# Capture Region For a Realistic TPN Guidance Law

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**This work obtains the capture region for a realistic true proportional navigation (RTPN) guidance law against a nonmaneuvering target where the assumption of constant closing velocity used in the guidance law is relaxed. This is a realistic departure from earlier treatments of true proportional navigation (TPN). A capture equation is obtained and a qualitative study carried out for both zero and non-zero miss-distances. The results obtained are shown to be significantly different from earlier studies, showing a degradation in the performance of the guidance law in the form of a reduced capture region. An example problem is computationally solved to obtain the exact capture region for various values of miss-distances.**

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## I. INTRODUCTION

Proportional navigation (PN) is the most commonly used guidance law for missiles in the homing phase of their flight. The resulting equations of motion are highly nonlinear even under simplifying assumptions. A substantial volume of literature is available on the performance evaluation of PN guidance laws from various points of view. These studies can be broadly classified as computational or analytical. In the computational studies, usually a linearized version of the state equations is assumed as the basic model and limited nonlinearities (such as missile acceleration saturation) are superposed on them. In analytical studies too, usually linearized versions of the state equations are used. The exceptions to this are the papers by Guelman [1–3], Yang, Hsiao, and Yeh [4], Becker [5], etc., which use the nonlinear state equations for performance and trajectory analysis. However, these papers also make a number of simplifying assumptions to make the nonlinear equations analytically tractable. These analytically convenient forms of PN guidance law are different from the ones actually implemented on-board. No studies are available in the literature on the performance of these realistic PN guidance laws in a nonlinear framework. This work relaxes an important simplifying assumption, made while defining PN, on the closing velocity between the missile and the target and obtains performance results for a realistic version of the true proportional navigation (TPN) guidance law. Both zero and non-zero miss-distance cases are considered. Also a uniform basis for comparison of the results available in the literature with the results obtained in this work is proposed to demonstrate the relative merits of the different formulations. The results on capture region are found to be substantially different from those obtained earlier. A computational study of the same problem has been carried out in [6]. A similar problem was investigated in a recently published paper [7], which considers the case of zero miss-distance only. However, the paper [7] does not provide a comparison with the earlier results [1].

In PN, the commanded acceleration is proportional to the line of sight (LOS) rate, i.e.,

$$a_M = C\dot{\theta} \tag{1}$$

where $a_M$ is the commanded acceleration, $\dot{\theta}$ is the LOS rate and $C$ is a proportionality factor. The value of the proportionality factor $C$, and the direction in which the latax is applied is different for different types of PN guidance laws.

For example, in the paper by Guelman [1], PN guidance law is classified as TPN and pure proportional navigation (PPN). In TPN, the commanded lateral acceleration (latax) of the missile is applied perpendicular to the LOS. In PPN, the commanded acceleration is applied normal to the missile velocity vector.

The value of $C$ could be a constant or it may vary with time. In many papers on PN guidance law [1, 8, 10], the parameter $C$ is defined as

$$C = NV_c \tag{2}$$

where $N$ is the navigation gain, $V_c$ $(= -\dot{r})$ is the closing velocity between missile and target, and $r$ is the LOS separation between missile and target.

In almost all analytical treatments of missile trajectories governed by the PN guidance law, the closing velocity ($V_c$) used in the guidance law is assumed to be constant [1, 8, 9]. However, when PN is actually implemented in practice, only the current closing velocity $V_c$ $(= -\dot{r})$ is considered [10]. In fact, the homing seeker uses a Doppler radar which continuously provides information about the current value of $\dot{r}$. Thus it appears that there is a difference in the way in which PN guidance law is analyzed and in the way in which it is implemented. This implies that the analytical results obtained may not be completely valid in a realistic situation.

One of the most complete solutions to a class of PN guidance law has been obtained by Guelman [1] for a nonmaneuvering target. Guelman obtained the closed-form solutions to the trajectory equations governed by the TPN guidance law. Further, these solutions were used to describe the performance of TPN guidance law in terms of capture region, boundedness of LOS rate, etc., for various navigation constants. However, all this has been done by assuming $C$ (and thus $V_c$) to be constant during the engagement.

The objective of this work is to determine the performance of TPN guidance law for nonmaneuvering targets when the above-mentioned restriction is not imposed and the TPN guidance law is defined in its implemented version. A motivation for this work arises from the fact that the very nature of TPN causes considerable change in the missile velocity (thus affecting the closing velocity) due to the non–zero component of missile lateral acceleration along the missile velocity vector. The issue of optimality of the PN guidance law is not addressed.

## II. PROBLEM FORMULATION

### A. Engagement Model and State Equations

A target $T$ and a missile $M$ are assumed to be point mass models on a plane moving with velocities $V_T$ and $V_M$, respectively, as shown in Fig. 1. The target is assumed to be a nonmaneuvering one. The position of the target $T$ is assumed to be the center of the relative coordinate system, the positive X-axis of which is along the straight-line trajectory of the target. The

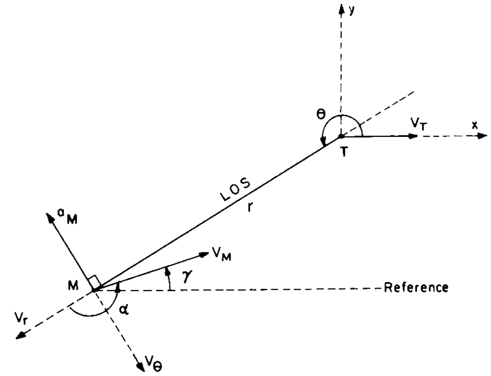


Fig. 1. Missile-target engagement geometry.

equations of motion of the missile-target engagement are obtained as follows

$$V_r(t) = \dot{r} = V_M \cos\alpha - V_T \cos\theta \tag{3}$$

$$V_\theta(t) = r\dot{\theta} = V_M \sin\alpha + V_T \sin\theta \tag{4}$$

where $V_r(t)$ and $V_\theta(t)$ are the relative velocity of the target with respect to the missile along the line of sight and perpendicular to the line of sight, respectively, at the time instant $t$. Note that the LOS angle $\theta$ is measured anti-clockwise from the target velocity direction. Also note that the closing velocity

$$V_c(t) = -V_r(t). \tag{5}$$

The basic principle of PN guidance law is that the commanded missile lateral acceleration (latax) is proportional to the LOS rate. Further, in the case of TPN guidance law, the commanded lateral acceleration $a_M$ is applied perpendicular to the LOS.

Thus, according to the PN law

$$a_M = C\dot{\theta} \tag{6}$$

where $C$ could be a constant or it may vary with time as a function of the state of the system.

In Guelman's paper [1], $C$ is assumed to be a constant. Furthermore $C$ could either be an arbitrary constant or it could be a function of the initial closing velocity $V_{c_0}$, i.e.,

$$C = NV_{c_0}. \tag{7}$$

However in most of the literature on PN guidance law which discuss its implementation [10] the parameter $C$ is defined as

$$C = NV_c(t) \tag{8}$$

where $V_c(t)$ is closing velocity at any time $t$.

Thus, here, the TPN guidance law is modified by using (8) instead of (7) in (6). Thus

$$a_M = NV_c(t)\dot{\theta}. \tag{9}$$

We call this guidance law realistic true proportional navigation (RTPN) guidance law.

Now, from Fig. 1, we have

$$\dot{\gamma} = -(a_M \cos\alpha)/V_M \tag{10}$$

$$\dot{V}_M = -a_M \sin\alpha \tag{11}$$

$$\gamma = \alpha + \theta - 2n. \tag{12}$$

Differentiating (12) with respect to time and substituting (10), we get

$$\dot{\alpha} = -(a_M \cos\alpha)/V_M - \dot{\theta}. \tag{13}$$

From (3), (5), and (9), we get

$$a_M = -N(V_M \cos\alpha - V_T \cos\theta)\dot{\theta}. \tag{14}$$

Substituting (14) and (4) in (13) we obtain

$$\dot{\alpha} = [1/(rV_M)][N(V_M \cos\alpha - V_T \cos\theta)\cos\alpha - V_M] \times [V_M \sin\alpha + V_T \sin\theta]. \tag{15}$$

Similarly, substituting (14) and (3) in (11), we get

$$\dot{V}_M = (N/r)(V_M \cos\alpha - V_T \cos\theta) \times (V_M \sin\alpha + V_T \sin\theta)\sin\alpha. \tag{16}$$

Thus, the state equations for the missile-target engagement with RTPN guidance law are given by (3), (4), (15), and (16). Note that a simple model is adopted here for analysis as this facilitates comparison with previous works which also make similar simplifying assumptions. It also helps to illustrate the salient features of the analysis without losing clarity due to the complexity of the model.

## B. Capture Equation

The missile captures the target when the separation between them becomes zero, i.e., when the value of the state $r$ becomes zero. In this section, we derive a differential equation which is dependent only on $r$.

Differentiating (3) and (4) we get

$$\dot{V}_r(t) = \dot{V}_M \cos\alpha - V_M \sin\alpha\dot{\alpha} + V_T \sin\theta\dot{\theta} \tag{17}$$

$$\dot{V}_\theta(t) = \dot{V}_M \sin\alpha + V_M \cos\alpha\dot{\alpha} + V_T \cos\theta\dot{\theta}. \tag{18}$$

Substituting the values of $\dot{V}_M$ and $\dot{\alpha}$ from (11) and (13) in the above equations and using (3) and (4), we get

$$\dot{V}_r(t) = V_\theta(t)\dot{\theta} \tag{19}$$

$$\dot{V}_\theta(t) = -a_M - V_r(t)\dot{\theta}. \tag{20}$$

Replacing $V_r(t)$ by $\dot{r}$ and $V_\theta(t)$ by $r\dot{\theta}$, we get

$$\ddot{r} - r(\dot{\theta})^2 = 0 \tag{21}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -a_M. \tag{22}$$

Substituting $a_M$ from (9) in (22), we get

$$r\ddot{\theta} + (2\dot{r} + NV_c(t))\dot{\theta} = 0. \tag{23}$$

Equations (21) and (23) describe the trajectory of the missile in the polar plane. From here onwards, we omit the argument $t$ for convenience. Substituting (9) in (20) we obtain

$$\dot{V}_\theta = (N-1)V_r\dot{\theta}. \tag{24}$$

Using (19) and (24) we arrive at the following relationship

$$kV_r\dot{V}_r = V_\theta\dot{V}_\theta \tag{25}$$

where, $k = N - 1$. Intregrating, we get

$$kV_r^2 - V_\theta^2 = A \tag{26}$$

where

$$A = kV_{r_0}^2 - V_{\theta_0}^2. \tag{27}$$

The subscript 0 denotes the initial values.

Multiplying $r$ on both sides of (19) and using (4), we get

$$r\dot{V}_r = V_\theta^2, \tag{28}$$

which when substituted in (26) yields

$$k(\dot{r})^2 - r(\ddot{r}) = A \tag{29}$$

which can be rewritten as

$$r(\ddot{r}) - k(\dot{r})^2 + A = 0 \tag{30}$$

with $A$ as given in (27).

The above equation depends on $r$ only. We call this equation the capture equation since it describes the behavior of $r$ with respect to time and it is this behavior which indicates whether capture is possible or not.

It is instructive to compare this equation with the analogous capture equation obtained by Guelman [1] for the simplified guidance law given by (6) where $C$ is an arbitrary constant. In this case the differential equation describing the behavior of $r$ is as follows

$$r(\ddot{r}) + (\dot{r})^2 + 2C\dot{r} = a \tag{31}$$

where

$$a = V_{r_0}^2 + 2CV_{r_0} + V_{\theta_0}^2. \tag{32}$$

This forms the corresponding capture equation for Guelman's TPN. Another version of the capture equation for Guelman's TPN is obtained by assuming

$$C = -NV_{r_0}. \tag{33}$$

Substituting which we get,

$$r(\ddot{r}) + (\dot{r})^2 - 2NV_{r_0}\dot{r} = a \tag{34}$$

where

$$a = V_{\theta_0}^2 + (1 - 2N)V_{r_0}^2. \tag{35}$$

In [1], Guelman obtained the closed-form solution for $r$ by solving the capture equation (31) and proved that capture is possible only for those initial conditions in $(V_{\theta_0}, V_{r_0})$-space which satisfy the conditions $a < 0$ and $\dot{r}_0 < 0$. This, in fact, defines the interior of a circle in the $(V_{\theta_0}, V_{r_0})$-space with center at $(0, -C)$ and radius equal to $C$.

The capture equation (30), obtained for RTPN, does not yield a closed-form solution. However, it is possible to perform a qualitative analysis of the capture equation and identify the region in which there is the possibility of capture and the region in which capture is not possible. The exact capture region is identified through computational means.

## III. PERFORMANCE ANALYSIS

### A. Analysis of Capture Equation

Unlike the capture equation (31) obtained in Guelman's case [1], the capture equation (30) obtained here is not solvable in closed form (i.e., obtain $r$ as a function of $t$). However, it is possible to conduct a qualitative study of this equation to obtain significant performance results.

Let,

$$z = \dot{r} \tag{36}$$

then,

$$\ddot{r} = dz/dt = (dz/dr)(dr/dt) = z(dz/dr). \tag{37}$$

Substituting (37) in (30) we obtain

$$rz(dz/dr) - kz^2 + A = 0 \tag{38}$$

which can be rewritten as,

$$(zdz)/(kz^2 - A) = dr/r. \tag{39}$$

Assuming $k \neq 0$, the above equation can be integrated to yield

$$kz^2 - A = (kV_{r_0}^2 - A)(r/r_0)^{2k}. \tag{40}$$

Using (27) and (36) we get,

$$k(V_r^2 - V_{r_0}^2) = \{(r/r_0)^{2k} - 1\}V_{\theta_0}^2. \tag{41}$$

When $k = 0$, (39) can be integrated to yield,

$$(V_r^2 - V_{r_0}^2) = 2V_{\theta_0}^2 \ln(r/r_0). \tag{42}$$

LEMMA 1. *Capture (define by $r = 0$) cannot occur when $N \leq 1$, and $V_{\theta_0} \neq 0$.*

PROOF. For $N < 1$, $k < 0$, and hence the left-hand side (LHS) of (41) remains bounded by $(-\infty, |k|V_{r_0}^2]$. For capture to occur, $r = 0$, in which case the right-hand side (RHS) of (41) tends to $+\infty$ as $r \to 0$. This implies that $r$ never becomes zero and so capture cannot occur.

Similarly, for $N = 1$, $k = 0$, and as $r \to 0$, the RHS of (42) tends to $-\infty$, whereas the LHS is bounded by $[-V_{r_0}^2, \infty)$. This again implies that capture cannot occur. □

LEMMA 2. *If $N > 1$, then capture (defined by $r = 0$) cannot occur when*

$$V_{\theta_0}^2 > (N-1)V_{r_0}^2. \tag{43}$$

PROOF. When $N > 1$, $k > 0$ and the LHS of (41) is bounded by $[-kV_{r_0}^2, \infty)$. But at $r = 0$, the RHS of (41) becomes $-V_{\theta_0}^2$. Thus, capture cannot occur if

$$-V_{\theta_0}^2 < -kV_{r_0}^2 \tag{44}$$

which leads to the condition in (43). □

From (41) we can write

$$V_r = [(V_{\theta_0}^2/k)((r/r_0)^{2k} - 1) + V_{r_0}^2]^{1/2}. \tag{45}$$

Since $V_r = \dot{r}$ is a continuous function of $t$, if $V_{r_0} > 0$, then $V_r > 0$ for all subsequent time and so capture cannot occur as $r$ never becomes zero.

All the above results lead to the following Theorem.

THEOREM 1. *If capture (defined by $r = 0$) occurs then the initial conditions in $(V_{\theta_0}, V_{r_0})$-space must satisfy $V_{r_0} < 0$ and*

1) *If $N \leq 1$, then $V_{\theta_0} = 0$* (46)
2) *If $N > 1$, then $V_{\theta_0}^2 \leq (N-1)V_{r_0}^2$.* (47)

In the above analysis we defined capture as the condition $r = 0$. However, missiles usually carry warheads which have non-zero lethal radius. This implies that capture may be said to occur even when $r \neq 0$. Suppose we define capture to occur whenever

$$r \leq r_m, \tag{48}$$

where $r_m$ is called the allowable miss-distance, and $0 < r_m < r_0$, then the identification of the capture region in the $(V_{\theta_0}, V_{r_0})$-space is done as follows.

LEMMA 3. *If $N < 1$ then capture (defined by $r \leq r_m$) cannot occur for*

$$\{(r_0/r_m)^{2-2N} - 1\}V_{\theta_0}^2 > (1-N)V_{r_0}^2. \tag{49}$$

PROOF. When $N < 1$, $k < 0$ and so the LHS of (41) is bounded by $(-\infty, -kV_{r_0}^2]$. For $r \leq r_m$, the RHS of (41) must satisfy,

$$\text{RHS} \geq \{(r_0/r_m)^{-2k} - 1\}V_{\theta_0}^2. \tag{50}$$

This directly leads to the condition given in the theorem. □

Similarly, we can state the following two lemmas. The proofs are straightforward and are omitted.

LEMMA 4. *If $N = 1$, then capture (defined by $r \leq r_m$) cannot occur for*

$$2V_{\theta_0}^2 \ln(r_0/r_m) > V_{r_0}^2. \tag{51}$$

LEMMA 5. *If $N > 1$, then capture (defined by $r \leq r_m$) cannot occur for*

$$\{1 - (r_m/r_0)^{2N-2}\}V_{\theta_0}^2 > (N-1)V_{r_0}^2. \tag{52}$$

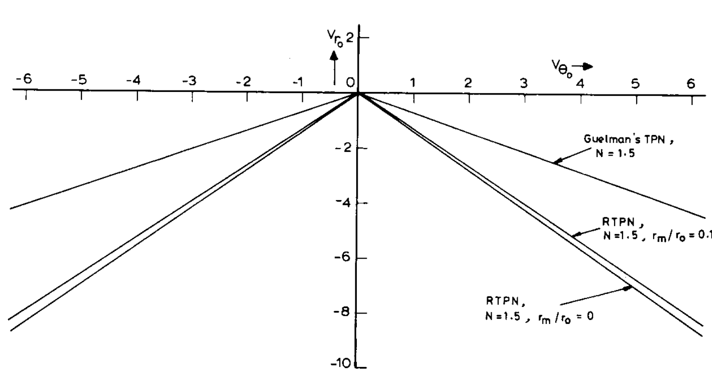


Fig. 2. Capture regions for fixed navigation gain.

All the above lemmas lead to the following Theorem.

THEOREM 2. *If capture (defined by $r \leq r_m$) occurs then the initial condition in the $(V_{\theta_0}, V_{r_0})$-space must satisfy $V_{r_0} < 0$ and*

1) *If $N = 1$, then* $(V_{r_0}^2/V_{\theta_0}^2) \geq 2\ln(r_0/r_m)$ (53)

2) *If $N \neq 1$, then* $(V_{r_0}^2/V_{\theta_0}^2) \geq \{1-(r_m/r_0)^{2N-2}\}/(N-1)$. (54)

B. Comparison with Guelman's Results

Until now we have identified the region in the $(V_{\theta_0}, V_{r_0})$-space in which there is possibility of capture with zero and non-zero miss-distances. We compare these results with the results in Guelman's analysis of TPN [1], on a uniform basis.

1) *Capture Region for a Fixed Navigation Gain*: In Guelman's paper [1], the guidance law is given by $a_M = C\dot{\theta}$ as expressed in (6), and $C$ is assumed to be a constant. Further, if we assume $C$ to be of the form given in (33), then it implies that $C$ is a constant during an engagement starting from a given initial condition $V_{r_0}$, but is different for engagements starting from different initial values of $V_{r_0}$. Note that $C$ does not depend on the value of $V_{\theta_0}$. Now, putting this value of $C$ from (33) in (32) we get the capture condition in the region $V_{r_0} < 0$, in case of Guelman's TPN, as

$$V_{\theta_0}^2 < (2N-1)V_{r_0}^2. \tag{55}$$

Assuming $N > 0.5$, we can write this as,

$$-V_{r_0} > |V_{\theta_0}|/\sqrt{2N-1}. \tag{56}$$

For a given value of $N$, the above equation represents a sector-like shape in the $(V_{\theta_0}, V_{r_0})$-space for different initial values of $V_r$ and $V_\theta$, in the region $V_{r_0} < 0$. This is shown in Fig. 2 for $N = 1.5$.

To obtain similar capture region for RTPN we use the conditions of Theorem 1, which shows that there is possibility of capture (with zero miss-distance) for $N > 1$, in the region $V_{r_0} < 0$, when

$$-V_{r_0} \geq |V_{\theta_0}|/\sqrt{N-1}. \tag{57}$$

The above equation also represents a sector-like shape for a given value of $N$ and is shown in Fig. 2 for $N = 1.5$.

When we consider an allowable miss-distance $r_m \neq 0$, the possible capture region for RTPN is given by Theorem 2. Note that there is a stable capture region here even for $N < 1$. In the region $V_{r_0} < 0$, when $N = 1$, the possible capture region is given by,

$$-V_{r_0} \geq \{2|V_{\theta_0}|\ln(r_0/r_m)\}^{1/2} \tag{58}$$

and when $N \neq 1$, it is given by

$$-V_{r_0} \geq |V_{\theta_0}|\{1-(r_m/r_0)^{2N-2}\}^{1/2}/\sqrt{(N-1)}. \tag{59}$$

This capture region is also sector-shaped and is shown in Fig. 2 for $N = 1.5$ and $(r_m/r_0) = 0.1$.

2) *Capture Region for a Variable Navigation Gain*: In Guelman's guidance law, if $C$ is assumed to be an arbitrary constant independent of the initial condition $V_{r_0}$ then the capture condition obtained is,

$$V_{r_0}^2 + 2CV_{r_0} + V_{\theta_0}^2 < 0 \tag{60}$$

which defines the interior of a circle in the $(V_{\theta_0}, V_{r_0})$-space with $(0, -C)$ as the center and $C$ as the radius. Fig. 3 shows this capture region for $C = 5$. In the earlier section (Section B1) since $C$ was assumed to be equal to $-NV_{r_0}$ for Guelman's TPN, it was reasonable to obtain RTPN by replacing $V_{r_0}$ with $V_r(t)$. But, now $C$ is a constant independent of $V_{r_0}$. Hence, it is clear that we must assume a new structure for Guelman's TPN law, such that it becomes dependent on $V_{r_0}$, while simultaneously keeping $C$ constant. This can be done by assuming a variable navigation gain,

$$N = -C/V_{r_0}. \tag{61}$$

For different values of $V_{r_0}$ (i.e., different initial conditions) the navigation gain will vary. The resulting

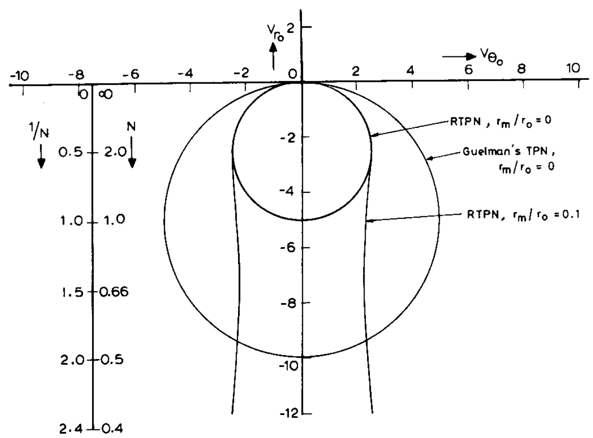


Fig. 3. Capture regions for variable navigation gain.

guidance law for RTPN can now be written as

$$a_M = (-C/V_{r_0})V_c(t)\dot{\theta}. \qquad (62)$$

Note that this reduces to Guelman's TPN guidance law when $V_c(t)$ is replaced with $V_{c_0}$.

Now, to obtain the capture region for RTPN we use the conditions of Theorem 1, with (61). This shows that there is possibility of capture (with zero miss-distance), for $N > 1$, when

$$V_{r_0}^2 + CV_{r_0} + V_{\theta_0}^2 \le 0 \qquad (63)$$

which defines a circle (and its interior) with radius $C/2$ and center at (0, $-C/2$) in the ($V_{\theta_0}$, $V_{r_0}$)-space. This is shown in Fig. 3 for $C = 5$.

When we consider an allowable miss-distance $r_m \neq 0$, the possible capture region for RTPN is given by the conditions in Theorem 2 in which we substitute (61). The possible capture region, when $V_{r_0} = -C$, is given by,

$$(V_{r_0} + C/2)^2 + pV_{\theta_0}^2 \le (C/2)^2 \qquad (64)$$

where

$$p = 1 - (r_m/r_0)^{2k} \qquad (65)$$

$$k = -(C/V_{r_0} + 1). \qquad (66)$$

When $V_{r_0} = -C$, the possible capture region is given by,

$$|V_{\theta_0}| \le C/\{2\ln(r_0/r_m)\}^{1/2}. \qquad (67)$$

This capture region for $r_m/r_0 = 0.1$ and $C = 5$ is shown in Fig. 3.

### C. Maximum Commanded Missile Acceleration

The latax commanded by the missile using RTPN can be expressed as a function of $r$ as follows. Equation (23) can be written as

$$r\ddot{\theta} = (N-1)\dot{r}\dot{\theta}. \qquad (68)$$

Letting,

$$y = \dot{\theta} \qquad (69)$$

we can rewrite (68) as,

$$dy/y = (N-2)\,dr/r \qquad (70)$$

which can be integrated to yield,

$$y = \dot{\theta} = \dot{\theta}_0(r/r_0)^{N-2}. \qquad (71)$$

Assuming $a_M$ to be of the form given in (9), we have for $N \neq 1$

$$|a_M| = (N|V_{\theta_0}|/r_0)[\{(r/r_0)^{2N-2} - 1\}V_{\theta_0}^2/(N-1) + V_{r_0}^2]^{1/2} \times (r/r_0)^{N-2} \qquad (72)$$

and for $N = 1$

$$|a_M| = (|V_{\theta_0}|/r_0)\{V_{r_0}^2 + 2V_{\theta_0}^2\ln(r/r_0)\}^{1/2}(r_0/r). \qquad (73)$$

It is easily seen that for $N < 2$, $|a_M| \to \infty$ as $r \to 0$. The missile acceleration not only remains bounded for $N > 2$, but also $|a_M| \to 0$ as $r \to 0$. Thus, the maximum acceleration commanded by the missile from an initial condition lying in the capture region and with allowable miss-distance $r_m = 0$ is given by

$$|a_M|_{\max} = |NV_{r_0}V_{\theta_0}|/r_0, \qquad \text{for} \quad N > 2$$

$$|a_M|_{\max} = (N|V_{\theta_0}|/r_0) \times [\{(r_m/r_0)^{2N-2} - 1\}V_{\theta_0}^2/(N-1) + V_{r_0}^2]^{1/2} \times (r_m/r_0)^{N-2} \qquad \text{for} \quad N < 2, \quad N \neq 1$$

$$|a_M|_{\max} = (|V_{\theta_0}|/r_0)\{V_{r_0}^2 + 2V_{\theta_0}^2\ln(r_m/r_0)\}^{1/2} \times (r_0/r_m), \qquad \text{for} \quad N = 1. \qquad (74)$$

## IV. COMPUTATIONAL RESULTS

In the previous section, the possible capture region for RTPN was obtained through a qualitative study of the capture equation. However, the exact capture region is obtained here for an example problem through computational means. This also substantiates the capture conditions obtained in the previous section.

## A. State Equations and Initial Conditions

The capture equation to be integrated can be written as

$$\dot{r} = x = f_1(r,x) \tag{75}$$

$$\dot{x} = (kx^2 - A)/r = f_2(r,x) \tag{76}$$

with initial conditions $r(0) = r_0$ and $x(0) = V_{r_0}$.

The state equations used are (3), (4), (15), and (16) for which the initial conditions are $r(0) = r_0$, $\theta(0) = \theta_0$, $\alpha(0) = \alpha_0$, and $V_M(0) = V_M$.

For a given set of initial conditions ($V_{\theta_0}$, $V_{r_0}$) we can determine a set of initial conditions for the capture equation or the state equations and integrate either of them numerically to determine if $r = 0$ (or $r \leq r_m$) occurs for some finite, positive value of $t$. Here we integrate both sets of equations and compare the results so as to have a control over computational accuracy.

The computational results are obtained for RTPN (zero and non-zero miss-distance) and for Guelman's TPN (non-zero miss-distance).

## B. Example Problem

In the subsequent section we consider the following example to obtain the capture region for RTPN

$$V_T = 10 \text{ m/s}; \qquad \theta_0 = 200°; \qquad r_0 = 100 \text{ m}.$$

The values of $V_{M_0}$ and $\alpha_0$ are chosen depending on the given initial conditions $V_{r_0}$ and $V_{\theta_0}$.

## C. Capture Regions

Here we only obtain the capture region for variable navigation gain since it permits easy comparison with Guelman's results. Capture region for variable navigation gain, using RTPN guidance law, are shown in Fig. 4. The capture circle as defined in Guelman's paper [1] for zero miss-distance is also shown here. We also plot the capture regions considering some non-zero miss-distances for RTPN guidance law. It is observed that the capture region for RTPN guidance law is much smaller compared with that of Guelman's TPN.

For the purpose of comparison, in Fig. 5 we also plot the capture region for Guelman's TPN for some non-zero miss-distances. These results were obtained computationally. It should be noted that in the graphs plotted here the $V_{\theta_0}$ and $V_{r_0}$ axes have been normalized by the constant target velocity $V_T$.

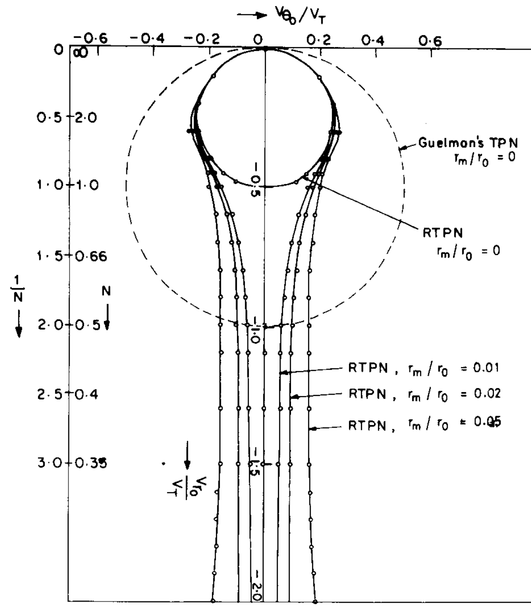


Fig. 4. Capture region for RTPN with various miss-distances.

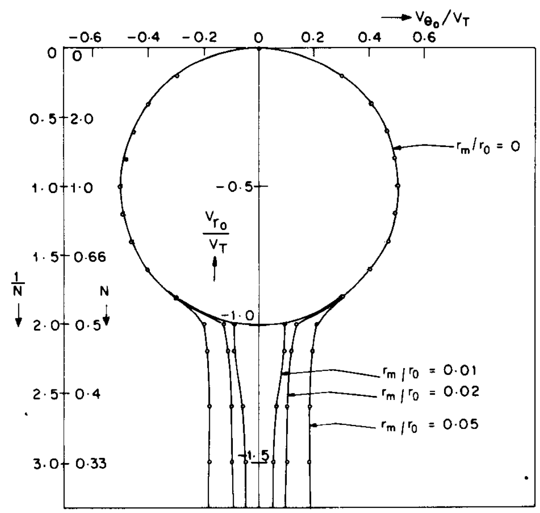


Fig. 5. Capture region for Guelman's TPN with various miss-distances.

In Fig. 4 we observe a circular capture region, using variable navigation constant $N$, for RTPN guidance law. Guelman's results are also shown in the same figure. It is observed that in RTPN, capture region remains circle shaped when $N$ is greater than one. But the region is smaller than that of Guelman's TPN guidance law. The line $V_{r_0} < 0$, $V_{\theta_0} = 0$ is a capture region in both the cases. If we consider some allowable miss-distance then the capture region expands. This is observed from Fig. 4 and Fig. 5 for both RTPN and Guelman's TPN. It is also observed that the capture

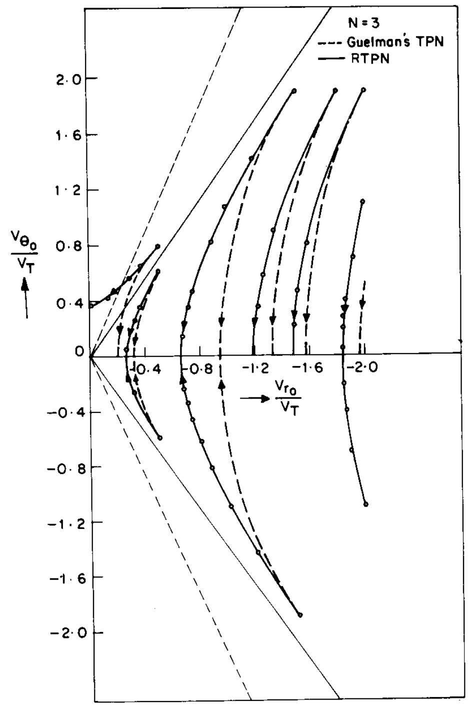


Fig. 6. Trajectory in ($V_{\theta_0}$, $V_{r_0}$)-space.

regions for Guelman's TPN expands more than that of RTPN for the same miss-distance.

The reason for the capture region obtained by using RTPN being smaller than that for Guelman's TPN guidance law is probably because the proportionality factor $C$ is not a constant for RTPN. In RTPN, it depends on the current closing velocity and navigation constant, whereas in Guelman's TPN, it depends on the initial closing velocity and the navigation gain. The value of closing velocity $V_c(t)$ reduces as engagement proceeds and the missile commands less latax with RTPN than in the case of Guelman's TPN. This latax is not sufficient for the initial conditions close to the boundary of Guelman's capture region. And therefore, RTPN fails to capture the target. Fig. 6 shows the trajectory of $V_\theta$ and $V_r$ in the ($V_{\theta_0}$, $V_{r_0}$)-space and substantiates the above arguments.

## V. CONCLUDING REMARKS

A survey of the literature on PN guidance law reveals that there is a significant difference between the way the PN law is implemented and the way it is actually used for performance analysis. This is mainly due to analytical convenience. This kind of dichotomy can be observed in all the variants of PN guidance law. In this paper we have chosen an implementable (realistic) version of TPN guidance law which was analyzed earlier by Guelman [1] using simplifying assumptions, and compared the performance results. It was observed that RTPN showed significant degradation in performance compared with Guelman's TPN.

Though this paper deals with TPN guidance law which has been criticized in the literature for its limitations in performance [11], it however raises a vital issue in the performance evaluation of guidance laws. The results in this paper demonstrates the need to evaluate guidance laws in the form in which it is implemented rather than the form which is convenient for analysis. Further, it also shows that analysis based upon simplifying assumptions may be grossly misleading. The study also points to the necessity of examining other guidance laws, which are proven to be better (e.g., PPN), in the same way. Moreover, this study can be extended to maneuvering targets and the results compared with those of augmented PN which might expand the capture region. The methods used in papers [12, 13] which use a robust linearization procedure to obtain the engagement results will be useful for this kind of study.

Another issue that these results raise is the issue of implementation of guidance laws. From these results it appears that the continuous use of closing velocity has a degrading effect on the performance. So, it should be examined whether the closing velocity term in the guidance law should be kept constant during the engagement. It is also observed that the capture region expands with increasing $N$. However, the question of latax saturation is closely related to these issues. In some parts of the capture region the LOS rate becomes unbounded. This may have a significant effect on the miss-distance. This could lead to an interesting study on the selection of an optimal value of $N$ and the use of the closing velocity ($V_c$) term.

## REFERENCES


[1] Guelman, M. (1976)
The closed-form solution of true proportional navigation.
*IEEE Transactions on Aerospace and Electronic Systems*, **AES-12**, 4 (July 1976), 472–482.

[2] Guelman, M. (1971)
A qualitative study of proportional navigation.
*IEEE Transactions on Aerospace and Electronic Systems*, **AES-7**, 4 (July 1971), 637–643.

[3] Guelman, M. (1971)
Proportional navigation with a maneuvering target.
*IEEE Transactions on Aerospace and Electronic Systems*, **AES-8**, 3 (May 1972), 364–371.

[4] Yang, C. D., Hsiao, F. B., and Yeh, F. B. (1989)
Generalized guidance law for homing missiles.
*IEEE Transactions on Aerospace and Electronic Systems*, **25**, 2 (Mar. 1989), 197–211.

[5] Becker, K. (1990)
Closed-form solution of pure proportional navigation.
*IEEE Transactions on Aerospace and Electronic Systems*, **26**, 3 (May 1990), 526–533.

[6] Dhar, A. (1992)
A performance study of an implemented version of true proportional navigation guidance law.
M.E. project report, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India, Jan. 1992.

[7] Yuan, P.-J., and Chern, J.-S. (1992)
Solutions of true proportional navigation for maneuvering and nonmaneuvering targets.
*Journal of Guidance, Control, and Dynamics*, **15**, 1 (Jan.–Feb. 1992), 268–271.

[8] Murtaugh, S. A., and Criel, H. E. (1966)
Fundamentals of proportional navigation.
*IEEE Spectrum*, (Dec. 1966), 75–85.

[9] Siouris, G. M. (1974)
Comparison between proportional and augmented proportional navigation.
*Nachrichtentechn 2*, **27**, 7 (July 1974), 278–280.

[10] Fossier, M. W. (1984)
The development of radar homing missiles.
*Journal of Guidance, Control and Dynamics*, **7**, 6 (Nov.–Dec. 1984), 641–651.

[11] Shukla, U. S., and Mahapatra, P. R. (1990)
The proportional navigation dilemma—Pure or true.
*IEEE Transactions on Aerospace and Electronic Systems*, **26**, 2 (Mar. 1990), 382–392.

[12] Shukla, U. S., and Mahapatra, P. R. (1988)
Generalised linear solution of proportional navigation.
*IEEE Transactions on Aerospace and Electronic Systems*, **24**, 3 (May 1988), 231–237.

[13] Mahapatra, P. R., and Shukla, U. S. (1989)
Accurate solution of proportional navigation for maneuvering targets.
*IEEE Transactions on Aerospace and Electronic Systems*, **25**, 1 (Jan. 1989), 81–88.

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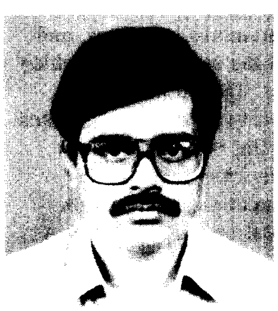

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