner sum in the integrand equals

$$\sum_{\ell=0}^{i_1+1} (-r)^\ell \binom{n-i_1/2+\ell-1}{\ell} = (1+r)^{\frac{1}{2}(i_1-2n)} \quad (4.18)$$

$$+ (-1)^{1+i_1} r^{2+i_1} \binom{n+i_1/2+1}{2+i} {}_2F_1 \left[\begin{matrix} 1, 2+i_1/2+n \\ 3+i \end{matrix}; -r \right],$$

where the ${}_2F_1$ terms is the Gaussian hypergeometric function. Since the hypergeometric term is nonsingular in r , we see that (4.17) is equal to

$$(-1)^{n-i_1/2} \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} (1+r)^{i_1/2-k} r^{2k-(i_1+1)}. \quad (4.19)$$

By (4.5), the integral above is equal to $\binom{i_1/2-k}{i_1+1-2k} = 0$. We have thus verified (4.9) for $x \in E_n \setminus D_n$.

Case (ii): Next we consider $x = (x_1, x_2) \in D_n$ provided that $x_1 < 2n - 1$. For this case, we have that the left side of (4.9) entrywise is equal to

$$(K_n^{-1}(x - (1, 1), \mathbf{b}) + K_n^{-1}(x - (1, 0), \mathbf{b})) \mathbb{1}_{x_1 > 0} + K_n^{-1}(x + (0, 1), \mathbf{b}) - K_n^{-1}(x + (1, 1), \mathbf{b})). \quad (4.20)$$

When $0 < x_1 < 2n - 1$, the integral contribution in (4.20) from using (3.6) and (3.7) is

$$\frac{1}{2\pi i} \int_{\Gamma_0} dr (1+r)^{n-k} r^{2k} \left(\frac{r^{-x_1+1}}{1-r} + r^{-x_1} + r^{-x_1-1} - \frac{r^{-x_1-1}}{1-r} \right) \quad (4.21)$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{1-r} r^{-x_1-1+2k} (r^2 + (1-r)r + (1-r) - 1) = 0,$$

which verifies (4.9) when $x = (x_1, x_2) \in D_n$ for $0 < x_1 < 2n$. When $x_1 = 0$, we have that (4.20) becomes

$$K_n^{-1}((0, 1), \mathbf{b}) - K_n^{-1}((1, 1), \mathbf{b}) = 0 \quad (4.22)$$

by explicit evaluation (both terms have a contribution from $k = 0$ in the sum in (3.6) and (3.7)). When $x = (x_1, x_2) \in D_n$ and $x_1 = 2n - 1$, the left side of (4.9) entrywise is equal to

$$K_n^{-1}((2n-2, 2n-2), \mathbf{b}) + K_n^{-1}((2n-2, 2n-1), \mathbf{b}) + K_n^{-1}((2n-1, 2n), \mathbf{b}) - K_n^{-1}(\mathbf{b}, \mathbf{b}) \quad (4.23)$$

if n is odd, and

$$K_n^{-1}((2n-2, 2n-2), \mathbf{b}) + K_n^{-1}((2n-2, 2n-1), \mathbf{b}) \quad (4.24)$$

$$+ K_n^{-1}((2n-1, 2n), \mathbf{b}) - K_n^{-1}((2n, 2n), \mathbf{b})$$

if n is even. From Propositions 4.4 and 4.3, we have that both (4.23) and (4.24) are equal to 0.

Case (iii): Next we consider the left side of (4.9) entrywise when $x = \mathbf{b}$ when n is odd (the even case is immediate from Proposition 4.3). Expanding out these terms using (3.8), (3.9), (3.7), and (3.6) gives

$$\begin{aligned}
 & K_n^{-1}((2n - 1, 2n), \mathbf{b}) + K_n^{-1}((2n - 1, 2n - 1), \mathbf{b}) \\
 &= \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{r^{2n-2k}} \left(1 + \frac{r}{1-r}\right) \\
 &= \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{r^{2n-2k}} \frac{1}{1-r} \\
 &= \sum_{k=0}^{n-1} p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{r^{2n-2k}} \frac{1}{1-r} \\
 &= \sum_{k=0}^{n-1} p(n, k, 0) 2^{n-k} = \frac{1}{2}(1 + (-1)^n) - (-1)^n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}
 \end{aligned} \tag{4.25}$$

where the third equality follows from the absence of a residue at $r = 0$ when $k = n$, the fourth equality follows from (4.5), and we have used Theorem 4.1 for the fifth equality. We also need the left side of (4.9) entrywise when $x = (2n, 2n)$ which is given by

$$K_n^{-1}((2n - 1, 2n), \mathbf{b}) + K_n^{-1}((2n - 1, 2n - 1), \mathbf{b}) - K_n^{-1}(\mathbf{b}, \mathbf{b}) = 0, \tag{4.26}$$

where the sum of the first two terms follows from the proceeding computation and we have used the antisymmetry of K_n^{-1} . We have now verified (4.9) for $x \in E_n \cup \{\mathbf{b}\}$.

Case (iv): Finally, we verify (4.9) for $x \in O_n$. When $x = (x_1, x_2) \in O_n$, we have that the left side of (4.9) entrywise equals

$$-K_n^{-1}(x + (0, 1), \mathbf{b}) \mathbb{1}_{x_2 < 2n+1} - K_n^{-1}(x + (0, -1), \mathbf{b}) - K_n^{-1}(x + (1, 0), \mathbf{b}). \tag{4.27}$$

If $x_2 < 2n + 1$, using (3.9) and writing $x = (x_1, x_2) = (i_1, i_1 + 2i_2 + 1)$, we have that (4.27) becomes

$$\begin{aligned}
 & - \sum_{\ell=0}^{i_2+1} (-1)^{i_2+1} \binom{i_2+1}{\ell} h_n^{0,\mathbf{b}}(i_1 + \ell) - \sum_{\ell=0}^{i_2} (-1)^{i_2} \binom{i_2}{\ell} h_n^{0,\mathbf{b}}(i_1 + \ell) \\
 & - \sum_{\ell=0}^{i_2} (-1)^{i_2} \binom{i_2}{\ell} h_n^{0,\mathbf{b}}(i_1 + \ell + 1) = 0.
 \end{aligned} \tag{4.28}$$

Finally, if $x = (x_1, 2n + 1)$, then (4.27) becomes

$$-K_n^{-1}(x + (0, -1), \mathbf{b}) - K_n^{-1}(x + (1, 0), \mathbf{b}). \tag{4.29}$$

This means that we have

$$x + (0, -1) = \left(x_1, x_1 + 2\frac{2n - x_1}{2}\right)$$

and

$$x + (1, 0) = \left(x_1 + 1, x_1 + 1 + 2\frac{2n - x_1}{2}\right).$$

Using (3.9), we have that (4.29) becomes

$$\begin{aligned} & - \sum_{\ell=0}^{n-\frac{x_1}{2}} (-1)^{n-\frac{x_1}{2}} \binom{n-\frac{x_1}{2}}{\ell} h_n^{0,\mathbf{b}}(x_1 + \ell) - \sum_{\ell=0}^{n-\frac{x_1}{2}} (-1)^{n-\frac{x_1}{2}} \binom{n-\frac{x_1}{2}}{\ell} h_n^{0,\mathbf{b}}(x_1 + 1 + \ell) \\ & = \sum_{\ell=0}^{n+1-\frac{x_1}{2}} (-1)^{n+1-\frac{x_1}{2}} \binom{n+1-\frac{x_1}{2}}{\ell} h_n^{0,\mathbf{b}}(x_1 + \ell), \end{aligned} \quad (4.30)$$

where we have used (4.28). We need to show the right side of the above equation equals 0. Expanding the above term out using the formula for (3.6) gives

$$\begin{aligned} & \sum_{k=0}^n \frac{p(n, k, 0)}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k} (1+1/r)^{n+1-x_1/2} r^{2k}}{(1-r)r^i} - \sum_{k=0}^{n+1-x_1/2} \binom{n+1-\frac{x_1}{2}}{k} [k+1]_2 \\ & = \sum_{k=0}^n \frac{p(n, k, 0)}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{2n-k+1-x_1/2}}{(1-r)r^{n+1+x_1/2-2k}} - 2^{n-x_1/2} \\ & = \sum_{k=0}^n p(n, k, 0) \sum_{j=0}^{n+\frac{x_1}{2}-2k} \binom{2n+1-\frac{x_1}{2}-k}{j} - 2^{n-x_1/2}, \end{aligned} \quad (4.31)$$

where we have used (4.6) for the second equality. We use Theorem 4.2 on the first term of the last line of the above equation and so the above equation is equal to 0 as required.

We have thus verified (4.9) for all x , completing the proof. \square

5. Both vertices at the top boundary

We will first establish a formula for K_n^{-1} when both entries are top at the boundary using a series of lemmas in Section 5.

REMARK 5.1. We will repeatedly use the following property of perfect matchings. If a leaf ℓ is adjacent to a vertex v via an edge with weight 1, then removing both ℓ and v from the graph does not change the partition function.

We will make use of the following result below, which follows immediately from Theorem 3.3 by multiplying both sides of the equation in Theorem 3.3 by Z_G .

PROPOSITION 5.2 (Graphical Condensation [31]). *Let G be a plane graph with four vertices a, b, c, d that appear in that cyclic order on a face of G . Then*

$$Z_G Z_{G \setminus \{a,b,c,d\}} + Z_{G \setminus \{a,c\}} Z_{G \setminus \{b,d\}} = Z_{G \setminus \{a,b\}} Z_{G \setminus \{c,d\}} + Z_{G \setminus \{a,d\}} Z_{G \setminus \{b,c\}}. \quad (5.1)$$

Recall that $Z_n^{\{v_1, \dots, v_m\}}$ is the partition function of the dimer model on G_n with the vertices $\{v_1, \dots, v_m\}$ removed. Introduce for $0 \leq i \leq n-1$,

$$T_n(i) = |K_n^{-1}((2i, 2n+1), \mathbf{b})| = \frac{Z_n^{\{(2i, 2n+1), \mathbf{b}\}}}{Z_n}, \quad (5.2)$$

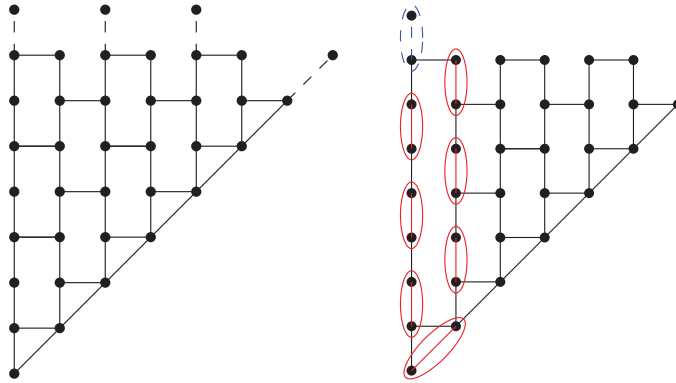


FIGURE 7 (colour online). The left figure shows adding pendant vertices to $\mathbf{a} = (0, 7)$, $\mathbf{1} = (2, 7)$, $\mathbf{2} = (4, 7)$ and \mathbf{b} from V_3 (here $n = 3$) for considering $Z_3^{\{\mathbf{a}, \mathbf{1}, \mathbf{2}, \mathbf{b}\}}$. These pendant edges are drawn in dashed. Note that removing a vertex is the equivalent of adding a pendant edge to that vertex. The right figure shows that adding a pendant vertex to \mathbf{a} forces the blue dashed dimer as well as the red solid dimers (see online version for colors).

and for $0 \leq i < j \leq n - 1$,

$$R_n(i, j) = |K_n^{-1}((2i, 2n + 1), (2j, 2n + 1))| = \frac{Z_n^{\{(2i, 2n+1), (2j, 2n+1)\}}}{Z_n}, \tag{5.3}$$

with $R_n(i, j) = -R_n(j, i)$ if $1 \leq j < i \leq n - 1$ and $R_n(i, i) = 0$ for $0 \leq i \leq n - 1$. We use the convention that $T_n(k) = 0$ for $k \geq n$, $R_0 = 1$ and that $R_n(k, l) = 0$ if $k \geq n$ or $l \geq n$.

LEMMA 5.3. For $1 \leq i, j \leq n - 1$, we have

$$R_n(i, j) = R_{n-1}(i - 1, j - 1) + T_n(i)T_{n-1}(j - 1) - T_{n-1}(i - 1)T_n(j). \tag{5.4}$$

Proof. For the purpose of the proof, write $\mathbf{i} = (2i, 2n + 1)$ for $1 \leq i \leq n - 1$ and let $\mathbf{a} = (0, 2n + 1)$. For $i < j$, we apply graphical condensation in Proposition 5.2[†] which gives

$$Z_n^{\{\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}\}} Z_n = Z_n^{\{\mathbf{a}, \mathbf{i}\}} Z_n^{\{\mathbf{j}, \mathbf{b}\}} - Z_n^{\{\mathbf{a}, \mathbf{j}\}} Z_n^{\{\mathbf{i}, \mathbf{b}\}} + Z_n^{\{\mathbf{a}, \mathbf{b}\}} Z_n^{\{\mathbf{i}, \mathbf{j}\}}. \tag{5.5}$$

since $\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}$ are in cyclic order on the face of TSSCPP. Note that removing \mathbf{a} from V_n transforms the graph to $V_{n-1} \setminus \{\mathbf{b}\}$ since the edges $((0, 0), (1, 1))$, $((0, 1 + 2k), (0, 2 + 2k))$, $((1, 2 + 2k), (1, 3 + 2k))$ for $0 \leq k \leq n - 1$ must be matched by Remark 5.1; see Figure 7.

Hence, we have $Z_n^{\{\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}\}} = Z_{n-1}^{\{\mathbf{i}-1, \mathbf{j}-1\}}$, $Z_n^{\{\mathbf{a}, \mathbf{i}\}} = Z_{n-1}^{\{\mathbf{i}-1, \mathbf{b}\}}$ and $Z_n^{\{\mathbf{a}, \mathbf{b}\}} = Z_{n-1}$. This means that the above equation is reduced to

$$Z_{n-1}^{\{\mathbf{i}-1, \mathbf{j}-1\}} Z_n = Z_{n-1}^{\{\mathbf{i}-1, \mathbf{b}\}} Z_n^{\{\mathbf{j}, \mathbf{b}\}} - Z_{n-1}^{\{\mathbf{j}-1, \mathbf{b}\}} Z_n^{\{\mathbf{i}, \mathbf{b}\}} + Z_{n-1} Z_n^{\{\mathbf{i}, \mathbf{j}\}}. \tag{5.6}$$

We divide the above equation by $Z_{n-1} Z_n$ and use the definitions of R_n and T_n to obtain the equation. A similar computation holds for $i > j$. \square

LEMMA 5.4. For $0 \leq i \leq n - 1$, we have $T_n(i) = (-1)^i p(n, i, 0)$ and $\text{sgn } K_n^{-1}((2i, 2n + 1), \mathbf{b}) = (-1)^n$.

Proof. Note that $(2i, 2n + 1) = (2i, 2(n - i) + 2i + 1)$. The lemma follows from using the formula for $K_n^{-1}((2i, 2n + 1), \mathbf{b})$ given in (3.8) and computing explicitly as follows:

[†]Here the face in Proposition 5.2 is the boundary face.

$$\begin{aligned}
 K_n^{-1}((2i, 2n + 1), \mathbf{b}) &= \sum_{\ell=0}^{2i} (-1)^{n-i+\ell} \binom{n-i+\ell-1}{\ell} h_n^{1,\mathbf{b}}(2i-\ell) \\
 &= \sum_{k=0}^n \frac{p(n, k, 0)}{2\pi i} \int_{\Gamma_0} dr \sum_{\ell=0}^{2i} (-1)^{n-i+\ell} \binom{n-i+\ell-1}{\ell} \frac{(1+r)^{n-k}}{r^{2i-2k-\ell+1}} \\
 &= (-1)^{n-i} \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k+i-n}}{r^{2i-2k+1}} \\
 &= (-1)^{n-i} p(n, i, 0),
 \end{aligned} \tag{5.7}$$

where the third equality uses the fact that the additional term proportional to ${}_2F_1\left[\begin{smallmatrix} 1, 1+i+n \\ 2+2i \end{smallmatrix}; -r\right]$ does not have a singularity in r and the fourth equality follows from (4.5). Since $\text{sgn } p(n, i, 0) = (-1)^i$, the result follows. \square

LEMMA 5.5. For $0 \leq i < j \leq n - 1$, $\text{sgn } K_n^{-1}((2i, 2n + 1), (2j, 2n + 1)) = 1$.

Proof. Introduce an auxiliary edge directed from $(2j, 2n + 1)$ to $(2i, 2n + 1)$ for $i < j$. This introduces a new face, but by this choice, the number of counterclockwise edges on the graph remains odd and so is a valid Kasteleyn orientation. Let \tilde{K}_n denote the modified Kasteleyn matrix with this additional edge. Then, by Theorem 3.3 and the antisymmetry of K_n

$$\begin{aligned}
 0 &\leq \mathbb{P}[\text{Edge } ((2i, 2n + 1), (2j, 2n + 1)) \text{ in the modified graph}] \\
 &= \tilde{K}_n((2i, 2n + 1), (2j, 2n + 1)) \tilde{K}_n^{-1}((2j, 2n + 1), (2i, 2n + 1)) \\
 &= -\tilde{K}_n^{-1}((2j, 2n + 1), (2i, 2n + 1)) = \tilde{K}_n^{-1}((2i, 2n + 1), (2j, 2n + 1)).
 \end{aligned} \tag{5.8}$$

Since we have $\tilde{K}_n^{-1}((2i, 2n + 1), (2j, 2n + 1)) = K_n^{-1}((2i, 2n + 1), (2j, 2n + 1))$, the result follows. \square

To get an explicit expression for $R_n(i, j)$, we use generating functions and require some further notation. Introduce

$$\mathbf{R}(u, v, w) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n R_n(i, j) u^i v^j w^n, \tag{5.9}$$

and

$$\mathbf{T}_n(u) = \sum_{i=0}^{\infty} T_n(i) u^i. \tag{5.10}$$

LEMMA 5.6. For $0 \leq i, j \leq n - 1$ and $n \geq 1$,

$$\begin{aligned}
 R_n(i, j) &= \frac{1}{(2\pi i)^3} \int_{\Gamma_0} \frac{du}{u} \int_{\Gamma_0} \frac{dv}{v} \int_{\Gamma_0} \frac{dw}{w} \frac{1}{1-uvw} \frac{1}{u^i v^j w^n} \\
 &\quad \times \left(\sum_{n=1}^{\infty} \mathbf{T}_n(u) \mathbf{T}_{n-1}(v) v w^n - \mathbf{T}_{n-1}(u) \mathbf{T}_n(v) u w^n \right).
 \end{aligned} \tag{5.11}$$

Proof. Multiply the recurrence in Lemma 5.3 by $u^i v^j w^n$ and taking sums gives

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (R_n(i, j) u^i v^j w^n - R_{n-1}(i-1, j-1) u^i v^j w^n) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (T_n(i) T_{n-1}(j-1) - T_{n-1}(i-1) T_n(j)) u^i v^j w^n. \end{aligned} \quad (5.12)$$

Using $R_n(0, i) = T_{n-1}(i-1)$ and $R_n(i, j) = -R_n(j, i)$, we have that

$$-\sum_{n=1}^{\infty} \sum_{i=1}^n R_n(i, 0) u^i w^n = \sum_{n=1}^{\infty} u \mathbf{T}_{n-1}(u) w^n \quad (5.13)$$

and

$$\sum_{n=1}^{\infty} \sum_{j=1}^n R_n(0, j) w^j w^n = \sum_{n=1}^{\infty} v \mathbf{T}_{n-1}(v) w^n. \quad (5.14)$$

Inserting the above two equations into (5.12) gives

$$\mathbf{R}(u, v, w)(1 - uvw) = 1 + \sum_{n=1}^{\infty} \mathbf{T}_n(u) \mathbf{T}_{n-1}(v) v w^n - \sum_{n=1}^{\infty} u \mathbf{T}_{n-1}(u) \mathbf{T}_n(u) w^n. \quad (5.15)$$

We divide both sides by $(1 - uvw)$ and it remains to extract out the coefficient of $u^i v^j w^n$ in the above equation to prove the lemma. \square

LEMMA 5.7. For $|u| < 1$,

$$\begin{aligned} \mathbf{T}_n(u) &= \frac{1}{(2\pi i)^3} \int_0^{\infty} dq \int_0^{\infty} dp \int_{\Gamma_0} dt \int_{\Gamma_0} dz \int_S ds e^{-p-q+\frac{p^2 s q}{t^2}+t} \\ &\times \frac{p^{2+n} q^{3+2n} (1+z)^{2n+2} (2+z)}{t^{3(1+n)} z^{1+n} (pq^2 s (1+z)^2 - t^3 u z)}, \end{aligned} \quad (5.16)$$

where the s -contour S is chosen so that $\frac{|t^3 z u|}{pq^2 |s(1+z)^2|} < 1$.

Proof. Note that $\mathbf{T}_n(u) = \sum_{k=0}^n T_n(k) u^k$ since $T_n(k) = 0$ for $k \geq n$ and so we use integral expressions to find an explicit expression using geometric sums. We now use the expressions for binomial coefficients and factorials given in (4.6), (4.7), and (4.8), as explained below, in the formula for $T_n(i)$ given in Lemma 5.4 by applying these integrals to terms in the expression for $(-1)^k p(n, k, 0)$ (3.5). We obtain

$$\begin{aligned} T_n(k) u^k &= \frac{1}{(2\pi i)^3} \int_0^{\infty} dq \int_0^{\infty} dp \int_{\Gamma_0} \frac{dz}{z} \int_{\Gamma_0} \frac{dt}{t} \int_{\Gamma_0} \frac{ds}{s} e^{-p-q+s+t} \\ &\times \frac{p^{n+k+1} q^{2n-k+1} (1+z)^{2n-2k} (2+z)}{s^k t^{3n-k+2} z^{n-k}} u^k. \end{aligned} \quad (5.17)$$

In the above equation, the integrals with respect to p and q are from the two factorial terms in the numerator of $(-1)^k p(n, k, 0)$ using (4.7), while the integrals respect to s and t are from the

two factorials in the denominator of $(-1)^k p(n, k, 0)$ using (4.8). For the integral with respect to z , this comes from rewriting the term $(3n - 3k + 2)C_{n-k}$ in $(-1)^k p(n, k, 0)$ as

$$(3n - 3k + 2)C_{n-k} = 2 \binom{2n - 2k}{n - k} + \binom{2n - 2k}{n - k + 1}, \quad (5.18)$$

and expressing each of these binomial terms as integrals using (4.5).

We choose the s -contour in (5.17) so that

$$\frac{p|tzu|}{q|s(1+z)^2|} < 1, \quad (5.19)$$

and evaluate the geometric sum

$$\begin{aligned} \sum_{k=1}^{\infty} T_n(k) u^k &= \frac{1}{(2\pi i)^3} \int_0^{\infty} dq \int_0^{\infty} dp \int_{\Gamma_0} dz \int_{\Gamma_0} dt \int_S ds \\ &= \frac{e^{-p-q+s+t} p^{1+n} q^{2+2n} (1+z)^{2n+2} (2+z)}{t^{3(1+n)} z^{1+n} (qs(1+z)^2 - ptuz)}. \end{aligned} \quad (5.20)$$

Making the change of variables $s \mapsto sp^2q/t^2$ gives the result. \square

In what follows below, we use the notation $d\mathbf{p} = dp_1 dp_2$ and similarly for the variables $\mathbf{q}, \mathbf{t}, \mathbf{z}, \mathbf{s}$ for compactness of notation.

LEMMA 5.8. For $0 \leq i, j \leq n - 1$,

$$\begin{aligned} R_n(i, j) &= \frac{1}{(2\pi i)^6} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{\Gamma_0^2} d\mathbf{s} \\ &\times e^{-p_1 - p_2 - q_1 - q_2 + \frac{p_1^2 s_1 q_1}{t_1^2} + \frac{p_2^2 s_2 q_2}{t_2^2} + t_1 + t_2} p_1^{1-i+n} p_2^{1-j+n} q_1^{1-2i+2n} q_2^{1-2j+2n} \frac{s_1 - s_2}{s_1^{1+i} s_2^{1+j} t_1^{3-3i+3n} t_2^{3-3j+3n}} \frac{s_1 - s_2}{s_1 s_2 - 1} \\ &\times \frac{(1+z_1)^{2n-2i} (1+z_2)^{2n-2j} (2+z_1)(2+z_2)}{z_1^{n-i+1} z_2^{n-j+1}}. \end{aligned} \quad (5.21)$$

Proof. This lemma follows from a computation starting with the formula for $R_n(i, j)$ given in Lemma 5.6. For brevity, we only present the computation of the first term $\mathbf{T}_n(u)\mathbf{T}_{n-1}(v)vw^n$; the computation for the other term is completely analogous.

Introduce

$$\tilde{R} = \frac{p_1 p_2 q_1^2 q_2^2 (1+z_1)^2 (1+z_2)^2}{t_1^3 t_2^3 z_1 z_2}. \quad (5.22)$$

We can use the integral expression for $\mathbf{T}_n(u)$ given in Lemma 5.7 to find that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{T}_n(u)\mathbf{T}_{n-1}(v)vw^n &= \frac{1}{(2\pi i)^6} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{S_1 \times S_2} d\mathbf{s} \\ &\times v \sum_{n=1}^{\infty} w^n \tilde{R}^n e^{\frac{-p_1 - q_1 + \frac{p_1^2 s_1 q_1}{t_1^2} + t_1 - p_2 - q_2 + \frac{p_2^2 s_2 q_2}{t_2^2} + t_2}{t_1^3 z_1 (p_1 q_1^2 s_1 (1+z_1)^2 - t_1^3 u z_1)} p_1^2 p_2 q_1^3 q_2 (1+z_1)^2 (2+z_1)(2+z_2)} \frac{(p_2 q_2^2 s_2 (1+z_2)^2 - t_2^3 v z_2)}{t_1^3 z_1 (p_1 q_1^2 s_1 (1+z_1)^2 - t_1^3 u z_1)}}, \end{aligned} \quad (5.23)$$

where the s_1 -contour S_1 is chosen so that $\frac{|t_1^3 z_1 u|}{p_1 q_1^2 |s_1(1+z_1)^2|} < 1$ and the s_2 -contour S_2 is chosen so that $\frac{|t_2^3 z_2 u|}{p_2 q_2^2 |s_2(1+z_2)^2|} < 1$. We substitute the above expression into the right side of the equation in Lemma 5.6 which gives

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \int_{\Gamma_0} \frac{du}{u} \int_{\Gamma_0} \frac{dv}{v} \int_{\Gamma_0} \frac{dw}{w} \frac{1}{1-uvw} \frac{1}{u^i v^j w^n} \sum_{n=1}^{\infty} \mathbf{T}_n(u) \mathbf{T}_{n-1}(v) v w^n \\ &= \frac{1}{(2\pi i)^3} \int_{\Gamma_0} \frac{du}{u} \int_{\Gamma_0} \frac{dv}{v} \int_{\Gamma_0} \frac{dw}{w} \frac{1}{1-uvw} \frac{1}{u^i v^j w^n} \frac{\tilde{R} w v}{1-\tilde{R} w} \\ & \times \frac{1}{(2\pi i)^6} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{S_1 \times S_2} d\mathbf{s} \\ & \times \frac{e^{-p_1-q_1+\frac{p_1^2 s_1 q_1}{t_1^2}+t_1-p_2-q_2+\frac{p_2^2 s_2 q_2}{t_2^2}+t_2} p_1^2 p_2 q_1^3 q_2 (1+z_1)^2 (2+z_1)(2+z_2)}{t_1^3 z_1 (p_1 q_1^2 s_1 (1+z_1)^2 - t_1^3 u z_1) (p_2 q_2^2 s_2 (1+z_2)^2 - t_2^3 v z_2)}. \end{aligned} \tag{5.24}$$

In the above expression, we rearrange the integrals and compute the w -integral first (no extra contribution is picked up). We push the contour through ∞ picking up residue contributions at $w = 1/R$ and $w = 1/(uv)$. The latter will not contribute since when computing the residue at $w = 1/(uv)$, there will be no residue at $u = 0$. This means that the above equation becomes

$$\begin{aligned} & \frac{1}{(2\pi i)^8} \int_{\Gamma_0} \frac{du}{u} \int_{\Gamma_0} \frac{dv}{v} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{S_1 \times S_2} d\mathbf{s} \\ & \times \frac{1}{\tilde{R}-uv} \frac{\tilde{R}^{n+1} v e^{-p_1-q_1+\frac{p_1^2 s_1 q_1}{t_1^2}+t_1-p_2-q_2+\frac{p_2^2 s_2 q_2}{t_2^2}+t_2} p_1^2 p_2 q_1^3 q_2 (1+z_1)^2 (2+z_1)(2+z_2)}{u^i v^j t_1^3 z_1 (p_1 q_1^2 s_1 (1+z_1)^2 - t_1^3 u z_1) (p_2 q_2^2 s_2 (1+z_2)^2 - t_2^3 v z_2)}. \end{aligned} \tag{5.25}$$

Next, we rearrange integrals to compute the u -integral by computing the residue contributions at $u = R/v$ and $u = p_1 q_1^2 s_1 (1+z_1)^2 / (t_1^3 z_1)$ and so we obtain

$$\begin{aligned} & \frac{1}{(2\pi i)^7} \int_{\Gamma_0} \frac{dv}{v} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{S'_1 \times S_2} d\mathbf{s} \\ & \times \frac{\tilde{R}^{n-i} e^{-p_1-q_1+\frac{p_1^2 s_1 q_1}{t_1^2}+t_1-p_2-q_2+\frac{p_2^2 s_2 q_2}{t_2^2}+t_2} p_1^2 p_2 q_1^3 q_2 (1+z_1)^2 (2+z_1)(2+z_2)}{v^{j-1-i} t_1^3 z_1 (p_1 q_1^2 s_1 (1+z_1)^2 - t_1^3 v z_1) (p_2 q_2^2 s_2 (1+z_2)^2 - t_2^3 v z_2)} \\ & + \frac{1}{(2\pi i)^7} \int_{\Gamma_0} \frac{dv}{v} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{S_1 \times S_2} d\mathbf{s} \\ & \times \frac{t_1^{3i} z_1^{i-1} e^{-p_1-q_1+\frac{p_1^2 s_1 q_1}{t_1^2}+t_1-p_2-q_2+\frac{p_2^2 s_2 q_2}{t_2^2}+t_2} p_1^2 p_2 q_1^3 q_2 (1+z_1)^2 (2+z_1)(2+z_2)}{(p_1 q_1^2 s_1 (1+z_1)^2)^i (p_1 q_1^2 s_1 (1+z_1)^2 v - R t_1^3 z_1) v^{j+1} t_1^3 z_1 (p_2 q_2^2 s_2 (1+z_2)^2 - t_2^3 v z_2)}, \end{aligned} \tag{5.26}$$

where S'_1 is chosen so that $\frac{|t_1^3 z_1 v|}{p_1 q_1^2 |s_1(1+z_1)^2|} < 1$. We compute each of the above terms separately. For the first term in (5.26), we rearrange the integrals to compute the v -integral and there is a residue at $v = 0$, provided that $j > i + 1$, and further residue contributions at $v = t_1^3 \tilde{R} z_1 / (p_1 q_1^2 s_1 (1+z_1)^2)$ and $v = t_2^3 z_2 / (p_2 q_2^2 s_2 (1+z_2)^2)$ due to exchanging contours. Summing up these contribution gives zero because there is no residue at $s_1 = 0$ after simplification (we omit the details of this computation). Thus, (5.24) is equal to the second term in (5.26). For

the second term in (5.26), we perform the same computational steps as mentioned for the first term and we arrive at

$$\begin{aligned} & \frac{1}{(2\pi i)^6} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{t} \iint_{\Gamma_0^2} d\mathbf{s} e^{-p_1 - q_1 + \frac{p_1^2 s_1 q_1}{t_1^2} + t_1 - p_2 - q_2 + \frac{p_2^2 s_2 q_2}{t_2^2} + t_2} \\ & \times \frac{p_1^{1-i+n} p_2^{1-j+n} q_1^{1-2i+2n} q_2^{1-2j+2n}}{s_1^{1+i} s_2^j t_1^{3-3i+3n} t_2^{3-3j+3n} (s_1 s_2 - 1)} \frac{(1+z_1)^{2n-2i}}{z_1^{n-i+1}} \frac{(1+z_2)^{2n-2j}}{z_2^{n-j+1}} (2+z_1)(2+z_2). \end{aligned} \tag{5.27}$$

We have now evaluated (5.24) and the second term on right side of the equation in Lemma 5.6 follows by symmetry. \square

6. Proof of the main result

We will now use the results from the previous sections to prove Theorem 3.2. For the proof, we will need the following result.

PROPOSITION 6.1. For $0 \leq j \leq 2n - 1$

$$-K_n^{-1}((0, 0), (j, j + 1)) = K_n^{-1}((j, j + 1), (0, 0)) = K_n^{-1}((j, j + 1), \mathbf{b}). \tag{6.1}$$

Proof. The first equality is from antisymmetry of K_n . For the second equality, note that $K_n^{-1}((j, j + 1), (0, 0))$ represents the ratio of a signed count on the graph formed from removing the vertices $(j, j + 1)$ and $(0, 0)$ along with their incident edges from the TSSCPP of size n and Z_n . The sign arises from having an even number of counterclockwise edges around the face which surrounds the removed vertex $(j, j + 1)$. For each of these dimer configurations (which is part of the signed count) on $V_n \setminus \{(0, 0), (j, j + 1)\}$, remove all dimers incident to the vertices $\{(i, i) : 0 \leq i \leq 2n - 1\}$. Then, there is a unique way to extend to a dimer configuration on $V_n \setminus \{\mathbf{b}, (j, j + 1)\}$ as required. Note that this operation does not change the sign associated to each dimer configuration. \square

Proof of Theorem 3.2. We will first focus on the formula (3.13) for $t_n^{1,1}(i, j)$. The formulas for $t_n^{k,l}(i, j)$ for $(k, l) \neq (1, 1)$ are obtained by very similar computations. Once these are found, the rest of entries are obtained from the matrix equations $K_n \cdot K_n^{-1} = K_n^{-1} \cdot K_n = \mathbb{I}$.

To obtain $t_n^{1,1}(i, j)$, first observe that $t_n^{1,1}$ is a linear combination of binomial coefficients for $R_n(k, l)$ by using the formula $K_n \cdot K_n^{-1} = K_n^{-1} \cdot K_n = \mathbb{I}$. Indeed, note that from the equation $K_n \cdot K_n^{-1} = \mathbb{I}$ entrywise, we have

$$K_n^{-1}((i_1, i_1 + 2i_2 + 1), y) = -K_n^{-1}((i_1 - 1, i_1 + 2i_2 + 2), y) - K_n^{-1}((i_1, i_1 + 2i_2 + 3), y) \tag{6.2}$$

for $1 \leq i_1 \leq 2n, 0 \leq i_2 \leq n - \lceil i_1/2 \rceil - 1$ and $y = (j_1, j_1 + 2j_2 + 1)$ (a similar relation holds for y). Using the above equation, we see that

$$K_n^{-1}((i, i + 1), (j, j + 1)) = t_n^{1,1}(i, j) = \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^{k_1+k_2} \binom{n-k_1}{i-2k_1} \binom{n-k_2}{i-2k_2} R_n(k_1, k_2), \tag{6.3}$$

since $R_n(k_1, k_2) = K_n^{-1}((2k_1, 2n + 1), (2k_2, 2n + 1))$ by (5.3) and Lemma 5.5. It suffices to evaluate the right side of the above equation and show that it equals $t_n^{1,1}(i, j)$ as given in (3.13). To do so, note that by using the residue theorem

$$\frac{1}{2\pi i} \int_{\Gamma_0} \frac{dt}{t} \frac{1}{t^{3n+2-2k}} e^{\frac{p^2 s q}{t^2} + t} = \sum_{\ell=0}^{\infty} \frac{(p^2 s q)^\ell}{\ell! (3n + 2\ell + 2 - 3k)!}. \tag{6.4}$$

Using the above two equations, the formula for R_n found in Lemma 5.8, and writing binomial coefficients above as contour integrals in the \mathbf{r} variables using (4.5), we obtain

$$\begin{aligned} & \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \sum_{k_1=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{\ell_1! \ell_2!} \frac{(-1)^{k_1+k_2}}{(2\ell_1 + 3n + 2 - 3k_1)! (2\ell_1 + 3n + 2 - 3k_2)!} \\ & \times \frac{1}{(2\pi i)^6} \iint_{\mathbb{R}_+^2} d\mathbf{q} \iint_{\mathbb{R}_+^2} d\mathbf{p} \iint_{\Gamma_0^2} d\mathbf{z} \iint_{\Gamma_0^2} d\mathbf{s} \iint_{\Gamma_0^2} d\mathbf{r} \frac{(1+r_1)^{n-k_1}}{r_1^{i-2k_1+1}} \\ & \times \frac{(1+r_2)^{n-k_2}}{r_2^{j-2k_2+1}} p_1^{1+2\ell_1-k_1+n} p_2^{1+2\ell_2-k_2+n} \frac{q_1^{1-2k_1+\ell_1+2n} q_2^{1-2k_2+\ell_2+2n}}{s_1^{1+k_1-\ell_1} s_2^{1+k_2-\ell_2}} \\ & \times \frac{s_1 - s_2}{s_1 s_2 - 1} \frac{(1+z_1)^{2n-2k_1}}{z_1^{n-k_1+1}} \frac{(1+z_2)^{2n-2k_2}}{z_2^{n-k_2+1}} (2+z_1)(2+z_2) e^{-p_1-p_2-q_1-q_2}. \end{aligned} \tag{6.5}$$

In the above formula, we use the standard expressions for integrals to give closed forms. That is, we use (4.7) to convert the integrals with respect to the variables \mathbf{p} and \mathbf{q} and use (4.8) to convert the integral with respect to the variable \mathbf{s} . The integral with respect to \mathbf{z} is converted using (4.5); see the discussion after (5.17) for the computation in reverse. Taking the change of summation $k_\varepsilon \mapsto \ell_\varepsilon - k_\varepsilon$ for $\varepsilon \in \{1, 2\}$ gives the formula for $t_n^{1,1}(i, j)$ given in (3.13).

We now explain how to get the remaining formulas from (3.13). From expanding the equation $K_n \cdot K_n^{-1} = \mathbb{I}$ entrywise, we have the formula for $i \geq 2$

$$\begin{aligned} & -K_n^{-1}((i, i), (j, j+1)) + K_n^{-1}((i-2, i-2), (j, j+1)) + K_n^{-1}((i-1, i), (j, j+1)) \\ & K_n^{-1}((i-2, i-1), (j, j+1)) = \mathbb{I}_{(i,i)=(j,j+1)} = 0. \end{aligned} \tag{6.6}$$

We now rearrange the above equation to obtain

$$K_n^{-1}((i, i), (j, j+1)) = \sum_{r=0}^{i-1} K_n^{-1}((r, r+1), (j, j+1)) + [i+1]_2 K_n^{-1}((0, 0), (j, j+1)). \tag{6.7}$$

Since $K_n^{-1}((r, r+1), (j, j+1)) = t_n^{(1,1)}(r, j)$, we can push the sum through to the terms in the integral in (3.13) and evaluate. This computation and Proposition 6.1 gives (3.12). The formulas in (3.11) and (3.10) follow from similar computations.

It is not hard to see that (3.14), (3.15) and (3.16) follow from a straightforward applications of $K_n \cdot K_n^{-1} = K_n^{-1} \cdot K_n = \mathbb{I}$. □

7. Proofs of the combinatorial identities

We now give the proofs of the combinatorial results used in Section 4.

Proof of Theorem 4.1. A minor issue is that the natural upper limit for the sum is $k = n + 1$, not $k = n$, although this is what it seems at first glance. Moreover, $f_n(n + 1) = -(-1)^n/2$. Therefore, we need to prove that $\sum_k f_n(k) = 1/2$. Maple solves this using the Wilf-Zeilberger algorithm. Then we need to check that

$$f_n(k) - f_{n+1}(k) = h(n, k + 1) - h(n, k),$$

where the certificate is

$$h(n, k) = (-1)^k 2^{n-k+1} (n+2-k)(n-k+1)(5n-3k+8) \\ \times \frac{(2n+2-k)!(n+1+k)!(2n+2-2k)!}{(k-1)!(n+2-k)!^2(3n+5-k)!}.$$

This guarantees that $\sum_k f_n(k)$ is constant. We now check that $f_0(0) + f_0(1) = 1 - 1/2 = 1/2$, completing the proof. \square

Before proving Theorem 4.2, we need another identity in the sequel, which we prove now.

THEOREM 7.1. *Let $n \in \mathbb{N}$ and $0 \leq i \leq n-1$. Define*

$$f'_{n,i}(k) = (-1)^{k+1} \frac{(3n-3k+2)(n+k+1)!(2n-k+1)!(2n-i-k)!}{k!(3n-k+2)!(n+i-2k+1)!(n+k-2i)!} C_{n-k}.$$

Then $\sum_{k=0}^n f'_{n,i}(k) = 0$.

Proof. Let $F'_{n,i}$ be the required sum. Then Zeilberger's algorithm implemented in the *Mathematica* package *fastZeil* [39] gives the recurrence

$$2(i+1)(2n+1-i)(2n+1-2i)F'_{n,i} + (3n+2-i)(n-1-i)(n+2+i)F'_{n,i+1} \\ = (-1)^n \frac{12(n+1)^2(n-i)!}{(2n-2i)!(i-n)!}. \quad (7.1)$$

To prove (7.1), we define the certificate,

$$R(k, i) = \frac{6k(i-n)(-2k+2n+1)(-k+2n+2)(-2i+k+n)}{(-3k+3n+2)(i-2k+n+2)},$$

and verify that

$$2(i+1)(i-2n-1)(2i-2n-1)f'_n(k, i) + (i-3n-2)(i-n+1)(i+n+2)f'_n(k, i+1) \\ = \Delta_k(f'_n(k, i)R(k, i)),$$

where Δ_k is the forward difference operator defined by $\Delta_k(f(k)) = f(k+1) - f(k)$.

In (7.1), note that $1/(i-n)! = 0$ and all other terms on the right-hand side are well-defined for $0 \leq i \leq n-1$. Therefore, the right-hand side is zero for our region of interest. Because of the factor $(n-1-i)$ in the second term on the left-hand side, setting $i = n-1$ also gives $F'_{n,n-1} = 0$. (This can also be checked independently using Zeilberger's algorithm.) Then, the recurrence above guarantees that $F'_{n,i} = 0$ for $0 \leq i \leq n-1$. \square

Proof of Theorem 4.2. First, note that

$$\sum_{j=0}^{n+i-2k} \binom{2n+1-i-k}{j} + \sum_{j=0}^{n+k-2i} \binom{2n+1-i-k}{j} \\ = \sum_{j=0}^{n+i-2k} \binom{2n+1-i-k}{j} + \sum_{j=0}^{n+k-2i} \binom{2n+1-i-k}{2n+1-i-k-j}$$

$$\begin{aligned}
&= \sum_{j=0}^{n+i-2k} \binom{2n+1-i-k}{j} + \sum_{j=n+1-2k+i}^{2n+1-i-k} \binom{2n+1-i-k}{j} \\
&= \sum_{j=0}^{2n+1-i-k} \binom{2n+1-i-k}{j} = 2^{2n+1-i-k}.
\end{aligned} \tag{7.2}$$

Using (7.2), add the left-hand sides of (4.1) and (4.2) to get

$$\begin{aligned}
&\sum_{k=0}^n \frac{1}{2^{n-k}} f_n(k) \left(\sum_{j=0}^{n+i-2k} \binom{2n+1-i-k}{j} + \sum_{j=0}^{n+k-2i} \binom{2n+1-i-k}{j} \right) \\
&= \sum_{k=0}^n \frac{1}{2^{n-k}} f_n(k) 2^{2n+1-i-k} = 2^{n-i} (1 + (-1)^n),
\end{aligned}$$

where we have used Theorem 4.1 in the last step. So we have shown that the sum of the left-hand sides is what we want.

We now prove (4.1). Our first task is to prove $G_{n,i} = 2G_{n,i+1}$. Using Wegschaider's algorithm [45] in the *Mathematica* package *MultiSum*, we get the recurrence

$$g_{n,i}(k, j) - 2g_{n,i+1}(k, j) = \Delta_j [g_{n,i+1}(k, j) - g_{n,i}(k, j)], \tag{7.3}$$

where Δ_j is the forward difference operator defined above. This is easily verified by a computation. Now, we perform the j -sum on both sides of (7.3). Since the right-hand side telescopes, we obtain

$$\begin{aligned}
&\sum_{j=0}^{n+i-2k} (g_{n,i}(k, j) - 2g_{n,i+1}(k, j)) \\
&= g_{n,i+1}(k, n+i+1-2k) - g_{n,i}(k, n+i+1-2k) \\
&= (-1)^{k+1} \frac{(3n-3k+2)(n+k+1)!(2n-k+1)!(2n-i-k)!}{k!(3n-k+2)!(n+i-2k)!(n+k-2i)!} C_{n-k}.
\end{aligned}$$

We perform the k -sum on both sides of the above equation and compensate for the $j = n+i+1-2k$ term to obtain that $G_{n,i} - 2G_{n,i+1}$ is equal to

$$\begin{aligned}
&\sum_{k=0}^n (-1)^{k+1} \frac{(3n-3k+2)(n+k+1)!(2n-k+1)!(2n-i-k)!}{k!(3n-k+2)!(n+i-2k)!(n+k-2i)!} C_{n-k} \\
&- 2 \sum_{k=0}^n (-1)^k \frac{(3n-3k+2)(n+k+1)!(2n-k+1)!(2n-i-k)!}{k!(3n-k+2)!(n+i-2k+1)!(n+k-2i-1)!} C_{n-k} \\
&= \sum_{k=0}^n f'_{n,i}(k),
\end{aligned}$$

where f' is defined above. Now, using Theorem 7.1, we obtain that this is equal to 0.

All that remains to be done is to prove that $G_{n,n} = 1$. But this is easily performed since the j -sum becomes

$$\sum_{j=0}^{2n-2k} \binom{n-k+1}{j} = 2^{n-k+1},$$

and we now use Theorem 4.1. This completes the proof. \square

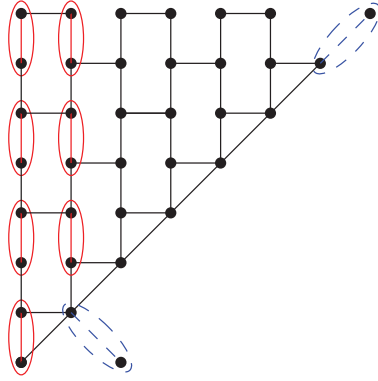


FIGURE 8 (colour online). The blue dashed dimers represent the removal of the vertices $(1,1)$ and \mathbf{b} from the graph. The red solid dimers are those that are induced when the vertex $(1,1)$ is removed (see online version for colors).

8. Boundary recurrences

Recall that $g_n^{\mathbf{b}}$ is given in (3.18) and define, for $0 \leq i < j \leq 2n - 1$,

$$g_n(i, j) = |K_n^{-1}((i, i), (j, j))| = -K_n^{-1}((i, i), (j, j)) = \frac{Z_n^{\{(i, i), (j, j)\}}}{Z_n}, \quad (8.1)$$

with $g_n(j, i) = -g_n(i, j)$ and $g_n(i, i) = 0$. The relevant signs given above are evaluated by using a similar argument given in Lemma 5.5; we omit this computation.

We obtain some formulas immediately.

LEMMA 8.1. We have that for $2 \leq j \leq 2n - 1$,

$$g_n^{\mathbf{b}}(1) = \frac{Z_{n-1}}{Z_n}, \quad (8.2)$$

$$g_n(1, j) = \frac{Z_{n-1}}{Z_n} g_{n-1}^{\mathbf{b}}(j - 2), \quad (8.3)$$

and $g_n^{\mathbf{b}}(0) = 1$.

Proof. The first two equations are immediate from writing the definitions of $g_n^{\mathbf{b}}(1)$ and $g_n(1, j)$ as a ratio of partition functions, see also (3.18), and removing the vertex $(1,1)$ from the graph forces edges to be covered by dimers; see Figure 8.

For the last equation, for each dimer configuration of Z_n on V_n , remove all dimers incident to the vertices $\{(i, i), 0 \leq i \leq 2n - 1\}$. For each of these dimer configurations, there is a unique way to extend to a dimer configuration on V_n and $V_n \setminus \{(0, 0), \mathbf{b}\}$. This shows that $Z_n = Z_n^{\{(0, 0), \mathbf{b}\}}$ as required. \square

We then have the following system of equations for $g_n(i, j)$ and $g_n^{\mathbf{b}}(j)$:

THEOREM 8.2. For $0 \leq i < j \leq 2n - 1$, we have the recurrence

$$g_n(i, j) = \begin{cases} 1 - g_n^{\mathbf{b}}(j) & \text{if } i = 0, j > 0, \\ \frac{Z_{n-1}}{Z_n} g_{n-1}^{\mathbf{b}}(j - 2) & \text{if } i = 1, j > 1, \\ g_{n-1}(i - 2, j - 2) - g_n^{\mathbf{b}}(j) g_{n-1}^{\mathbf{b}}(i - 2) + g_n^{\mathbf{b}}(i) g_{n-1}^{\mathbf{b}}(j - 2) & \text{if } i, j \geq 2. \end{cases} \quad (8.4)$$

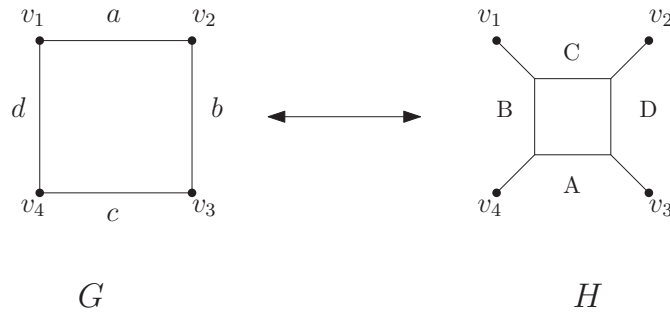


FIGURE 9. The spider move.



FIGURE 10. The edge contraction move.

THEOREM 8.3. For $0 \leq j \leq 2n - 1$, we have the recurrence

$$g_n^b(j) = \begin{cases} 1 & \text{if } j = 0, \\ \frac{Z_{n-1}}{Z_n} & \text{if } j = 1, \\ \frac{Z_{n-1}}{Z_n} \left(1 - g_{n-1}^b(j-2) + \sum_{r=0}^{n-1} (-1)^r \binom{n+1}{r+2} g_{n-1}(r, j-2) \right) & \text{if } j \geq 2. \end{cases} \quad (8.5)$$

These two recurrences determine $K_n^{-1}((i, i), (j, j))$ for $0 \leq i, j \leq 2n - 1$, and along with the equations $K_n^{-1}.K_n = K_n.K_n^{-1} = \mathbb{I}$ viewed entrywise fully determines the entries of K_n^{-1} .

To prove these results, we need two local moves for dimers.

(1) *Spider move*: Suppose we have a large square with edge weights a, b, c and d (clockwise labeling) on some graph G . This square can be deformed to smaller square with additional edges added between the vertices of the original square and the vertices of the smaller square as shown in Figure 9 to form a new graph called H . If the new edge weights in H around the smaller square, A, B, C and D are related to the old weights in G by $A = a/\Delta$, $B = b/\Delta$, $C = c/\Delta$ and $D = d/\Delta$, where $\Delta = ab + cd$ and the edge weights for the additional transformation from G to H . This transformation is called the *spider move* [41], and we have $Z_G = \Delta Z_H$.

(2) *Edge Contraction*: If a vertex is incident to two edges each having weight 1, contract the two incident edges. This does not change the partition function; see Figure 10.

Proof of Theorem 8.2. For the purpose of the proof, write \mathbf{i} to be the vertex (i, i) for $1 \leq i \leq 2n - 1$ and let \mathbf{a} denote the vertex $(0, 2n + 1)$.

We start with the proof of (8.4). The first two conditions are immediate by definition. For $0 \leq i < j < 2n - 1$, we apply graphical condensation, similar to (5.5) in the proof of Lemma 5.3, which gives

$$Z_n^{\{\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}\}} Z_n = Z_n^{\{\mathbf{a}, \mathbf{i}\}} Z_n^{\{\mathbf{j}, \mathbf{b}\}} - Z_n^{\{\mathbf{a}, \mathbf{j}\}} Z_n^{\{\mathbf{i}, \mathbf{b}\}} + Z_n^{\{\mathbf{a}, \mathbf{b}\}} Z_n^{\{\mathbf{i}, \mathbf{j}\}}. \quad (8.6)$$

We first consider the above equation when $i \geq 2$. As in the proof of Lemma 5.3, removing \mathbf{a} freezes off edges; see Figure 7. This means that $Z_n^{\{\mathbf{a}, \mathbf{b}\}} = Z_{n-1}$, $Z_n^{\{\mathbf{a}, \mathbf{i}\}} = Z_{n-1}^{\{\mathbf{i}-2, \mathbf{b}\}}$ and $Z_n^{\{\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}\}} = Z_{n-1}^{\{\mathbf{a}, \mathbf{i}, \mathbf{j}, \mathbf{b}\}}$. Dividing both sides of (8.6) by $Z_{n-1} Z_n$ gives the last condition in (8.4).

Next, when $i = 0$, note that removing both \mathbf{a} and $\mathbf{0}$ has no impact on the forced edges, which means that $Z_n^{\{\mathbf{a},0,\mathbf{j},\mathbf{b}\}} = Z_{n-1}^{\{\mathbf{j}-2,\mathbf{b}\}}$. Dividing (8.6) by $Z_{n-1}Z_n$ in this case gives

$$g_{n-1}(0, j-2) = g_n^{\mathbf{b}}(j) - g_{n-1}^{\mathbf{b}}(j-2) + g_n(0, b) \quad (8.7)$$

and so

$$g_{n-1}(0, j-2) + g_{n-1}^{\mathbf{b}}(j-2) = g_n^{\mathbf{b}}(j) + g_n(0, j). \quad (8.8)$$

The equality in (8.8) is equal to 1 when n is even which follows immediately from the last equation in Lemma 8.1. We now consider (8.8) when n is odd. To do so, set $i = 0$ and $j = 1$ in (8.6) to get

$$Z_n^{\{\mathbf{a},0,1,\mathbf{b}\}} Z_n = Z_n^{\{\mathbf{a},0\}} Z_n^{\{1,\mathbf{b}\}} - Z_n^{\{\mathbf{a},1\}} Z_n^{\{0,\mathbf{b}\}} + Z_n^{\{\mathbf{a},\mathbf{b}\}} Z_n^{\{0,1\}}. \quad (8.9)$$

Note that due to edges being forced $Z_n^{\{\mathbf{a},0,1,\mathbf{b}\}} = Z_{n-1}$ and $Z_n^{\{\mathbf{a},1\}} = 0$, where the latter follows since the dimer covering of $\mathbf{V}_n \setminus \{\mathbf{a}, (1, 1)\}$ is zero as the induced dimers from removing \mathbf{a} from the graph are incompatible with removing $(1, 1)$ from the graph; compare Figures 7 and 8. Dividing the above equation by $Z_{n-1}Z_n$ gives

$$1 = g_n^{\mathbf{b}}(1) + g_n(0, 1). \quad (8.10)$$

We have shown (8.8) is equal to 1 when n is odd, which verifies the third condition in (8.4). Finally, when $i = 1$ in (8.6), gives

$$Z_n^{\{\mathbf{a},1,\mathbf{j},\mathbf{b}\}} Z_n = Z_n^{\{\mathbf{a},1\}} Z_n^{\{\mathbf{j},\mathbf{b}\}} - Z_n^{\{\mathbf{a},\mathbf{j}\}} Z_n^{\{1,\mathbf{b}\}} + Z_n^{\{\mathbf{a},\mathbf{b}\}} Z_n^{\{1,\mathbf{j}\}}. \quad (8.11)$$

Due to their being no dimer covering of $\mathbf{V}_n \setminus \{\mathbf{a}, (1, 1)\}$ as mentioned above, the above equation becomes

$$0 = -Z_n^{\{\mathbf{a},\mathbf{j}\}} Z_n^{\{1,\mathbf{b}\}} + Z_n^{\{\mathbf{a},\mathbf{b}\}} Z_n^{\{1,\mathbf{j}\}}. \quad (8.12)$$

Using that $Z_n^{\{\mathbf{a},\mathbf{j}\}} = Z_{n-1}^{\{\mathbf{j},\mathbf{b}\}}$ and dividing by $Z_{n-1}Z_n$ and using Lemma 8.1 gives the fourth condition in (8.4). \square

Proof of Theorem 8.3. The first two conditions follow from the third and first conditions in Lemma 8.1. The last condition in (8.5) is more involved and we illustrate the steps to find a recurrence of the partition function first.

We perform an edge contraction on the edges $((0, 2n), (0, 2n+1))$ and $((0, 2n+1), (1, 2n+1))$. This means that there is an edge between $(1, 2n)$ and $(0, 2n)$ as well as a square face with coordinates $(0, 2n-1), (1, 2n-1), (1, 2n)$ and $(0, 2n)$; see the left graph in Figure 11. Label this graph H_0 . To the square above, we apply the spider move and edge contraction on the bottom two edges protruding from the new (smaller) square; see the right figure in Figure 11. These operations have:

- (1) deleted the edge $((0, 2n), (1, 2n))$ on H_0 ;
- (2) added an edge between $((0, 2n-2), (1, 2n-2))$ on H_0 ;
- (3) changed the edge weights on the edges $((0, 2n-2), (1, 2n-2)), ((1, 2n-2), (1, 2n-1)), ((0, 2n-1), (1, 2n-1))$ and $((1, 2n-2), (1, 2n-1))$ to $1/2$ on H_0 .

Label this new graph H_1 . This operation gives

$$Z_{H_0} = 2Z_{H_1}. \quad (8.13)$$

We now proceed iteratively and describe the step from H_{k-1} to H_k by applying the square move for $2 \leq k \leq n-1$. On the graph H_{k-1} , we apply the spider move on the square face whose center is given by $(1/2, 2n-2k+3/2)$ and applying edge contraction on the two bottom edges protruding from the new (smaller) square; see Figure 12.

We call this new graph H_k . These operations have:

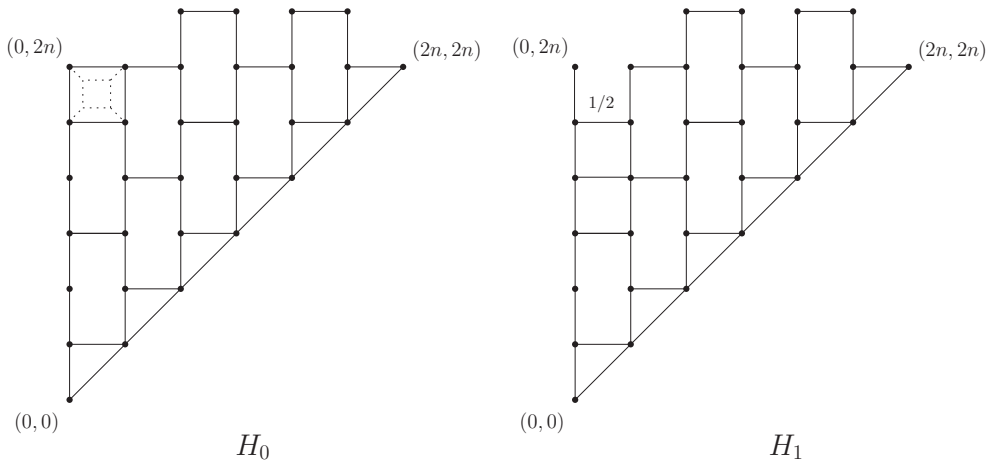


FIGURE 11. The left graph H_0 is obtained after applying an edge contraction. Now, a square appears where we can apply the spider move. After applying the spider move and contracting edges, we obtain the right graph H_1 .

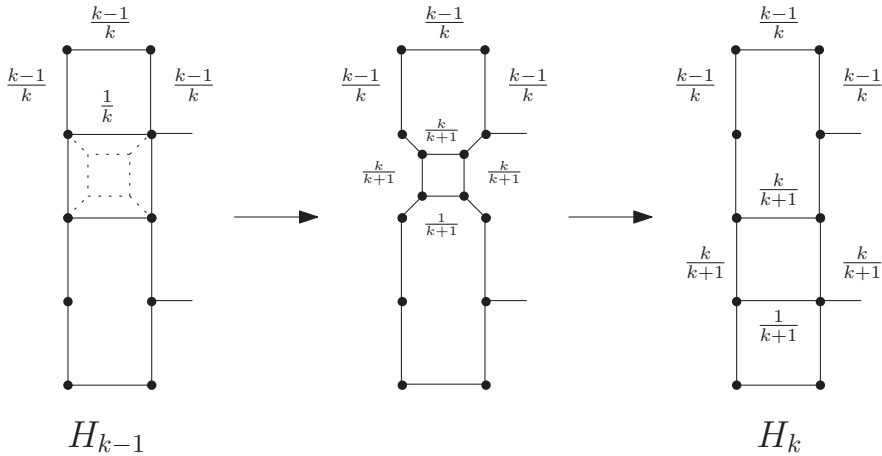


FIGURE 12. The local moves taking H_{k-1} to H_k . In each of the three graphs, the top left vertex is $(0, 2n - 2k + 3)$. All unlabeled edges have weight 1.

- (1) deleted the edge $((0, 2n - 2k + 2), (1, 2n - 2k + 2))$ on H_{k-1} ;
- (2) added an edge between $((0, 2n - 2k), (1, 2n - 2k))$ on H_{k-1} ;
- (3) changed the edge weights of the edges, $((1, 2n - 2k), (1, 2n + 1 - 2k))$, $((0, 2n - 2k + 1), (1, 2n - 2k + 1))$ and $((1, 2n - 2k), (1, 2n - 2k + 1))$ to $k/(k + 1)$ and changed the edge weight of the edge $((0, 2n - 2k), (1, 2n - 2k))$ to $1/(k + 1)$ in H_{k-1} .

Label this new graph H_k . This operation gives

$$Z_{H_{k-1}} = \frac{k + 1}{k} Z_{H_k}. \tag{8.14}$$

From (8.13) and (8.14), we have

$$Z_n = n Z_{H_{n-1}}. \tag{8.15}$$

From the above operations, the edges $((1, 2n - 2k), (1, 2n - 2k + 1))$ have weight $\frac{k}{k+1}$ for $1 \leq k \leq n - 1$. Since the graph H_{n-1} contains a pendant edge $((0, 2n - 1), (0, 2n))$, this can be

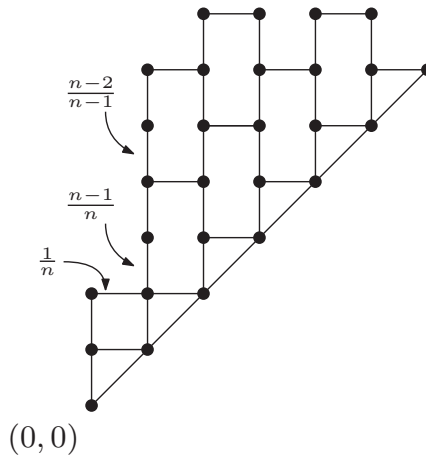


FIGURE 13. The situation after removing pendant edges from H_{n-1} . All unlabelled edges have weight 1.

removed inducing another pendant edge. Iteratively removing these pendant edges $((0, 2n - 1 - 2k), (0, 2n - 2k))$ for $1 \leq k \leq n - 2$ from H_{n-1} leaves us with the graph in Figure 13.

For the vertex $\mathbf{0} = (0, 0)$, either there is a dimer on the edge $(\mathbf{0}, (0, 1))$ or there is a dimer on the edge $(\mathbf{0}, (1, 1))$. For the former, this forces a dimer on the edge $((0, 2), (1, 2))$ (which has weight $1/n$) and forces dimers on the edges $((1, 1), (2, 2))$ and $((1, 2k + 1), (1, 2k + 2))$ for $1 \leq k \leq n - 1$, leaving us with the graph with vertices $V_{n-1} \setminus \{\mathbf{0}, \mathbf{b}\}$. It follows from Lemma 8.1 that the number of dimer covers on this graph is Z_{n-1} . For the latter, this induces a dimer on the edge $((0, 1), (0, 2))$. In this case, only one of the edges $((1, 2 + 2k), (2, 2 + 2k))$ for $0 \leq k \leq n - 1$ can be covered by a dimer and since the edges $((1, 2n - 2k), (1, 2n - 2k + 1))$ have weight $\frac{k}{k+1}$ for $1 \leq k \leq n - 1$, we find using (8.15) that

$$Z_n = Z_{n-1} + \sum_{k=0}^{n-1} (n - k) Z_{n-1}^{\{\mathbf{0}, (0, 2k+1)\}}. \tag{8.16}$$

We can now prove the third equation in (8.5). To so, we consider the same steps that lead to (8.16) but replace the graph of V_n by the induced graph of V_n with the vertices (j, j) and \mathbf{b} for $j \geq 2$ removed. We then arrive at

$$Z_n^{\{\mathbf{j}, \mathbf{b}\}} = Z_{n-1}^{\{\mathbf{j}-2, \mathbf{0}\}} + \sum_{k=0}^{n-1} (n - k) Z_{n-1}^{\{(0, 2k), \mathbf{j}\}}. \tag{8.17}$$

Dividing the above equation by Z_n and using the first condition in (8.4) gives

$$g_n^{\mathbf{b}}(j) = \frac{Z_{n-1}}{Z_n} (1 - g_{n-1}^{\mathbf{b}}(j - 2)) + \frac{Z_{n-1}}{Z_n} \sum_{k=0}^{n-1} (n - k) \frac{Z_{n-1}^{\{(0, 2k), \mathbf{j}-2\}}}{Z_{n-1}}. \tag{8.18}$$

Since the vertex $(0, 2k)$ for $0 \leq k \leq n - 1$ is on the leftmost boundary, this can be moved to the boundary $\{(j, j) : 0 \leq j \leq 2n - 1\}$ by applying the matrix equation $K_n \cdot K_n^{-1} = \mathbb{I}$ entrywise. This gives

$$g_n^{\mathbf{b}}(j) = \frac{Z_{n-1}}{Z_n} (1 - g_{n-1}^{\mathbf{b}}(j - 2)) + \frac{Z_{n-1}}{Z_n} \sum_{k=0}^{n-1} (n - k) \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{Z_{n-1}^{\{\mathbf{r}, \mathbf{j}-2\}}}{Z_{n-1}}. \tag{8.19}$$

Rearranging the sums and evaluating the k -sum gives the final equation in (8.5). □

We now prove the sum rule.

Proof of Theorem 3.4. Substituting (3.9) and (3.6) into (3.18), we have that

$$\begin{aligned} \sum_{j=0}^{2n} g_n^{\mathbf{b}}(j) &= \sum_{j=0}^{2n} (-1)^{j+1} K_n^{-1}((j, j), \mathbf{b}) = \sum_{j=0}^{2n} (-1)^{j+1} h_n^{0, \mathbf{b}}(j) \\ &= \sum_{j=0}^{2n} -(-1)^{j+1} [j + 1]_2 + \sum_{j=0}^{2n} (-1)^{j+1} \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{(1-r)r^{j-2k}}. \end{aligned} \tag{8.20}$$

Both sums in j can be evaluated on the right side of the above equation giving

$$\sum_{j=0}^{2n} g_n^{\mathbf{b}}(j) = n + 1 - \sum_{k=0}^n p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k}}{(1-r)(1+r)r^{2n-2k}}, \tag{8.21}$$

where we have only kept the term with a residue contribution at $r = 0$ when evaluating the geometric sum. Note that the second term on the right side of the above equation does not have a residue contribution at $r = 0$ when $k = n$ and so we have

$$\begin{aligned} \sum_{j=0}^{2n} g_n^{\mathbf{b}}(j) &= n + 1 - \sum_{k=0}^{n-1} p(n, k, 0) \frac{1}{2\pi i} \int_{\Gamma_0} dr \frac{(1+r)^{n-k-1}}{(1-r)r^{2n-2k}} \\ &= n + 1 - \frac{1}{2} \sum_{k=0}^{n-1} p(n, k, 0) 2^{n-k}, \end{aligned} \tag{8.22}$$

where the last line follows from pushing the contour through ∞ picking up the residue at $r = 1$. The result then follows from Theorem 4.1 and from the fact that $p(n, n, 0) = (-1)^n$. \square

9. Heuristics for the limit Shape

To obtain the conjectured limit shape formula, we only consider the asymptotics of $K_n^{-1}((x_1, x_2), \mathbf{b})$ for $x_2 \in 2\mathbb{Z} + 1$ for (x_1, x_2) rescaled as given in Conjecture 3.5. Strictly speaking, this term does not contain any probabilistic information but we expect to see a similar structure when analyzing other entries of the inverse Kasteleyn matrix when both terms are close to the limit shape curves. To find these asymptotics, we express the formula for $K_n^{-1}((x_1, x_2), \mathbf{b})$ as a single contour integral and apply the method of steepest descent. This will give a function whose double roots parameterize the limit shape curves. This is a fairly standard approach in the asymptotics of random tilings, see, for instance, [22, Lecture 15] for a good exposition, and so we only give a brief outline of the main steps. Even though we can give the computations below in full detail, we cannot analyze the rest of the entries of the inverse Kasteleyn matrix with this method, as they are currently not in the best form for asymptotic analysis.

Since we have $(x_1, x_2) = (i_1, i_1 + 2i_2 + 1)$, we use the expression gathered from (3.8) and (3.7). Note, we can truncate the k -sum to $\lfloor \frac{i_1}{2} \rfloor$ since there are no residues at $r = 0$ for $k > \lfloor \frac{i_1}{2} \rfloor$. We obtain

$$\begin{aligned}
K_n^{-1}((x_1, x_2), \mathbf{b}) &= (-1)^{i_2} \sum_{k=0}^{\lfloor \frac{i_1}{2} \rfloor} \frac{p(n, k, 0)}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{(1+r)^{n-k}}{r^{i_1-2k}} \sum_{\ell=0}^{i_2} \binom{i_2-1+\ell}{\ell} (-r)^\ell \\
&= (-1)^{i_2} \sum_{k=0}^{\lfloor \frac{i_1}{2} \rfloor} \frac{p(n, k, 0)}{2\pi i} \int_{\Gamma_0} \frac{dr}{r} \frac{(1+r)^{n-k-i_2}}{r^{i_1-2k}},
\end{aligned} \tag{9.1}$$

where we have used

$$\sum_{\ell=0}^{i_2} \binom{i_2-1+\ell}{\ell} (-r)^\ell = (1+r)^{-i_2} - (-r)^{i_1+1} \binom{i_1+i_2}{1+i_1} {}_2F_1 \left[\begin{matrix} 1, 1+i_1+i_2 \\ 2+i_2 \end{matrix}; -r \right], \tag{9.2}$$

and the fact that the latter term will have no residue at $r=0$ in the middle equation in (9.1). Computing the integral in the last equation in (9.1), using (3.5) and writing factorials as gamma functions gives

$$\begin{aligned}
&(-1)^{i_2} \sum_{k=0}^{\lfloor \frac{i_1}{2} \rfloor} \frac{(-1)^k}{k!(n-k)!} \frac{\Gamma(n+k+2)\Gamma(2n-k+2)\Gamma(2n-2k+1)}{\Gamma(3n-k+3)\Gamma(n-k+2)} \\
&\times (3n-3k+2) \frac{\Gamma(n-k+i_2+1)}{\Gamma(i_1-2k+1)\Gamma(n+k-i_1-i_2+1)}.
\end{aligned} \tag{9.3}$$

Using

$$\frac{(-1)^k}{k!(n-k)!} = \int_{\Gamma_k} dw \frac{(-1)^n}{\prod_{r=0}^n (w-r)}, \quad 0 \leq k \leq n, \tag{9.4}$$

and the residue theorem, we have that (9.3) is equal to

$$\begin{aligned}
&\frac{(-1)^{i_2}}{2\pi i} \int_{\Gamma_{0, \dots, \lfloor i_1/2 \rfloor}} dw \frac{(-1)^n}{\prod_{r=0}^n (w-r)} \frac{\Gamma(n+w+2)\Gamma(2n-w+2)\Gamma(2n-2w+1)}{\Gamma(3n-w+3)\Gamma(n-w+2)} \\
&\times \frac{(3n-3w+2)\Gamma(n-w+i_2+1)}{\Gamma(i_1-2w+1)\Gamma(n+w-i_1-i_2+1)}.
\end{aligned} \tag{9.5}$$

Using the rescaling $x_1 = [(X+2)n]$ and $x_2 = [(\sqrt{3}Y+2)n]$ given in the statement of Conjecture 3.5 gives that $i_1 = [(X+2)n]$ and $i_2 = \frac{1}{2}([(\sqrt{3}Y+2)n] - [(X+2)n] - 1)$. In (9.5), we make the change of variables $w \mapsto wn$ and apply Stirling's formula, see [40, Proposition 7.3] for the exact form. Then we obtain

$$\frac{(-1)^{i_2+n}}{2\pi i} \int_{\tilde{\Gamma}} dw h(w) e^{ns(w)+O(1/n)}, \tag{9.6}$$

where $\tilde{\Gamma}$ is a positively oriented contour that surrounds the points $0, \frac{1}{n}, \dots, \lfloor \frac{i_1}{2} \rfloor \frac{1}{n}$, $h(w)$ is a rational function in w and

$$\begin{aligned}
s(w) &= (1+w) \log(1+w) + (2-w) \log(2-w) + (2-2w) \log(2-2w) \\
&+ \left(1-w - \frac{\sqrt{3}Y-X}{2} \right) \log \left(1-w - \frac{\sqrt{3}Y-X}{2} \right) - w \log(-w) \\
&- (3-w) \log(3-w) - 2(1-w) \log(1-w) - (X+2-2w) \log(X+2-2w) \\
&- \left(1+w - (2+X) - \frac{\sqrt{3}Y-X}{2} \right) \log \left(1+w - (2+X) - \frac{\sqrt{3}Y-X}{2} \right).
\end{aligned} \tag{9.7}$$

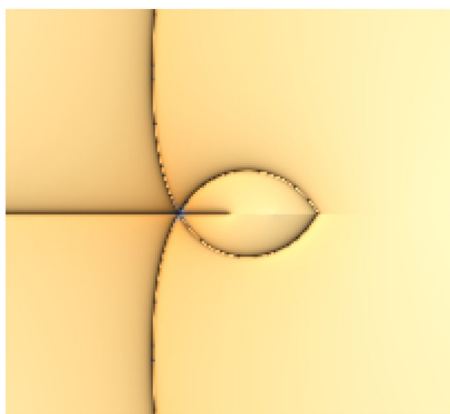


FIGURE 14 (colour online). The contours of steepest descent and ascent for $s(w)$. The contour of steepest descent leaves the double critical point with angle $+\pi/3$ and passes through the point $(X+2) > 0$ and then returns to the double critical point. The crossing points of the real axis can be determined explicitly.

The exact form of $h(w)$ is not important (in fact, it can be computed explicitly). What is important is that it does not influence the saddle point function $s(w)$, nor does it contain any additional poles when we deform the contours. The roots of $s(w)$ can be determined by solving $s'(w) = 0$ in w which gives

$$\frac{2X - Y^2 + 4 \pm \sqrt{Y^2(X^2 + Y^2 - 4)}}{4 - Y^2}. \quad (9.8)$$

This has double roots when $Y = 0$, which corresponds to the top boundary of the rescaled TSSCPP, and when $X^2 + Y^2 = 4$, which is precisely the conjectured limit shape curve. We focus on the latter and set $Y = -\sqrt{4 - X^2}$. Due to the rescaling of the TSSCPP, we have that $-2 < X < -\sqrt{3}$. The contour $\tilde{\Gamma}$ can be deformed to pass through the double root, following the contours of steepest descent; see Figure 14 for a description. No other poles are crossed when performing this deformation and so the main contribution comes locally around the double critical point. This leads to Airy function type asymptotics, which can easily be computed; see [26] for an example. We omit the details and the explicit computation since we have already evaluated the limit shape curves.

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