Matching complexes of $3 \times n$ grid graphs

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Submitted: Jun 14, 2021; Accepted: Oct 18, 2021; Published: Nov 5, 2021

Abstract

The *matching complex* of a graph G is a simplicial complex whose simplices are matchings in G. In the last few years the matching complexes of grid graphs have gained much attention among the topological combinatorists. In 2017, Braun and Hough obtained homological results related to the matching complexes of $2 \times n$ grid graphs. Further in 2019, Matsushita showed that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article we prove that the matching complexes of $3 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. We also give the comprehensive list of the dimensions of spheres appearing in the wedge.

Mathematics Subject Classifications: 05E45, 55P15

1 Introduction

A *matching* in a (simple) graph G is a collection of pairwise disjoint edges of G. The *matching complex* of G, denoted $M(G)$, is a simplicial complex whose vertex set is the edge set of G and simplices are all the matchings in G. The matching complexes first appeared in the 1979 work of Garst [4], where the matching complexes of complete bipartite graphs (also known as the chessboard complexes) were studied while dealing with the Tits coset complexes. In 1992, Bouc [1] studied the matching complexes of complete graphs

*Partially supported by the Infosys Foundation.

in connection with the Brown complexes and Quillen complexes. Thereafter, these complexes arose in connection with several areas of mathematics. For a broader perspective, see the 2003 survey article of Wachs [10].

Let $[n]$ denotes the set $\{1, 2, \ldots, n\}$. For two positive integers m, n, the $m \times n$ (rectangular) grid graph $\Gamma_{m,n}$ is a graph with vertex set $V(\Gamma_{m,n})$ and edge set $E(\Gamma_{m,n})$ defined as follows:

$$V(\Gamma_{m,n}) = \{(i,j) \in \mathbb{N}^2 : i \in [m], j \in [n]\}, \quad \text{and}$$
$$E(\Gamma_{m,n}) = \{((i,j),(i',j')) : |i-i'| + |j-j'| = 1\}.$$

The matching complex of $\Gamma_{1,n}$ was computed by Kozlov in [8]. In the same article, he also computed the matching complexes of cycle graphs. In 2005, Jonsson [6] studied the homotopical depth and topological connectivity of matching complexes of grid graphs and stated that "it is probably very hard to determine the homotopy type of" these complexes. In 2017, Braun and Hough [2] obtained homological results related to matching complexes of $2 \times n$ grid graphs. Matsushita [9], in 2019, extended their results by showing that the matching complexes of $2 \times n$ grid graphs are homotopy equivalent to a wedge of spheres. In this article, we compute the homotopy type of matching complexes of $3 \times n$ grid graphs. The main results of this article are summarised below.

Theorem 1. *For $n \geqslant 1$, the matching complex of $\Gamma_{3,n}$ is homotopy equivalent to a wedge of spheres. Moreover, if $n \in \{9k, 9k+1, \ldots, 9k+8\}$ for some $k \geqslant 0$, then*

$$M(\Gamma_{3,n}) \simeq \bigvee_{i=n-1}^{n+k-1} (\vee_{b_i} \mathbb{S}^i),$$

where b_i's are some positive integers and $\simeq$ denotes the homotopy equivalence of spaces.

For a graph G, a subset $I \subseteq V(G)$ is said to be *independent* if there are no edges in the induced subgraph $G[I]$, *i.e.*, $E(G[I]) = \emptyset$. The *independence complex* of G, denoted $\mathrm{Ind}(G)$, is a simplicial complex whose vertex set is $V(G)$ and simplices are all the independent subsets of G. The *line graph* of a graph G, denoted $L(G)$, is a graph with $V(L(G)) = E(G)$ and two distinct vertices $(a_1, b_1), (a_2, b_2) \in V(L(G))$ are adjacent if and only if $\{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset$. Note that the matching complex of G is same as the independence complex of its line graph, *i.e.*, $M(G) = \mathrm{Ind}(L(G))$.

Let G_n denotes the line graph of the grid graph $\Gamma_{3,n}$. To compute the homotopy type of $M(\Gamma_{3,n})$, we determine the homotopy type of $\mathrm{Ind}(G_n)$. The main idea used in this article for the computation of $\mathrm{Ind}(G_n)$ is to make a step by step careful choice to reduce the graph G_n and arrive at different classes of graphs (a total of nine). All these new nine classes of graphs have been defined in Section 3.1. To obtain the Theorem 1, we use simultaneous inductive arguments on the independence complexes of these ten classes of graphs. For a quick overview of the relations between all these ten classes of graphs, we refer the reader to see Figure 3.1.

Flow of the article: In the following section, we list out various definitions and results that are used in this article. Section 3 is subdivided into three major subsections. The first two subsections deal with the base cases for the graph G_n along with nine more associated classes of graphs. In the next subsection, Section 3.3, we provide and prove recursive formulae to compute the homotopy type of the independence complexes of these ten classes of graphs. The main result of Section 4 is Theorem 19, which gives the exact dimensions of the spheres occurring in the homotopy type of the independence complexes of the above mentioned ten classes of graphs.

2 Preliminaries

An (*abstract*) *simplicial complex* $\mathcal{K}$ is a collection of finite sets such that if $\tau \in \mathcal{K}$ and $\sigma \subset \tau$, then $\sigma \in \mathcal{K}$. The elements of $\mathcal{K}$ are called the *simplices* (or *faces*) of $\mathcal{K}$. If $\sigma \in \mathcal{K}$ and $|\sigma| = k+1$, then σ is said to be *k-dimensional*. The set of 0-dimensional simplices of $\mathcal{K}$ is denoted by $V(\mathcal{K})$, and its elements are called *vertices* of $\mathcal{K}$. A *subcomplex* of a simplicial complex $\mathcal{K}$ is a simplicial complex whose simplices are contained in $\mathcal{K}$. In this article, we always assume empty set as a simplex of any simplicial complex and we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to Kozlov's book [7].

For a simplex $\sigma \in \mathcal{K}$, define

$$\begin{aligned} \mathrm{lk}(\sigma, \mathcal{K}) &:= \{\tau \in \mathcal{K} : \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \mathcal{K}\}, \\ \mathrm{del}(\sigma, \mathcal{K}) &:= \{\tau \in \mathcal{K} : \sigma \nsubseteq \tau\}. \end{aligned}$$

The simplicial complexes $\mathrm{lk}(\sigma, \mathcal{K})$ and $\mathrm{del}(\sigma, \mathcal{K})$ are called *link* of σ in $\mathcal{K}$ and *(face) deletion* of σ in $\mathcal{K}$ respectively. The join of two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$, denoted as $\mathcal{K}_1 * \mathcal{K}_2$, is a simplicial complex whose simplices are disjoint union of simplices of $\mathcal{K}_1$ and $\mathcal{K}_2$. Let Δ^S denotes a $(|S|-1)$-dimensional simplex with vertex set S. The *cone* on $\mathcal{K}$ with apex a, denoted as $C_a(\mathcal{K})$, is defined as

$$C_a(\mathcal{K}) := \mathcal{K} * \Delta^{\{a\}}.$$

For $a, b \notin V(\mathcal{K})$, the suspension of $\mathcal{K}$, denoted as $\Sigma(\mathcal{K})$, is defined as

$$\Sigma(\mathcal{K}) := \mathcal{K} * \{a\} \cup \mathcal{K} * \{b\}).$$

Observe that for any vertex $v \in V(\mathcal{K})$, we have

$$\mathcal{K} = C_v(\mathrm{lk}(v, \mathcal{K})) \cup \mathrm{del}(v, \mathcal{K}) \text{ and } C_v(\mathrm{lk}(v, \mathcal{K})) \cap \mathrm{del}(v, \mathcal{K}) = \mathrm{lk}(v, \mathcal{K}).$$

Clearly, $C_v(\mathrm{lk}(v, \mathcal{K}))$ is contractible. Therefore, from [5, Example 0.14], we have the following.

Lemma 2. *Let* $\mathcal{K}$ *be a simplicial complex and* v *be a vertex of* $\mathcal{K}$*. If* $\mathrm{lk}(v, \mathcal{K})$ *is contractible in* $\mathrm{del}(v, \mathcal{K})$ *then*

$$\mathcal{K} \simeq \mathrm{del}(v, \mathcal{K}) \vee \Sigma(\mathrm{lk}(v, \mathcal{K})),$$

where $\vee$ *denotes the wedge of spaces.*

A (*simple*) *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is called the set of vertices and $E(G) \subseteq \binom{V(G)}{2}$, the set of (unordered) edges of G. The vertices $v_1, v_2 \in V(G)$ are said to be adjacent, if $(v_1, v_2) \in E(G)$.

The following observation directly follows from the definition of independence complexes of graphs.

Lemma 3. *Let $G_1 \sqcup G_2$ denotes the disjoint union of two graphs G_1 and G_2. Then*

$$\mathrm{Ind}(G_1 \sqcup G_2) \simeq \mathrm{Ind}(G_1) * \mathrm{Ind}(G_2).$$

Let G and H be two graphs. A map $f : V(G) \to V(H)$ is said to be a *graph homomorphism* if $(f(v), f(w)) \in E(H)$ for all $(v, w) \in E(G)$. A graph homomorphism is called an *isomorphism* if it is bijective and its inverse map is also a graph homomorphism. Two graphs G and H are said to be *isomorphic* if there is an isomorphism between them and we denote it by $G \cong H$.

For a subset $A \subseteq V(G)$, the set of *neighbours* of A is $N_G(A) = \{x \in V(G) : (x, a) \in E(G)$ for some $a \in A\}$. The *closed neighbourhood* set of $A \subseteq V(G)$, is $N_G[A] = N_G(A) \cup A$. If $A = \{v\}$ is a singleton set, then we write $N_G(v)$ (resp. $N_G[v]$) for $N_G(\{v\})$ (resp. $N_G[\{v\}]$). A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of the graph G. For a nonempty subset $U \subseteq V(G)$, the *induced subgraph* $G[U]$, is the subgraph of G with $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) : a, b \in U\}$. The graph $G[V(G) \setminus A]$ is denoted by $G - A$, for $A \subsetneq V(G)$. For a subset $B \subseteq E(G)$, we let $G - B$ to be the graph with the vertex set $V(G - B) = V(G)$ and the edge set $E(G - B) = E(G) \setminus B$.

Lemma 4. *Let G be a graph and $\{a, b\}$ be a 1-simplex in $\mathrm{Ind}(G)$. If $\mathrm{Ind}(G - N_G[\{a, b\}])$ is contractible, then*

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(\tilde{G}),$$

where $V(\tilde{G}) = V(G)$ and $E(\tilde{G}) = E(G) \cup \{(a, b)\}$.

Proof. Let $\sigma = \{a, b\}$. Observe that $\mathrm{del}(\sigma, \mathrm{Ind}(G)) = \mathrm{Ind}(\tilde{G})$ and $\mathrm{lk}(\sigma, \mathrm{Ind}(G)) = \mathrm{Ind}(\tilde{G}) = \mathrm{Ind}(G - N_G[\{a, b\}])$. Since $\mathrm{Ind}(G - N_G[\{a, b\}])$ is contractible, the result follows from Theorem 2. □

Lemma 5. *[3, Lemma 2.4] Let G be a graph and $u, u' \in V(G), u \neq u'$ such that $N_G(u) \subseteq N_G(u')$. Then*

$$\mathrm{Ind}(G) \simeq \mathrm{Ind}(G - \{u'\}).$$

Lemma 6. *[3, Lemma 2.5] Let G be graph and v be a simplicial vertex[1] of G. Let $N_G(v) = \{w_1, w_2, \ldots, w_k\}$. Then*

$$\mathrm{Ind}(G) \simeq \bigvee_{i=1}^{k} \Sigma(\mathrm{Ind}(G - N_G[w_i])).$$

For $r \geqslant 1$, the *path graph* P_r is a graph with $V(P_r) = [r]$ and $E(P_r) = \{(i, i+1) : i \in [r-1]\}$.

Lemma 7. *[8, Proposition 4.6] For $r \geqslant 1$,*

$$\mathrm{Ind}(P_r) \simeq \begin{cases} \mathbb{S}^{k-1} & \text{if } r = 3k, \\ \mathrm{pt} & \text{if } r = 3k+1, \\ \mathbb{S}^{k} & \text{if } r = 3k+2. \end{cases}$$

[1]A vertex v of G is called *simplicial* if $G[N_G(v)]$ is a complete graph, *i.e.* any two distinct vertices are adjacent.

For $r \geqslant 3$, the *cycle graph* C_r is the graph with $V(C_r) = [r]$ and $E(C_r) = \{(i, i+1) : i \in [r-1]\} \cup \{(1, r)\}$.

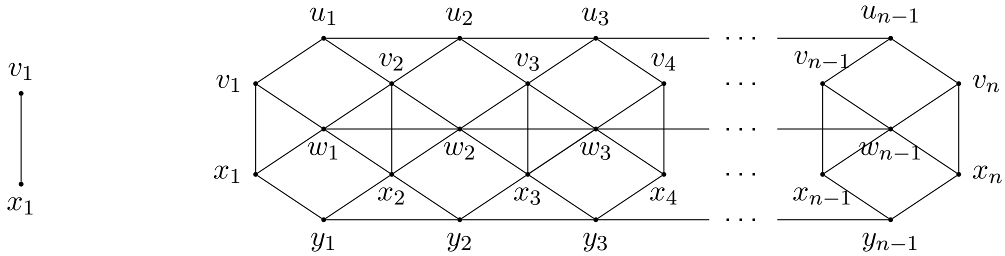
Lemma 8. *[8, Proposition 5.2] For $r \geqslant 3$,*

$$\mathrm{Ind}(C_r) \simeq \begin{cases} \mathbb{S}^{k-1} \bigvee \mathbb{S}^{k-1} & \text{if } r = 3k, \\ \mathbb{S}^{k-1} & \text{if } r = 3k \pm 1. \end{cases}$$

We now proceed towards the main graph of this article. Recall that for $m, n \in \mathbb{N}$, the $m \times n$ grid graph is denoted by $\Gamma_{m,n}$ and $G_n = L(\Gamma_{3,n})$ denotes the line graph of $\Gamma_{3,n}$ (see Figure 2.1 for example). Formally we define G_n with,

$$\begin{aligned} V(G_n) =& \{u_i, v_j, w_i, x_j, y_i : i \in [n-1], j \in [n]\}, \\ E(G_1) =& \{(v_1, x_1)\}, \\ E(G_2) =& \{(v_1, x_1), (w_1, v_1), (w_1, v_1), (w_1, x_1), (w_1, x_2), (u_1, v_1), \\ & (u_1, v_2), (v_2, x_2), (x_1, y_1), (y_1, x_2)\}, \text{ and} \\ E(G_n) =& \{(u_i, v_i), (v_i, w_i), (w_i, x_i), (x_i, y_i), (y_i, x_{i+1}), (x_{i+1}, w_i), (w_i, v_{i+1}), \\ & (v_{i+1}, u_i) : i \in [n-1]\} \sqcup \{(u_i, u_{i+1}), (w_i, w_{i+1}), (y_i, y_{i+1}) : i \in [n-2]\} \sqcup \\ & \{(v_i, x_i) : i \in [n]\}. \end{aligned}$$

for $n \geqslant 3$.



(a) G_1 (b) $G_n,\ n \geqslant 2$

Figure 2.1

3 Homotopy type of independence complexes of G_n and associated classes of graphs

In this section, we define nine new graph classes viz. $\{B_n\}_{n\in\mathbb{N}}$, $\{A_n\}_{n\in\mathbb{N}}$, $\{D_n\}_{n\in\mathbb{N}}$, $\{J_n\}_{n\in\mathbb{N}}$, $\{O_n\}_{n\in\mathbb{N}}$, $\{M_n\}_{n\in\mathbb{N}}$, $\{Q_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$, $\{H_n\}_{n\in\mathbb{N}}$ and compute the homotopy type of their independence complexes along with that of G_n. The n-th member of each of these graph classes contains a copy of G_n as an induced subgraph but not of G_{n+1}.

We divide this section into three subsections. In the first subsection, we define the above said nine classes of graphs and compute the homotopy type of independence complexes of these graphs along with G_n for $n = 1$. In the next subsection, we compute the homotopy type of independence complexes of all these graphs for $n = 2$. The final subsection is devoted towards proving recursive formulae for the independence complexes of all ten graph classes, thereby computing their homotopy types. In particular, we show that the independence complex of each of the graphs from these graph classes is a wedge of spheres up to homotopy.

For a better understanding of these inductive results, the recursive formulae of the homotopy equivalences obtained in Section 3.3 are depicted in Figure 3.1. Each node in Figure 3.1 denotes a graph class defined in Section 3.1 and an edge $X \xrightarrow{r\times(\Sigma^m,-k)} Y$ indicates that $\vee_r \Sigma^m(\mathrm{Ind}(Y_{n-k}))$ appears in the homotopy type formula of $\mathrm{Ind}(X_n)$ obtained in Section 3.3. For simplicity of notations, $1 \times (\Sigma^m, -k)$ is denoted by $(\Sigma^m, -k)$ and Σ^1 is denoted by Σ.

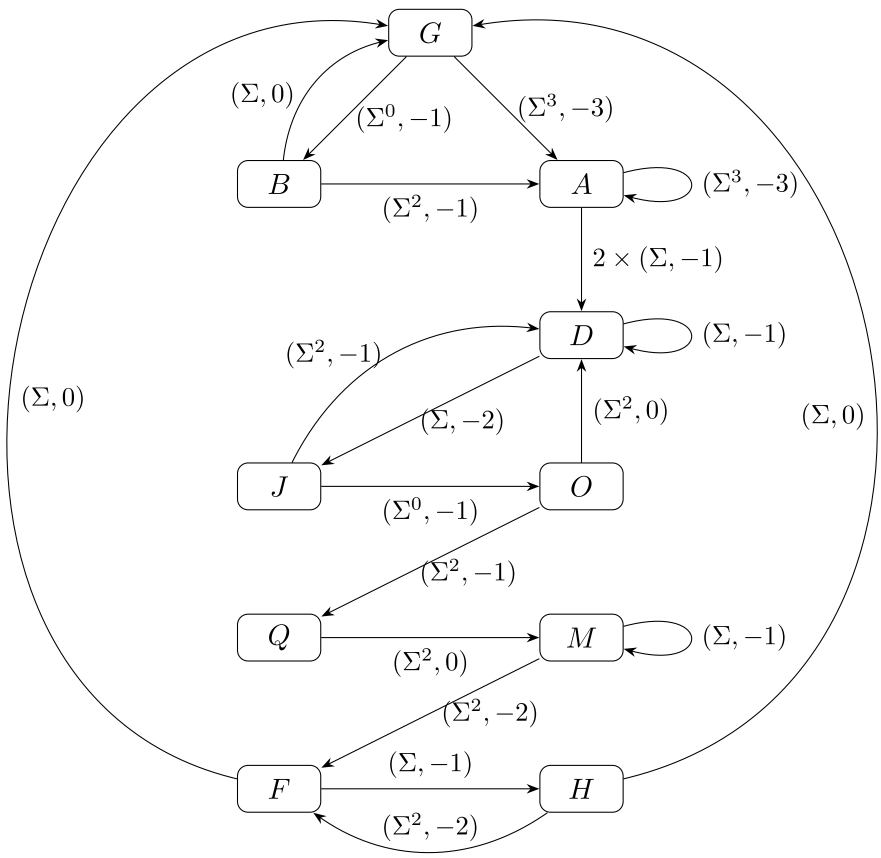


Figure 3.1: Summary of homotopy type formulae obtained in Section 3.3

3.1 Graph definitions and $n = 1$ computations

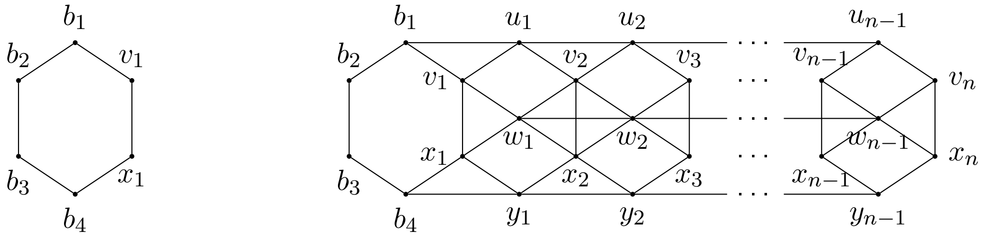
3.1.1 G_n

Since G_1 is an edge (see Figure 2.1a), $\mathrm{Ind}(G_1) \simeq \mathbb{S}^0$.

3.1.2 B_n

For $n \geqslant 1$, we define the graph B_n as follows:
$V(B_n) = V(G_n) \sqcup \{b_1, b_2, b_3, b_4\}$,
$E(B_1) = E(G_1) \sqcup \{(b_1, v_1), (b_1, b_2), (b_2, b_3), (b_3, b_4), (b_4, x_1)\}$ and for $n \geqslant 2$,

$E(B_n) = E(G_n) \sqcup \{(b_1, u_1), (b_1, v_1), (b_1, b_2), (b_2, b_3), (b_3, b_4), (b_4, x_1), (b_4, y_1)\}$.

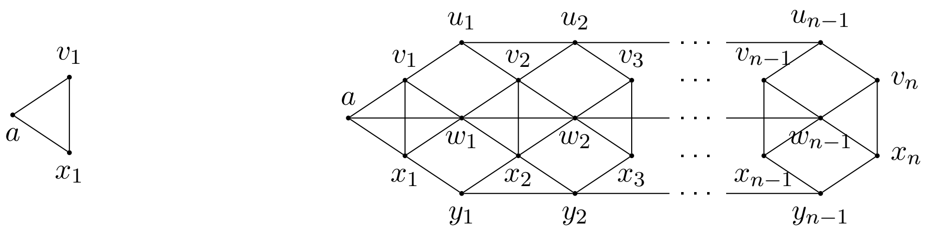


(a) B_1 (b) $B_n,\ n \geqslant 2$

Since $B_1 \cong C_6$, Theorem 8 implies that $\mathrm{Ind}(B_1) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$.

3.1.3 A_n

For $n \geqslant 1$, we define the graph A_n as follows: $V(A_n) = V(G_n) \sqcup \{a\}$,
$E(A_1) = E(G_1) \sqcup \{(a, x_1), (a, v_1)\}$ and for $n \geqslant 2$,
$E(A_n) = E(G_n) \sqcup \{(a, x_1), (a, v_1), (a, w_1)\}$.

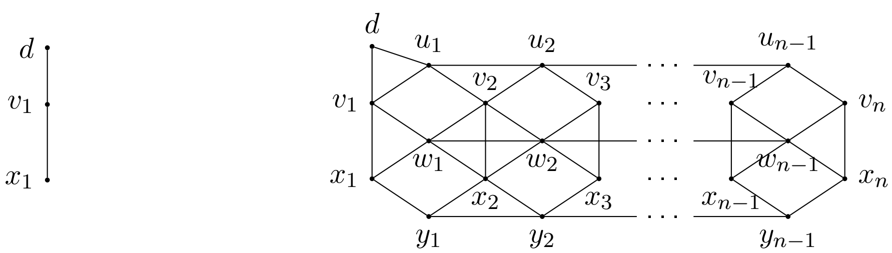


(a) A_1 (b) $A_n,\ n \geqslant 2$

Since $A_1 \simeq C_3$, Theorem 8 implies that $\mathrm{Ind}(A_1) \simeq \mathbb{S}^0 \vee \mathbb{S}^0$.

3.1.4 D_n

For $n \geqslant 1$, we define the graph D_n as follows:
$V(D_n) = V(G_n) \sqcup \{d\}$,
$E(D_1) = E(G_1) \sqcup \{(v_1, d)\}$ and for $n \geqslant 2$,
$E(D_n) = E(G_n) \sqcup \{(d, v_1), (d, u_1)\}$.



(a) D_1 (b) $D_n,\ n \geqslant 2$

Clearly, $\mathrm{Ind}(D_1) \simeq \mathbb{S}^0$.

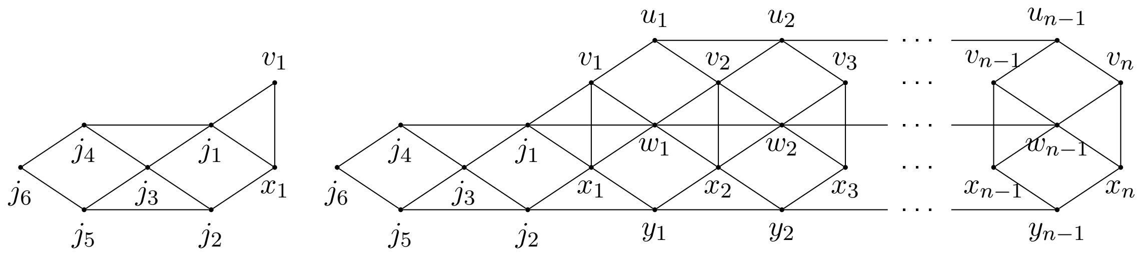
3.1.5 J_n

For $n \geqslant 1$, we define the graph J_n as follows:

$V(J_n) = V(G_n) \sqcup \{j_1, j_2, j_3, j_4, j_5, j_6\}$,

$E(J_1) = E(G_1) \sqcup \{(j_1, v_1), (j_1, x_1), (j_1, j_3), (j_1, j_4), (j_2, x_1), (j_2, j_3), (j_2, j_5), (j_3, j_4), (j_3, j_5), (j_4, j_6), (j_5, j_6)\}$ and for $n \geqslant 2$,

$E(J_n) = E(G_n) \sqcup \{(j_1, v_1), (j_1, x_1), (j_1, j_3), (j_1, j_4), (j_2, x_1), (j_2, j_3), (j_2, j_5), (j_3, j_4), (j_3, j_5), (j_4, j_6), (j_5, j_6), (j_2, y_1), (j_1, w_1)\}$.



(a) J_1 (b) $J_n,\ n \geqslant 2$

Since $N_{J_1}(j_6) \subseteq N_{J_1}(j_3)$, Theorem 5 implies that $\mathrm{Ind}(J_1) \simeq \mathrm{Ind}(J_1 - \{j_3\})$. Let J_1' be the graph $J_1 - \{j_3\}$. Using the fact that v_1 is a simplicial vertex in J_1' and $N_{J_1'}(v_1) = \{j_1, x_1\}$, from Theorem 6 we get that

$$\mathrm{Ind}(J_1') \simeq \Sigma(\mathrm{Ind}(J_1' - N_{J_1'}[j_1])) \vee \Sigma(\mathrm{Ind}(J_1' - N_{J_1'}[x_1])).$$

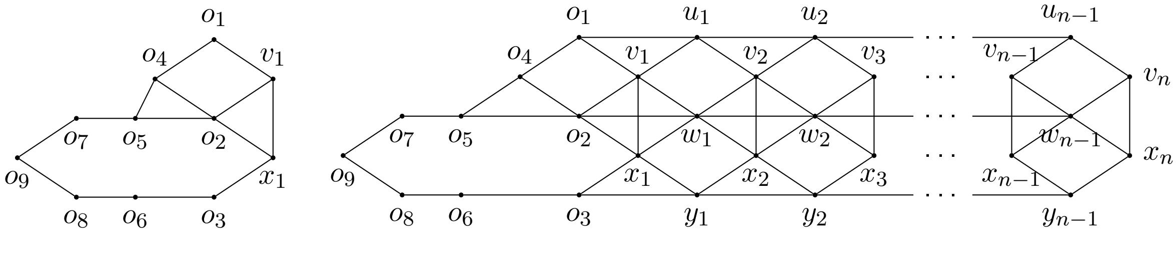
Observe that $J_1' - N_{J_1'}[j_1] \cong P_3 \cong J_1' - N_{J_1'}[x_1]$. Therefore using Theorem 7, we conclude that $\mathrm{Ind}(J_1) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$.

3.1.6 O_n

For $n \geqslant 1$, we define the graph J_n as follows:

$$\begin{aligned} V(O_n) =& V(G_n) \sqcup \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9\}, \\ E(O_1) =& E(G_1) \sqcup \{(o_1, v_1), (o_1, o_4), (o_2, v_1), (o_2, x_1), (o_2, o_4), (o_2, o_5), (o_3, x_1), (o_3, o_6), \\ & (o_4, o_5), (o_5, o_7), (o_6, o_8), (o_7, o_9), (o_8, o_9)\} \text{ and} \\ E(O_n) =& E(G_n) \sqcup \{(o_1, u_1), (o_1, v_1), (o_1, v_4), (o_2, v_1), (o_2, x_1), (o_2, o_4), (o_2, o_5), (o_2, w_1), \\ & (o_3, x_1), (o_3, y_1), (o_3, o_6), (o_4, o_5), (o_5 o_8), (o_7, o_9), (o_8, o_9)\} \end{aligned}$$

for $n \geqslant 2$.

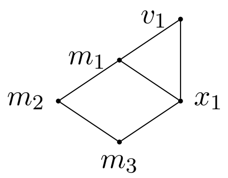


(a) O_1 (b) $O_n,\ n \geqslant 2$

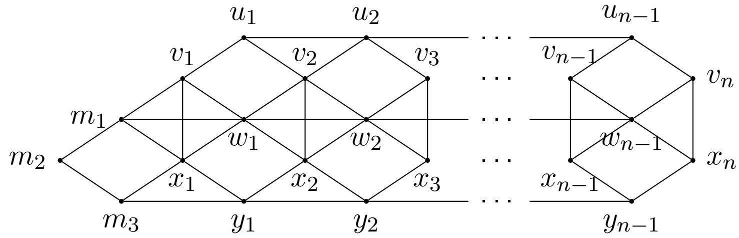
Since $N_{O_1}(o_1) \subseteq N_{O_1}(o_2)$, Theorem 5 implies that $\mathrm{Ind}(O_1) \simeq \mathrm{Ind}(O_1 - \{o_2\})$. Observe that $O_1 - \{o_2\} \cong C_{10}$, thus by Lemma 8 we get that $\mathrm{Ind}(O_1) \simeq \mathbb{S}^2$.

3.1.7 M_n

For $n \geqslant 1$, we define the graph M_n as follows:



(a) M_1



(b) $M_n,\ n \geqslant 2$

$V(M_n) = V(G_n) \sqcup \{m_1, m_2, m_3\}$,
$E(M_1) = E(G_1) \sqcup \{(m_1, v_1), (m_1, x_1), (m_1, m_2), (m_2, m_3), (m_3, x_1)\}$ and for $n \geqslant 2$,
$E(M_n) = E(G_n) \sqcup \{(m_1, v_1), (m_1, x_1), (m_1, w_1), (m_1, m_2), (m_2, m_3), (m_3, x_1), (m_3, y_1)\}$.

Since $N_{M_1}(m_2) \subseteq N_{M_1}(x_1)$ and $M_1 - \{x_1\} \cong P_4$, Theorem 5 and Theorem 7 imply that

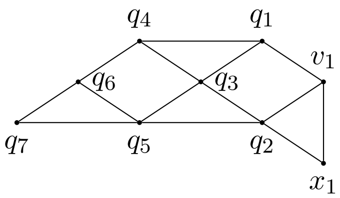
$$\mathrm{Ind}(M_1) \simeq \mathrm{Ind}(M_1 - \{x_1\}) \simeq \mathrm{Ind}(P_4) \simeq \mathrm{pt}.$$

3.1.8 Q_n

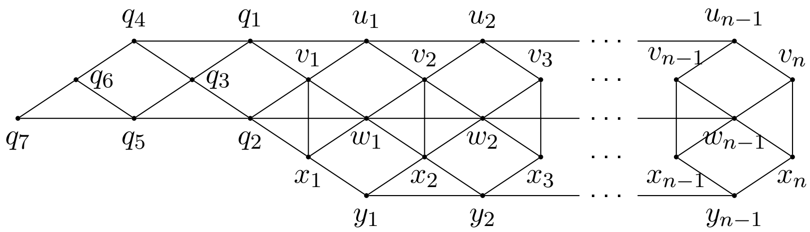
For $n \geqslant 1$, we define the graph Q_n as follows:

$$\begin{aligned}
V(Q_n) &= V(G_n) \sqcup \{q_1, q_2, q_3, q_4, q_5, q_6, q_7\},\\
E(Q_1) &= E(G_1) \sqcup \{(q_1, v_1), (q_1, q_3), (q_1, q_4), (q_2, v_1), (q_2, x_1), (q_2, q_3), (q_2, q_5),\\
&\quad (q_3, q_4), (q_3, q_5), (q_4, q_6), (q_5, q_6), (q_5, q_7), (q_6, q_7)\} \text{ and}\\
E(Q_n) &= E(G_n) \sqcup \{(q_1, u_1), (q_1, v_1), (q_1, q_3), (q_1, q_4), (q_2, v_1), (q_2, x_1),\\
&\quad (q_2, w_1), (q_2, q_3), (q_2, q_5), (q_3, q_4), (q_3, q_5), (q_4, q_6), (q_5, q_6), (q_5, q_7), (q_6, q_7)\}
\end{aligned}$$

for $n \geqslant 2$.



(a) Q_1



(b) $Q_n,\ n \geqslant 2$

Since q_7 is a simplicial vertex in Q_1 and $N_{Q_1}(q_7) = \{q_5, q_6\}$, Theorem 6 implies that

$$\mathrm{Ind}(Q_1) \simeq \Sigma(\mathrm{Ind}(Q_1 - N_{Q_1}[q_5])) \vee \Sigma(\mathrm{Ind}(Q_1 - N_{Q_1}[q_6])).$$

Observe that $Q_1 - N_{Q_1}[q_5] \cong P_4$, therefore $\mathrm{Ind}(Q_1 - N_{Q_1}[q_5])$ is contractible by Theorem 7. Since $N_{Q_1 - N_{Q_1}[q_6]}(q_3) \subseteq N_{Q_1 - N_{Q_1}[q_6]}(v_1)$, Theorem 5 implies that

$$\mathrm{Ind}(Q_1 - N_{Q_1}[q_6]) \simeq \mathrm{Ind}(Q_1 - N_{Q_1}[q_6] - \{v_1\}).$$

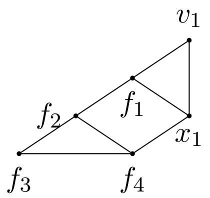
Since $Q_1 - N_{Q_1}[q_6] - \{v_1\} \cong P_4$, we conclude that $\mathrm{Ind}(Q_1)$ is contractible.

3.1.9 F_n

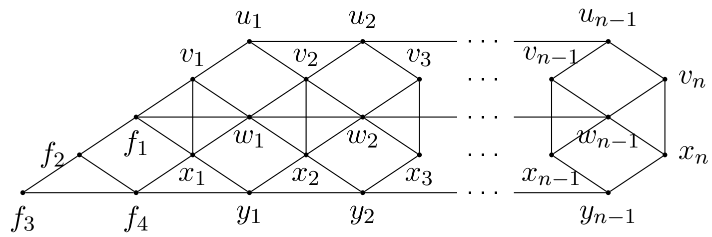
For $n \geqslant 1$, we define the graph F_n as follows:

$$V(F_n) = V(G_n) \sqcup \{f_1, f_2, f_3, f_4\},$$
$$E(F_1) = E(G_1) \sqcup \{(f_1, v_1), (f_1, x_1), (f_1, f_2), (f_2, f_3), (f_2, f_4), (f_3, f_4), (f_4, x_1)\} \text{ and}$$
$$E(F_n) = E(G_n) \sqcup \{(f_1, v_1), (f_1, x_1), (f_1, f_2), (f_1, w_1), (f_2, f_3), (f_2, f_4), (f_3, f_4), (f_4, x_1), (f_4, y_1)\}$$

for $n \geqslant 2$.



(a) F_1



(b) $F_n,\ n \geqslant 2$

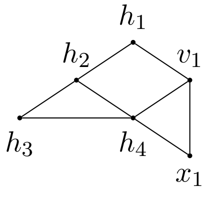
Observe that f_3 is a simplicial vertex in F_1 and $N_{F_1}(f_3) = \{f_2, f_4\}$. Using Theorem 6, we get that

$$\text{Ind}(F_1) \simeq \Sigma(\text{Ind}(F_1 - N_{F_1}[f_2])) \vee \Sigma(\text{Ind}(F_1 - N_{F_1}[f_4])).$$

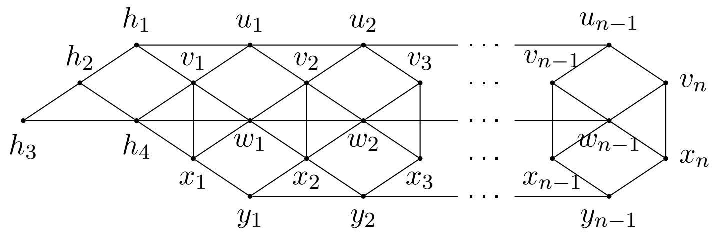
Since $F_1 - N_{F_1}[f_2] = F_1 - \{f_1, f_2, f_3, f_4\} \cong P_2 \cong F_1 - \{x_1, f_2, f_3, f_4\} = F_1 - N_{F_1}[f_4]$, $\text{Ind}(F_1) \simeq \Sigma(\text{Ind}(P_2)) \vee \Sigma(\text{Ind}(P_2)) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$.

3.1.10 H_n

For $n \geqslant 1$, we define the graph H_n as follows:
$V(H_n) = V(G_n) \sqcup \{h_1, h_2, h_3, h_4\}$,
$E(H_1) = E(G_1) \sqcup \{(h_1, v_1), (h_1, h_2), (h_2, h_3), (h_2, h_4), (h_3, h_4), (h_4, v_1), (h_4, x_1)\}$ and for $n \geqslant 2$,
$E(H_n) = E(G_n) \sqcup \{(h_1, v_1), (h_1, u_1), (h_1, h_2), (h_2, h_3), (h_2, h_4), (h_3, h_4), (h_4, v_1), (h_4, x_1), (h_4, w_1)\}$.



(a) H_1



(b) $H_n,\ n \geqslant 2$

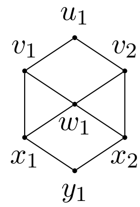
Since h_3 is a simplicial vertex in H_1 and $N_{H_1}(h_3) = \{h_2, h_4\}$, Theorem 6 implies that

$$\mathrm{Ind}(H_1) \simeq \Sigma(\mathrm{Ind}(H_1 - N_{H_1}[h_2])) \vee \Sigma(\mathrm{Ind}(H_1 - N_{H_1}[h_4])).$$

Note that $H_1 - N_{H_1}[h_2] = H_1 - \{h_1, h_2, h_3, h_4\} \cong P_2$ and $H_1 - N_{H_1}[h_4] = H_1 - \{v_1, x_1, h_2, h_3, h_4\} \cong P_1$, we get that $\mathrm{Ind}(H_1) \simeq \Sigma(\mathrm{Ind}(P_2)) \vee \Sigma(\mathrm{Ind}(P_1)) \simeq \mathbb{S}^1 \vee \mathrm{pt} \simeq \mathbb{S}^1$.

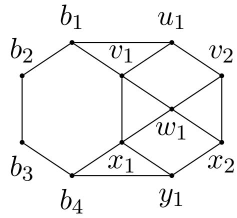
3.2 Case $n = 2$ computation

3.2.1 G_2



In G_2, $N_{G_2}(u_1) \subseteq N_{G_2}(w_1)$ (see figure on the right), thus by Lemma 5, $\mathrm{Ind}(G_2) \simeq \mathrm{Ind}(G_2 - \{w_1\})$. Since $G_2 - \{w_1\} \cong C_6$, Lemma 8 implies that $\mathrm{Ind}(G_2 - \{w_1\}) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$. Therefore $\mathrm{Ind}(G_2) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$.

3.2.2 B_2



We consider the vertex b_4 in B_2 (see figure on the right) and analyse $\mathrm{del}(b_4, \mathrm{Ind}(B_2))$ and $\mathrm{lk}(b_4, \mathrm{Ind}(B_2))$.

First note that $\mathrm{lk}(b_4, \mathrm{Ind}(B_2)) = \mathrm{Ind}(B_2 - N_{B_2}[b_4])$. Let B_2' denote the graph $B_2 - N_{B_2}[b_4]$ (see Figure 3.11c). Since $N_{B_2'}(b_2) \subseteq N_{B_2'}(v_1)$, Lemma 5 gives that $\mathrm{lk}(b_4, \mathrm{Ind}(B_2)) \simeq \mathrm{Ind}(B_2' - \{v_1\})$. Let B_2'' denote the graph $B_2' - \{v_1\} - \{(u_1, v_2)\}$.

Claim: $\mathrm{Ind}(B_2'') \simeq \mathrm{Ind}(B_2' - \{v_1\})$.

Since $(u_1, v_2) \notin E(B_2'')$, $\{u_1, v_2\} \in \mathrm{Ind}(B_2'')$. Observe that b_2 is an isolated vertex in $B_2'' - N_{B_2''}[\{u_1, v_2\}]$ and hence Lemma 3 implies that $\mathrm{Ind}(B_2'' - N_{B_2''}[\{u_1, v_2\}])$ is contractible. Therefore by Lemma 4, $\mathrm{Ind}(B_2'') \simeq \mathrm{Ind}(B_2'' \cup \{(u_1, v_2)\}) = \mathrm{Ind}(B_2' - \{v_1\})$.

Observe that, $N_{B_2''}(u_1) = N_{B_2''}(b_2)$, so Theorem 5 implies that $\mathrm{Ind}(B_2'') \simeq \mathrm{Ind}(B_2'' - \{b_2\})$. Since $V(B_2'' - \{b_2\}) \cap N_{B_2 - \{b_4\}}(b_3) = \emptyset$, $\mathrm{Ind}(B_2'' - \{b_2\}) * \{b_3\} \subseteq \mathrm{Ind}(B_2 - \{b_4\}) = \mathrm{del}(b_4, \mathrm{Ind}(B_2))$. Hence the inclusion map $\mathrm{Ind}(B_2'' - \{b_2\}) \hookrightarrow \mathrm{del}(b_4, \mathrm{Ind}(B_2))$ is null homotopic. Therefore the following composition of maps is null homotopic

$$\mathrm{lk}(b_4, \mathrm{Ind}(B_2)) \xrightarrow{\simeq} \mathrm{Ind}(B_2' - \{v_1\}) \xrightarrow{\simeq} \mathrm{Ind}(B_2'') \xrightarrow{\simeq} \mathrm{Ind}(B_2'' - \{b_2\}) \hookrightarrow \mathrm{del}(b_4, \mathrm{Ind}(B_2)).$$

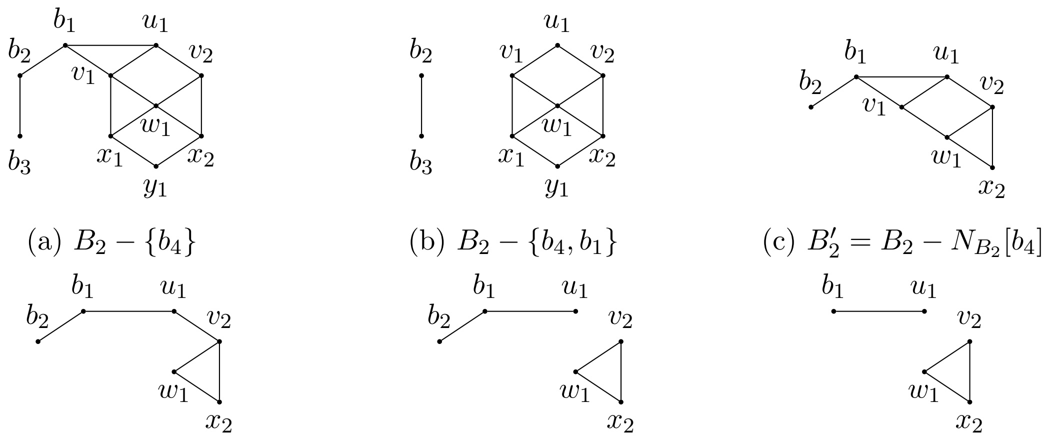
Hence $\mathrm{lk}(b_4, \mathrm{Ind}(B_2))$ is contractible in $\mathrm{del}(b_4, \mathrm{Ind}(B_2))$ and therefore by Lemma 2,

$$\mathrm{Ind}(B_2) \simeq \mathrm{del}(b_4, \mathrm{Ind}(B_2)) \vee \Sigma(\mathrm{lk}(b_4, \mathrm{Ind}(B_2))). \tag{1}$$

Note that $B_2'' - \{b_2\} \cong P_2 \sqcup A_1$, hence $\mathrm{lk}(b_4, \mathrm{Ind}(B_2)) \simeq \mathrm{Ind}(B_2'' - \{b_2\}) \simeq \mathrm{Ind}(P_2 \sqcup A_1)$. Also, $\mathrm{del}(b_4, \mathrm{Ind}(B_2)) = \mathrm{Ind}(B_2 - \{b_4\})$ and $N_{B_2 - \{b_4\}}(b_3) \subseteq N_{B_2 - \{b_4\}}(b_1)$, Lemma 5 implies that $\mathrm{del}(b_4, \mathrm{Ind}(B_2)) \simeq \mathrm{Ind}(B_2 - \{b_4, b_1\})$. However, $B_2 - \{b_4, b_1\}$ is isomorphic to $P_2 \sqcup G_2$ (see Figure 3.11b), therefore $\mathrm{del}(b_4, \mathrm{Ind}(B_2)) \simeq \Sigma(\mathrm{Ind}(G_2))$. Hence from Equation (1), we get the following homotopy equivalence.

$$\mathrm{Ind}(B_2) \simeq \Sigma(\mathrm{Ind}(G_2)) \vee \Sigma^2(\mathrm{Ind}(A_1)). \tag{2}$$

Since $\mathrm{Ind}(A_1) \simeq \mathbb{S}^0 \vee \mathbb{S}^0$ (see Section 3.1.3) and $\mathrm{Ind}(G_2) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ (see Section 3.2.1), we get that $\mathrm{Ind}(B_2) \simeq (\mathbb{S}^2 \vee \mathbb{S}^2) \vee \Sigma^2(\mathbb{S}^0 \vee \mathbb{S}^0) \simeq \vee_4 \mathbb{S}^2$.

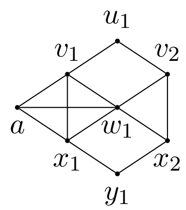


(a) $B_2 - \{b_4\}$ (b) $B_2 - \{b_4, b_1\}$ (c) $B_2' = B_2 - N_{B_2}[b_4]$

(d) $B_2' - \{v_1\}$ (e) B_2'' (f) $B_2'' - \{b_2\}$

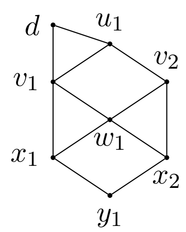
Figure 3.11

3.2.3 A_2



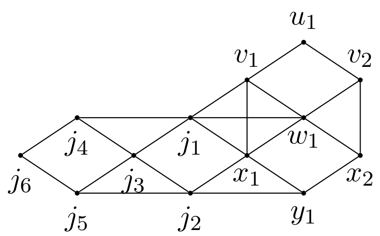
Note that $N_{A_2}(u_1) \subseteq N_{A_2}(w_1)$ (see figure on the right), therefore by Lemma 5, $\text{Ind}(A_2) \simeq \text{Ind}(A_2 - \{w_1\})$. Let $A_2' = A_2 - \{w_1\}$. Since a is a simplicial vertex in A_2' and $N_{A_2'}(a) = \{x_1, v_1\}$, Lemma 6 implies that $\text{Ind}(A_2') \simeq \Sigma(\text{Ind}(A_2' - N_{A_2'}[x_1])) \vee \Sigma(\text{Ind}(A_2' - N_{A_2'}[v_1]))$. The graph $A_2' - N_{A_2'}[x_1] = A_2 - \{w_1, a, x_1, v_1, y_1\} \cong P_3$. Therefore $\text{Ind}(A_2' - N[x_1]) \simeq \mathbb{S}^0$. Also, $A_2' - N_{A_2'}[x_1] \cong A_2' - N_{A_2'}[v_1]$. Hence $\text{Ind}(A_2) \simeq \text{Ind}(A_2') \simeq \Sigma(\mathbb{S}^0) \vee \Sigma(\mathbb{S}^0) = \mathbb{S}^1 \vee \mathbb{S}^1$.

3.2.4 D_2



Note that $N_{D_2}(y_1) \subseteq N_{D_2}(w_1)$ (see figure on the right), therefore by Lemma 5, $\text{Ind}(D_2) \simeq \text{Ind}(D_2 - \{w_1\})$. Let $D_2' = D_2 - \{w_1\}$. Since d is a simplicial vertex in D_2' and $N_{D_2'}(d) = \{u_1, v_1\}$, Lemma 6 implies that $\text{Ind}(D_2') \simeq \Sigma(\text{Ind}(D_2' - N_{D_2'}[u_1])) \vee \Sigma(\text{Ind}(D_2' - N_{D_2'}[v_1]))$. The graph $D_2' - N_{D_2'}[u_1] = D_2 - \{w_1, d, u_1, v_1, v_2\} \cong P_3$. Therefore $\text{Ind}(D_2' - N_{D_2'}[u_1]) \simeq \mathbb{S}^0$. Also, $D_2' - N_{D_2'}[u_1] \cong D_2' - N_{D_2'}[v_1]$, therefore $\text{Ind}(D_2' - N_{D_2'}[v_1]) \simeq \mathbb{S}^0$. Hence $\text{Ind}(D_2) \simeq \text{Ind}(D_2') \simeq \Sigma(\mathbb{S}^0) \vee \Sigma(\mathbb{S}^0) = \mathbb{S}^1 \vee \mathbb{S}^1$.

3.2.5 J_2



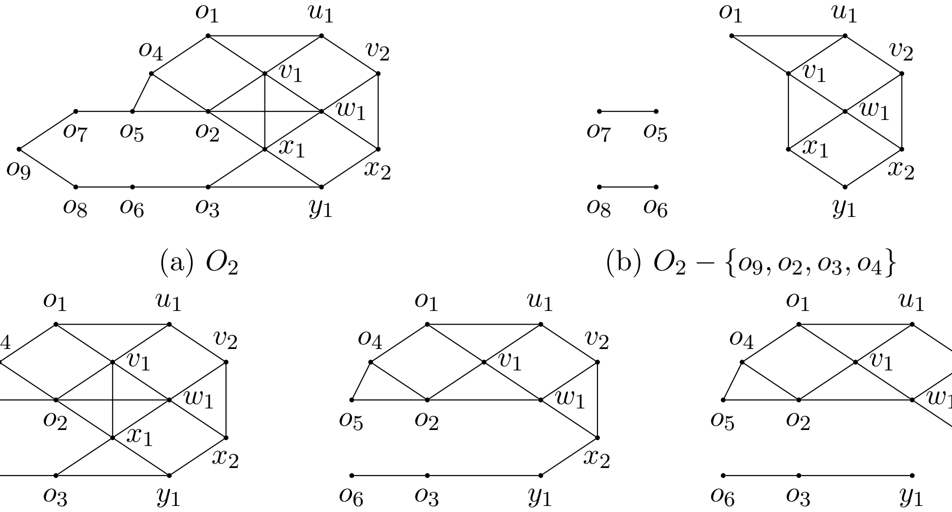
Since $N_{J_2}(j_6) \subseteq N_{J_2}(j_3)$ (see figure on the right), from Lemma 5, $\text{Ind}(J_2) \simeq \text{Ind}(J_2 - \{j_3\})$. Observe that $J_2 - \{j_3, x_1\} \cong O_1$ and therefore from Section 3.1.6 we see that $\text{del}(x_1, \text{Ind}(J_2 - \{j_3\})) \simeq \text{Ind}(O_1) \simeq \mathbb{S}^2$.

Note that, $\text{lk}(x_1, \text{Ind}(J_2 - \{j_3\})) \simeq \text{Ind}(J_2 - \{x_1, j_3, j_1, j_2, v_1, w_1, y_1\}) \cong \text{Ind}(P_2 \sqcup P_3) \simeq \mathbb{S}^1$. Since the fundamental group of $\mathbb{S}^2$ is trivial, $\text{lk}(x_1, \text{Ind}(J_2 - \{j_3\}))$ is contractible in $\text{del}(x_1, \text{Ind}(J_2 - \{j_3\}))$. Therefore Theorem 2 implies that $\text{Ind}(J_2) \simeq \mathbb{S}^2 \vee \Sigma(\mathbb{S}^1) \simeq \mathbb{S}^2 \vee \mathbb{S}^2$.

3.2.6 O_2

Since $N_{O_2-\{o_9\}}(o_7) = \{o_5\} \subseteq N_{O_2-\{o_9\}}(o_2) \cap N_{O_2-\{o_9\}}(o_4)$ and $N_{O_2-\{o_9\}}(o_8) = \{o_6\} \subseteq N_{O_2-\{o_9\}}(o_3)$, from Theorem 5 we get that $\mathrm{del}(o_9, \mathrm{Ind}(O_2)) = \mathrm{Ind}(O_2 - \{o_9\}) \simeq \mathrm{Ind}(O_2 - \{o_9, o_2, o_4, o_3\})$. Observe that $O_2 - \{o_9, o_2, o_4, o_3\} \cong D_2 \sqcup P_2 \sqcup P_2$ (see Figure 3.12b), and therefore $\mathrm{del}(o_9, \mathrm{Ind}(O_2)) \simeq \Sigma^2(\mathrm{Ind}(D_2))$.

We see that $\mathrm{lk}(o_9, \mathrm{Ind}(O_2)) = \mathrm{Ind}(O_2 - N_{O_2}[o_9]) = \mathrm{Ind}(O_2 - \{o_7, o_8, o_9\})$. Let O_2' be the graph $O_2 - \{o_7, o_8, o_9\}$ (see Figure 3.12c). Since $N_{O_2'}(o_6) \subseteq N_{O_2'}(x_1)$, from Theorem 5, $\mathrm{Ind}(O_2') \simeq \mathrm{Ind}(O_2' - \{x_1\})$. Denote the graph $O_2' - \{x_1\}$ by O_2'' (see Figure 3.12d). Since $O_2'' - N_{O_2''}[\{y_1, x_2\}]$ contains an isolated vertex o_6, we see that $\mathrm{Ind}(O_2'' - N_{O_2''}[\{y_1, x_2\}])$ is a cone over o_6 and therefore contractible. Hence from Theorem 4, $\mathrm{Ind}(O_2'') \simeq \mathrm{Ind}(O_2'' - \{(y_1, x_2)\})$.



(a) O_2 (b) $O_2 - \{o_9, o_2, o_3, o_4\}$

(c) $O_2' = O_2 - \{o_7, o_8, o_9\}$ (d) $O_2'' = O_2' - \{x_1\}$ (e) O_2'''

Figure 3.12

Denote the graph $O_2'' - \{(y_1, x_2)\}$ by O_2'''. Since $N_{O_2'''}(o_6) = \{o_3\} = N_{O_2'''}(y_1)$, from Theorem 5 we get that $\mathrm{Ind}(O_2''') \simeq \mathrm{Ind}(O_2''' - \{o_6\})$. Clearly $O_2''' - \{o_6\} \cong Q_1 \sqcup P_2$, and therefore $\mathrm{lk}(o_9, \mathrm{Ind}(O_2)) \simeq \Sigma(\mathrm{Ind}(Q_1))$. Note that $V(O_2''' - \{o_6\}) \cap N_{O_2-\{o_9\}}(o_8) = \emptyset$ and therefore $\mathrm{Ind}(O_2''' - \{o_6\}) * \{o_8\} \subseteq \mathrm{Ind}(O_2 - \{o_9\}) = \mathrm{del}(o_9, \mathrm{Ind}(O_2))$. Hence the inclusion map $\mathrm{Ind}(O_2''' - \{o_6\}) \hookrightarrow del(o_9, \mathrm{Ind}(O_2))$ is null homotopic. Thus the composite map

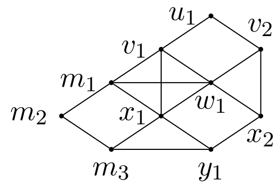
$$\mathrm{lk}(o_9, \mathrm{Ind}(O_2)) = \mathrm{Ind}(O_2') \xrightarrow{\simeq} \mathrm{Ind}(O_2'') \xrightarrow{\simeq} \mathrm{Ind}(O_2''') \xrightarrow{\simeq} \mathrm{Ind}(O_2''' - \{o_6\}) \hookrightarrow \mathrm{del}(o_9, \mathrm{Ind}(O_2))$$

is null homotopic.

Therefore Theorem 2 implies that $\mathrm{Ind}(O_2) \simeq \mathrm{del}(o_9, \mathrm{Ind}(O_2)) \vee \Sigma(\mathrm{lk}(o_9, \mathrm{Ind}(O_2))) \simeq \Sigma^2(\mathrm{Ind}(D_2)) \vee \Sigma^2(\mathrm{Ind}(Q_1))$. Since $\mathrm{Ind}(D_2) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ (cf. Section 3.2.4) and $\mathrm{Ind}(Q_1)$ is contractible (cf. Section 3.1.8), $\mathrm{Ind}(O_2) \simeq \mathbb{S}^3 \vee \mathbb{S}^3$.

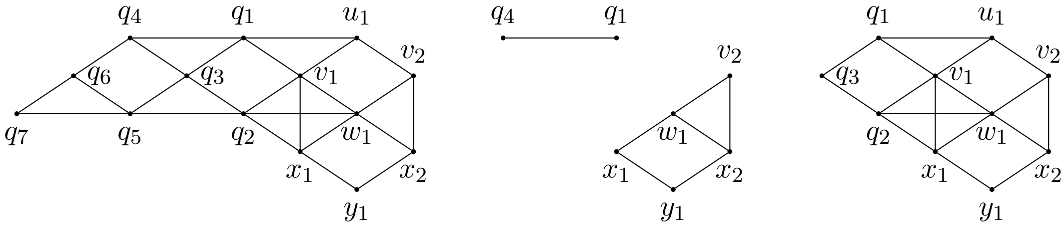
3.2.7 M_2

Since $N_{M_2}(m_2) \subseteq N_{M_2}(x_1)$ (see figure on the right), Theorem 5 implies that $\mathrm{Ind}(M_2) \simeq \mathrm{Ind}(M_2 - \{x_1\})$. Since $N_{M_2-\{x_1\}}(u_1) \subseteq N_{M_2-\{x_1\}}(w_1)$ and $M_2 - \{x_1\} \cong C_8$, Theorem 5 and Theorem 8 implies that $\mathrm{Ind}(M_2) \simeq \mathbb{S}^2$.



3.2.8 Q_2

Since q_7 is a simplicial vertex in Q_2 (see Figure 3.13a), Lemma 6 implies that $\mathrm{Ind}(Q_2) \simeq \Sigma(\mathrm{Ind}(Q_2 - N_{Q_2}[q_5])) \vee \Sigma(\mathrm{Ind}(Q_2 - N_{Q_2}[q_6]))$.



(a) Q_2 (b) $Q_2 - N_{Q_2}[q_5] - \{u_1, v_1\}$ (c) $Q_2 - N_{Q_2}[q_6]$

Figure 3.13

Note that

$$N_{Q_2-N_{Q_2}[q_5]}(q_4) = \{q_1\} \subseteq N_{Q_2-N_{Q_2}[q_5]}(u_1) \cap N_{Q_2-N_{Q_2}[q_5]}(u_2),$$

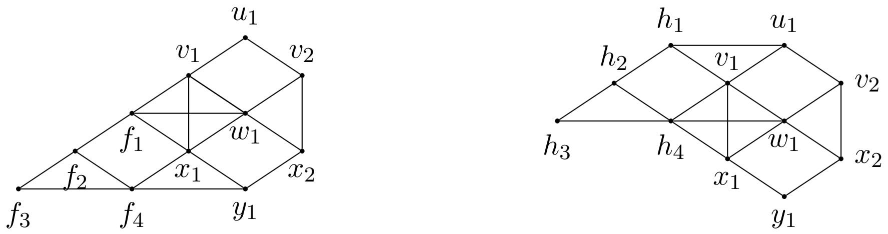
therefore Lemma 5 implies that $\mathrm{Ind}(Q_2 - N_{Q_2}[q_5]) \simeq \mathrm{Ind}(Q_2 - N_{Q_2}[q_5] - \{u_1, v_1\})$. See that $Q_2 - N_{Q_2}[q_5] - \{u_1, v_1\} \cong M_1 \sqcup P_2$ (see Figure 3.13b). Hence $\mathrm{Ind}(Q_2 - N_{Q_2}[q_5]) \simeq \Sigma(\mathrm{Ind}(M_1))$. Also $Q_2 - N_{Q_2}[q_6] \cong M_2$ (see Figure 3.13c), and therefore $\mathrm{Ind}(Q_2) \simeq \Sigma^2(\mathrm{Ind}(M_1)) \vee \Sigma(\mathrm{Ind}(M_2))$. Since $\mathrm{Ind}(M_1)$ is contractible (cf. Section 3.1.7) and $\mathrm{Ind}(M_2) \simeq \mathbb{S}^2$ (cf. Section 3.2.7, we get that $\mathrm{Ind}(Q_2) \simeq \mathbb{S}^3$.

3.2.9 F_2

Since f_3 is a simplicial vertex in F_2, from Theorem 6 we have

$$\mathrm{Ind}(F_2) \simeq \Sigma(\mathrm{Ind}(F_2 - N_{F_2}[f_2])) \vee \Sigma(\mathrm{Ind}(F_2 - N_{F_2}[f_4])).$$

Observe that $F_2 - N_{F_2}[f_2] = F_2 - \{f_1, f_2, f_3, f_4\} \cong G_2$ and $F_2 - N_{F_2}[f_4] = F_2 - \{x_1, y_1, f_2, f_3, f_4\} \cong H_1$ (see Figure 3.14a). Since $\mathrm{Ind}(G_2) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ (cf. Section 3.2.1) and $\mathrm{Ind}(H_1) \simeq \mathbb{S}^1$ (cf. Section 3.1.10), $\mathrm{Ind}(F_2) \simeq \Sigma(\mathbb{S}^1 \vee \mathbb{S}^1) \vee \Sigma(\mathbb{S}^1) \simeq \vee_3 \mathbb{S}^2$.



(a) F_2 (b) H_2

Figure 3.14

3.2.10 H_2

Observe that h_3 is a simplicial vertex in H_2 (see Figure 3.14b). Thus using Theorem 6, we get

$$\mathrm{Ind}(H_2) \simeq \Sigma(\mathrm{Ind}(H_2 - N_{H_2}[h_2])) \vee \Sigma(\mathrm{Ind}(H_2 - N_{H_2}[h_4])).$$

Observe that $H_2 - N_{H_2}[h_2] \cong G_2$ and $H_2 - N_{H_2}[h_4] \cong P_5$. Therefore, $\mathrm{Ind}(H_2) \simeq \Sigma(\mathrm{Ind}(G_2)) \vee \Sigma(\mathrm{Ind}(P_5)) \simeq \Sigma(\mathbb{S}^1 \vee \mathbb{S}^1) \vee \Sigma(\mathbb{S}^1) = \vee_3\mathbb{S}^2$.

3.3 General case computation

The main outcome of this subsection is that the independence complex of any graph among the ten classes of graphs (defined in Section 3.1) is a wedge of spheres up to homotopy. We prove this by induction on the subscript of the graphs, *i.e.,* n. The cases $n = 1, 2$ follow from the Section 3.1 and Section 3.2. Fix $n \geqslant 3$, and inductively assume that for any $k < n$, the independence complex of any graph, among the ten classes of graphs, with subscript k is a wedge of spheres up to homotopy.

3.3.1 G_n

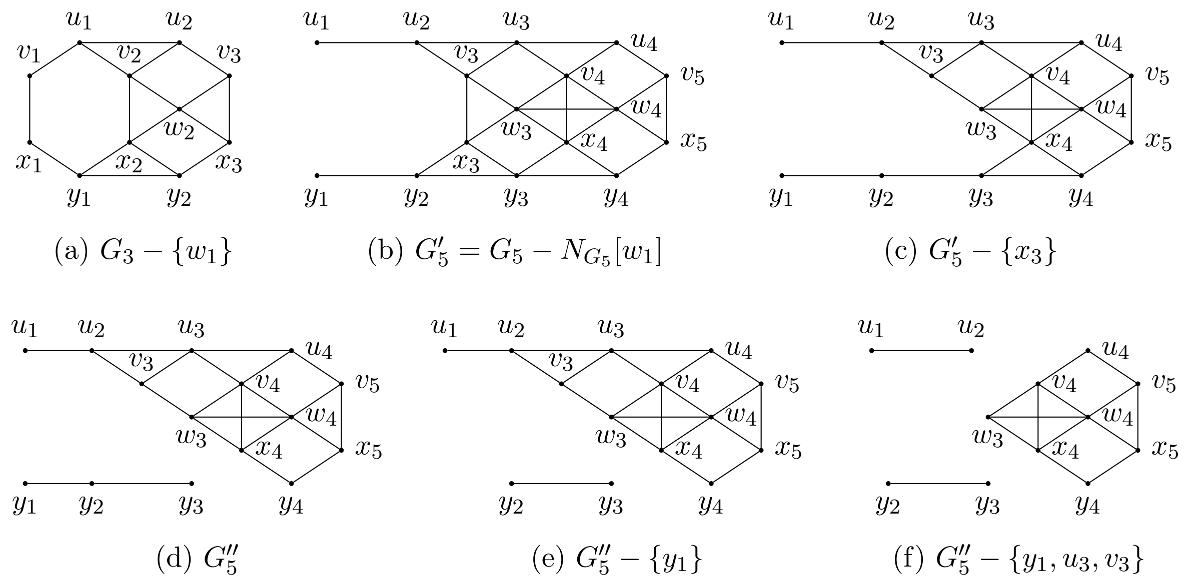
For $n \geqslant 3$, we show that

$$\mathrm{Ind}(G_n) \simeq \begin{cases} \vee_5\mathbb{S}^2 & \text{if } n = 3, \\ \mathrm{Ind}(B_{n-1}) \vee \Sigma^3(\mathrm{Ind}(A_{n-3})) & \text{if } n \geqslant 4. \end{cases} \tag{3}$$

We do this by analysing $\mathrm{del}(w_1, \mathrm{Ind}(G_n))$ and $\mathrm{lk}(w_1, \mathrm{Ind}(G_n))$. As $\mathrm{del}(w_1, \mathrm{Ind}(G_n)) = \mathrm{Ind}(G_n - \{w_1\})$ and $G_n - \{w_1\} \cong B_{n-1}$ (cf. Figure 3.15a), we have $\mathrm{del}(w_1, \mathrm{Ind}(G_n)) \simeq \mathrm{Ind}(B_{n-1})$, for $n \geqslant 3$. Further note that, $\mathrm{lk}(w_1, \mathrm{Ind}(G_n)) = \mathrm{Ind}(G_n - N_{G_n}[w_1])$.

For $n = 3$, $\mathrm{del}(w_1, \mathrm{Ind}(G_3)) \simeq \mathrm{Ind}(B_2)$ thus by Section 3.2.2, $\mathrm{Ind}(B_2) \simeq \vee_4\mathbb{S}^2$. Also, $G_3 - N_{G_3}[w_1] \cong P_6$, therefore Lemma 7 implies that $\mathrm{lk}(w_1, \mathrm{Ind}(G_3)) \simeq \mathbb{S}^1$. Since the fundamental group of $\vee_4\mathbb{S}^2$ is trivial, $\mathrm{lk}(w_1, \mathrm{Ind}(G_3))$ is contractible in $\mathrm{del}(w_1, \mathrm{Ind}(G_3))$. Hence, from Lemma 2, $\mathrm{Ind}(G_3) \simeq \mathrm{del}(w_1, \mathrm{Ind}(G_3)) \vee \Sigma(\mathrm{lk}(w_1, \mathrm{Ind}(G_3))) \simeq \vee_4\mathbb{S}^2 \vee \Sigma(\mathbb{S}^1) \simeq \vee_5\mathbb{S}^2$.

We now analyse $\mathrm{lk}(w_1, \mathrm{Ind}(G_n))$ for $n \geqslant 4$. Let G'_n be the graph $G_n - N_{G_n}[w_1]$ (cf. Figure 3.15b). Since $N_{G'_n}(y_1) \subseteq N_{G'_n}(x_3)$, Theorem 5 implies that $\mathrm{Ind}(G'_n) \simeq \mathrm{Ind}(G'_n - \{x_3\})$. Observe that both the graphs $G'_n - \{x_3\} - N_{G'_n - \{x_3\}}[\{y_3, x_4\}]$ and $G'_n - \{x_3\} - N_{G'_n - \{x_3\}}[\{y_3, y_4\}]$ have y_1 as an isolated vertex (see Figure 3.15c), therefore



(a) $G_3 - \{w_1\}$ (b) $G_5' = G_5 - N_{G_5}[w_1]$ (c) $G_5' - \{x_3\}$

(d) G_5'' (e) $G_5'' - \{y_1\}$ (f) $G_5'' - \{y_1, u_3, v_3\}$

Figure 3.15

the independence complexes of these graphs are contractible. Hence, Theorem 4 implies that $\mathrm{Ind}(G_n' - \{x_3\}) \simeq \mathrm{Ind}(G_n' - \{x_3\} - \{(y_3, x_4), (y_3, y_4)\})$.

Let $G_n'' = G_n' - \{x_3\} - \{(y_3, x_4), (y_3, y_4)\}$. Since $N_{G_n''}(y_3) \subseteq N_{G_n''}(y_1)$ (see Figure 3.15d), Theorem 5 implies $\mathrm{Ind}(G_n'') \simeq \mathrm{Ind}(G_n'' - \{y_1\})$. Note that $V(G_n'' - \{y_1\}) \cap N_{G_n - \{w_1\}}(x_1) = \emptyset$, therefore $\mathrm{Ind}(G_n'' - \{y_1\}) * \{w_1\} \subseteq \mathrm{Ind}(G_n - \{w_1\}) = \mathrm{del}(w_1, \mathrm{Ind}(G_n))$. Hence the inclusion map $\mathrm{Ind}(G_n'' - \{y_1\}) \hookrightarrow \mathrm{del}(w_1, \mathrm{Ind}(G_n))$ is null homotopic. Thus the following composition of maps is null homotopic:

$$\mathrm{lk}(w_1, \mathrm{Ind}(G_n)) \xrightarrow{\simeq} \mathrm{Ind}(G_n' - \{x_3\}) \xrightarrow{\simeq} \mathrm{Ind}(G_n'') \xrightarrow{\simeq} \mathrm{Ind}(G_n'' - \{y_1\}) \hookrightarrow \mathrm{del}(w_1, \mathrm{Ind}(G_n)).$$

Therefore by Theorem 2,

$$\mathrm{Ind}(G_n) \simeq \mathrm{del}(w_1, \mathrm{Ind}(G_n)) \vee \Sigma(\mathrm{lk}(w_1, \mathrm{Ind}(G_n))).$$

As shown earlier, $\mathrm{del}(w_1, \mathrm{Ind}(G_n)) \simeq \mathrm{Ind}(B_{n-1})$, therefore to prove Equation (3), it suffices to show that $\mathrm{lk}(w_1, \mathrm{Ind}(G_n)) \simeq \Sigma^2(\mathrm{Ind}(A_{n-3}))$. From the above discussion, we know that $\mathrm{lk}(w_1, \mathrm{Ind}(G_n)) \simeq \mathrm{Ind}(G_n'' - \{y_1\})$. Since $N_{G_n'' - \{y_1\}}(u_1) \subseteq N_{G_n'' - \{y_1\}}(u_3) \cap N_{G_n'' - \{y_1\}}(v_3)$, Lemma 5 implies $\mathrm{Ind}(G_n'' - \{y_1\}) \simeq \mathrm{Ind}(G_n'' - \{y_1\} - \{u_3, v_3\})$. Moreover, $G_n'' - \{y_1, u_3, v_3\}$ is isomorphic to $P_2 \sqcup P_2 \sqcup A_{n-3}$ (see Figure 3.15f). Thus, by Theorem 3, $\mathrm{lk}(w_1, \mathrm{Ind}(G_n)) \simeq \mathrm{Ind}(G_n'' - \{y_1, u_3, v_3\}) \simeq \Sigma^2(\mathrm{Ind}(A_{n-3}))$.

Corollary 9. *For $n \geqslant 1$, $\mathrm{Ind}(G_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.1, Section 3.2.1, induction hypothesis, and Equation (3). □

3.3.2 B_n

Let $n \geqslant 3$ and consider the vertex b_4 in B_n. Let $B_n' = B_n - N_{B_n}[b_4]$. Since both the graphs $B_n' - \{v_1\} - N_{B_n' - \{v_1\}}[\{u_1, u_2\}]$ and $B_n' - \{v_1\} - N_{B_n' - \{v_1\}}[\{u_1, v_2\}]$ have b_2 as an isolated vertex, their independence complexes are contractible. Let $B''_n = B'_n - \{v_1\} - \{(u_1, u_2), (u_1, v_2)\}$. Now using the similar arguments as in the case of B_2, we get that $\mathrm{Ind}(B_n) \simeq \mathrm{Ind}(B''_n)$. Moreover, the similar arguments imply that

$$\mathrm{Ind}(B_n) \simeq \Sigma(\mathrm{Ind}(G_n)) \vee \Sigma^2(\mathrm{Ind}(A_{n-1})). \tag{4}$$

Corollary 10. *For $n \geqslant 1$, $\mathrm{Ind}(B_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.2, Section 3.2.2, Theorem 9, induction hypothesis, and Equation (4). □

3.3.3 A_n

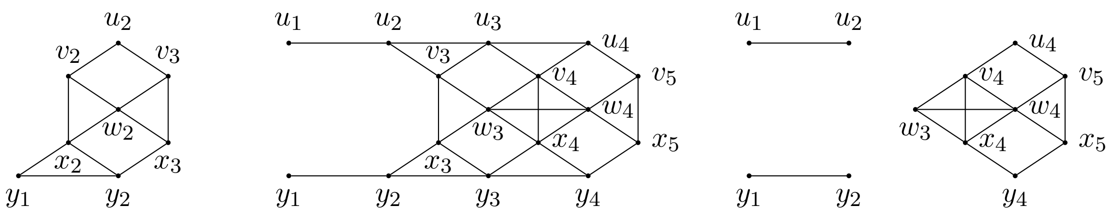
For $n \geqslant 3$, we show that

$$\mathrm{Ind}(A_n) \simeq \begin{cases} \vee_5 \mathbb{S}^2 & \text{if } n = 3, \\ \vee_2 \Sigma(\mathrm{Ind}(D_{n-1})) \vee \Sigma^3(\mathrm{Ind}(A_{n-3})) & \text{if } n \geqslant 4. \end{cases} \tag{5}$$

In A_n, a is a simplicial vertex with $N_{A_n}(a) = \{v_1, w_1, x_1\}$. Therefore by Lemma 6,

$$\mathrm{Ind}(A_n) \simeq \Sigma(\mathrm{Ind}(A_n - N_{A_n}[v_1])) \vee \Sigma(\mathrm{Ind}(A_n - N_{A_n}[w_1])) \vee \Sigma(\mathrm{Ind}(A_n - N_{A_n}[x_1])).$$

Clearly $A_n - N_{A_n}[x_1] \cong D_{n-1} \cong A_n - N_{A_n}[v_1]$ (see Figure 3.16a for $n = 3$ case). Therefore $\mathrm{Ind}(A_n) \simeq \vee_2 \mathrm{Ind}(D_{n-1}) \vee \mathrm{Ind}(A_n - N_{A_n}[w_1])$. Since $A_3 - N_{A_3}[w_1] \cong P_6$, Lemma 7 and Section 3.2.4 implies $\mathrm{Ind}(A_3) \simeq \Sigma(\mathbb{S}^1 \vee \mathbb{S}^1) \vee \Sigma(\mathbb{S}^1 \vee \mathbb{S}^1) \vee \Sigma(\mathbb{S}^1) = \vee_5 \mathbb{S}^2$.



(a) $A_3 - N_{A_3}[v_1]$ (b) $A'_5 = A_5 - N_{A_5}[w_1]$ (c) $A'_5 - \{v_3, u_3\} - \{x_3, y_3\}$

Figure 3.16

It now suffices to show that $\mathrm{Ind}(A_n - N_{A_n}[w_1]) \simeq \Sigma^2(\mathrm{Ind}(A_{n-3}))$, for $n \geqslant 4$. Let $A'_n = A_n - N_{A_n}[w_1]$. Since $N_{A'_n}(u_1) = \{u_2\} \subseteq N_{A'_n}(u_3) \cap N_{A'_n}(v_3)$ and $N_{A'_n}(y_1) = \{y_2\} \subseteq N_{A'_n}(x_3) \cap N_{A'_n}(y_3)$ (see Figure 3.16b), Lemma 5 implies that $\mathrm{Ind}(A'_n) \simeq \mathrm{Ind}(A'_n - \{v_3, u_3, x_3, y_3\})$. Moreover, $A'_n - \{v_3, u_3, x_3, y_3\} \cong P_2 \sqcup P_2 \sqcup A_{n-3}$ (see Figure 3.16c). Therefore, $\mathrm{Ind}(A_n - N_{A_n}[w_1]) \simeq \Sigma^2(\mathrm{Ind}(A_{n-3}))$.

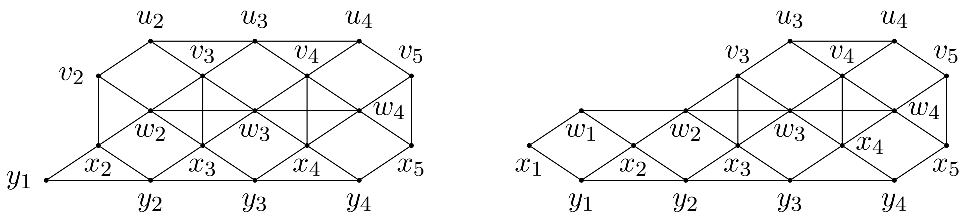
Corollary 11. *For $n \geqslant 1$, $\mathrm{Ind}(A_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.3, Section 3.2.3, induction hypothesis, and Equation (5). □

3.3.4 D_n

Let $n \geqslant 3$. Clearly d is a simplicial vertex of D_n and $N_{D_n}(d) = \{u_1, v_1\}$. Therefore by Lemma 6, $\text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_n - N_{D_n}[u_1])) \vee \Sigma(\text{Ind}(D_n - N_{D_n}[v_1]))$. Since $D_n - N_{D_n}[v_1] \cong D_{n-1}$ and $D_n - N_{D_n}[u_1] \cong J_{n-2}$ (see Figure 3.17), we have the following homotopy equivalence

$$\text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_{n-1})) \vee \Sigma(\text{Ind}(J_{n-2})). \tag{6}$$



(a) $D_5 - N_{D_5}[v_1] \cong D_4$ (b) $D_5 - N_{D_5}[u_1] \cong J_3$

Figure 3.17

Corollary 12. *For $n \geqslant 1$, $\text{Ind}(D_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.4, Section 3.2.4, induction hypothesis, and Equation (6). □

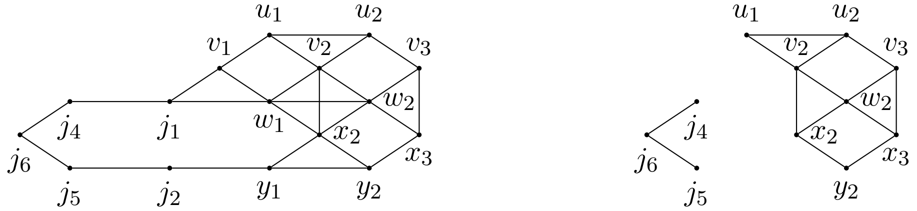
3.3.5 J_n

For $n \geqslant 3$, we show that

$$\text{Ind}(J_n) \simeq \text{Ind}(O_{n-1}) \vee \Sigma^2(\text{Ind}(D_{n-1})). \tag{7}$$

Since $N_{J_n}(j_6) \subseteq N_{J_n}(j_3)$, Theorem 5 implies that $\text{Ind}(J_n) \simeq \text{Ind}(J_n - \{j_3\})$. Observe that $J_n - \{j_3, x_1\} \cong O_{n-1}$ (see Figure 3.18a) and therefore $\text{del}(x_1, \text{Ind}(J_n - \{j_3\})) = \text{Ind}(J_n - \{j_3, x_1\}) \simeq \text{Ind}(O_{n-1})$.

Let J'_n be the graph $J_n - \{j_3\}$. We analyse $\text{lk}(x_1, \text{Ind}(J'_n))$. Clearly $\text{lk}(x_1, \text{Ind}(J'_n)) = \text{Ind}(J'_n - N_{J'_n}[x_1])$. Let $J''_n = J'_n - N_{J'_n}[x_1]$. Observe that the graph $J''_n \cong D_{n-1} \sqcup P_3$ (see Figure 3.18b). Since $N_{J''_n}(j_4) = \{j_6\} = N_{J''_n}(j_5)$, from Theorem 5 we get that $\text{Ind}(J''_n) \simeq \text{Ind}(J''_n - \{j_5\})$. Clearly $J''_n - \{j_5\}$ is isomorphic to $D_{n-1} \sqcup P_2$. Hence $\text{lk}(x_1, \text{Ind}(J'_n)) \simeq \text{Ind}(D_{n-1} \sqcup P_2) \simeq \Sigma(\text{Ind}(D_{n-1}))$.



(a) $J_3 - \{j_3, x_1\} \cong O_2$ (b) $J''_3 \cong D_2 \sqcup P_3$

Figure 3.18

Note that $V(J''_n - \{j_5\}) \cap N_{J_n-\{j_3,x_1\}}(j_2) = \emptyset$ and therefore $\text{Ind}(J''_n - \{j_5\}) * \{j_2\} \subseteq \text{Ind}(J_n - \{j_3, x_1\}) = \text{del}(x_1, \text{Ind}(J'_n))$. Hence the inclusion map $\text{Ind}(J''_n - \{j_5\}) \hookrightarrow \text{del}(x_1, \text{Ind}(J'_n))$ is null homotopic. Thus the composite map

$$\text{lk}(x_1, \text{Ind}(J'_n)) = \text{Ind}(J''_n) \xrightarrow{\simeq} \text{Ind}(J''_n - \{j_5\}) \hookrightarrow \text{del}(x_1, \text{Ind}(J'_n))$$

is null homotopic. Thus from Theorem 2 we get that $\text{Ind}(J'_n) \simeq \text{del}(x_1, \text{Ind}(J'_n)) \vee \Sigma(\text{lk}(x_1, \text{Ind}(J'_n))) \simeq \text{Ind}(O_{n-1}) \vee \Sigma^2(\text{Ind}(D_{n-1}))$.

Corollary 13. *For $n \geqslant 1$, $\text{Ind}(J_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.5, Section 3.2.5, induction hypothesis, and Equation (7). □

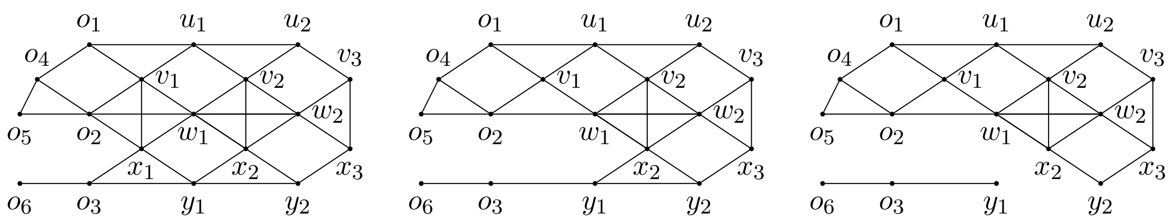
3.3.6 O_n

For $n \geqslant 3$, we show that

$$\text{Ind}(O_n) \simeq \Sigma^2(\text{Ind}(D_n)) \vee \Sigma^2(\text{Ind}(Q_{n-1})). \tag{8}$$

Let $O'_n = O_n - \{o_7, o_8, o_9\}$ (see Figure 3.19a). Using similar arguments as in the case of O_2, we get that $\text{del}(o_9, \text{Ind}(O_n)) = \Sigma^2(\text{Ind}(D_n))$ and $\text{lk}(o_9, \text{Ind}(O_n))) \simeq \text{Ind}(O''_n)$, where $O''_n = O'_n - \{x_1\}$ (see Figure 3.19b). Since $O''_n - N_{O''_n}[\{y_1, x_2\}]$ and $O''_n - N_{O''_n}[\{y_1, y_2\}]$ both contain o_6 as an isolated vertex, $\text{Ind}(O''_n - N_{O''_n}[\{y_1, x_2\}])$ and $\text{Ind}(O''_n - N_{O''_n}[\{y_1, y_2\}])$ both are cones with apex o_6 and therefore contractible. Hence from Theorem 4, $\text{Ind}(O''_n) \simeq \text{Ind}(O''_n - \{(y_1, x_2), (y_1, y_2)\})$. Denote the graph $O''_n - \{(y_1, x_2), (y_1, y_2)\}$ by O'''_n. Observe that the graph O'''_n is isomorphic to $Q_{n-1} \sqcup P_3$ (see Figure 3.19c) implying that $\text{lk}(o_9, \text{Ind}(O_n)) \simeq \Sigma(\text{Ind}(Q_{n-1}))$.

Now similar arguments as in the case of O_2 imply that $\text{lk}(o_9, \text{Ind}(O_n))$ is contractible in $\text{del}(o_9, \text{Ind}(O_n))$. The Equation (8) then follows from Theorem 2.



(a) O'_3 (b) O''_3 (c) O'''_3

Figure 3.19

Corollary 14. *For $n \geqslant 1$, $\text{Ind}(O_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.6, Section 3.2.6, Theorem 12, induction hypothesis, and Equation (8). □

3.3.7 M_n

For $n \geqslant 3$, we show that

$$\mathrm{Ind}(M_n) \simeq \Sigma(\mathrm{Ind}(M_{n-1})) \vee \Sigma^2(\mathrm{Ind}(F_{n-2})). \tag{9}$$

Since $N_{M_n}(m_2) \subseteq N_{M_n}(x_1)$, Theorem 5 implies that $\mathrm{Ind}(M_n) \simeq \mathrm{Ind}(M_n - \{x_1\})$. Let M_n' be the graph $M_n - \{x_1\}$. We compute the link and deletion of the vertex m_1 in $\mathrm{Ind}(M_n')$.

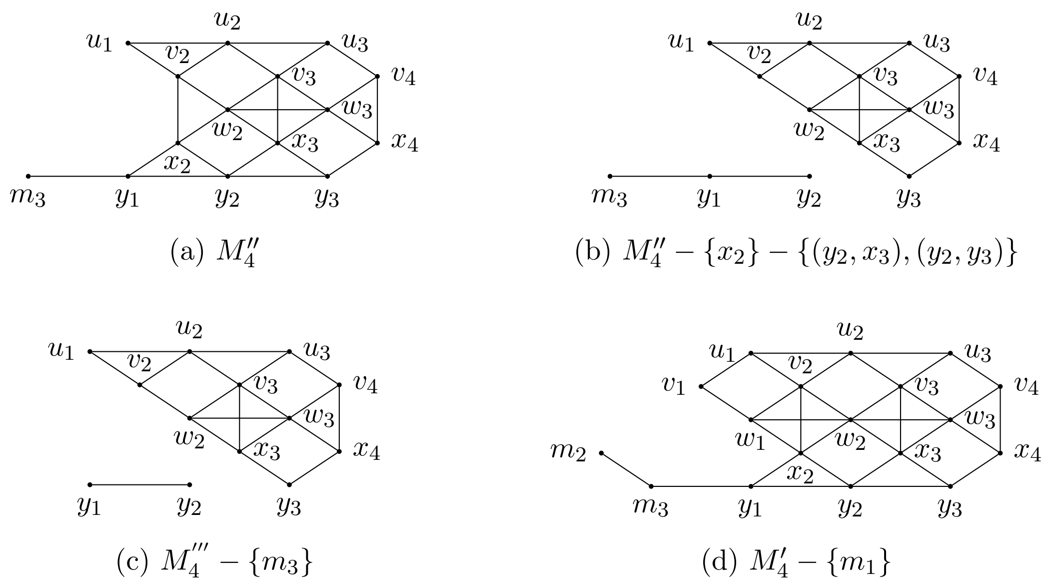


Figure 3.20

Note that $\mathrm{lk}(m_1, \mathrm{Ind}(M_n')) = \mathrm{Ind}(M_n' - N_{M_n'}[m_1])$. Let M_n'' be the graph $M_n' - N_{M_n'}[m_1]$ (see Figure 3.20a). Since $N_{M_n''}(m_3) \subseteq N_{M_n''}(x_2)$, from Theorem 5, we get that $\mathrm{Ind}(M_n'') \simeq \mathrm{Ind}(M_n'' - \{x_2\})$. Observe that both the graphs $M_n'' - \{x_2\} - N_{M_n'' - \{x_2\}}[\{y_2, x_3\}]$ and $M_n'' - \{x_2\} - N_{M_n'' - \{x_2\}}[\{y_2, y_3\}]$ contain an isolated vertex m_3 and therefore their independence complexes are contractible. Thus, using Theorem 4 we get that $\mathrm{Ind}(M_n'' - \{x_2\}) \simeq \mathrm{Ind}(M_n'' - \{x_2\} - \{(y_2, x_3), (y_2, y_3)\})$ (see Figure 3.20b). Denote the graph $M_n'' - \{x_2\} - \{(y_2, x_3), (y_2, y_3)\}$ by M_n'''. Since $N_{M_n'''}(y_2) \subseteq N_{M_n'''}(m_3)$, $\mathrm{Ind}(M_n''') \simeq \mathrm{Ind}(M_n''' - \{m_3\})$. Observe that $M_n''' - \{m_3\} \cong P_2 \sqcup F_{n-2}$ (see Figure 3.20c), therefore $\mathrm{lk}(m_1, \mathrm{Ind}(M_n')) \simeq \mathrm{Ind}(M_n''') \simeq \mathrm{Ind}(P_2 \sqcup F_{n-2}) \simeq \Sigma(\mathrm{Ind}(F_{n-2}))$.

We now compute the homotopy type of $\mathrm{del}(m_1, \mathrm{Ind}(M_n')) = \mathrm{Ind}(M_n' - \{m_1\})$ (see Figure 3.20d). Since $N_{M_n' - \{m_1\}}(m_2) \subseteq N_{M_n' - \{m_1\}}(y_1)$ and $M_n' - \{m_1, y_1\} \cong P_2 \sqcup M_{n-1}$, we get that $\mathrm{del}(m_1, \mathrm{Ind}(M_n')) \simeq \mathrm{Ind}(P_2 \sqcup M_{n-1}) \simeq \Sigma(\mathrm{Ind}(M_{n-1}))$.

From Theorem 2, it is now enough to show that the inclusion map $\mathrm{lk}(m_1, \mathrm{Ind}(M_n')) \hookrightarrow \mathrm{del}(m_1, \mathrm{Ind}(M_n'))$ is null homotopic. We know that $\mathrm{lk}(m_1, \mathrm{Ind}(M_n')) = \mathrm{Ind}(M_n'') \simeq \mathrm{Ind}(M_n''') \simeq \mathrm{Ind}(M_n''' - \{m_3\})$ and $\mathrm{del}(m_1, \mathrm{Ind}(M_n')) = \mathrm{Ind}(M_n' - \{m_1\})$. Note that $V(M_n''' - \{m_3\}) \cap N_{M_n' - \{m_1\}}(m_2) = \emptyset$. Thus $\mathrm{Ind}(M_n''' - \{m_3\}) * \{m_2\} \subseteq \mathrm{Ind}(M_n' - \{m_1\})$ implying that the map $\mathrm{Ind}(M_n''' - \{m_3\}) \hookrightarrow \mathrm{Ind}(M_n' - \{m_1\})$ is null homotopic. Hence the composition map

$$\mathrm{lk}(m_1, \mathrm{Ind}(M_n')) \xrightarrow{\simeq} \mathrm{Ind}(M_n''' - \{m_3\}) \hookrightarrow \mathrm{del}(m_1, \mathrm{Ind}(M_n'))$$

is null homotopic.

Corollary 15. *For $n \geqslant 1$, $\mathrm{Ind}(M_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.7, Section 3.2.7, induction hypothesis, and Equation (9). □

3.3.8 Q_n

For $n \geqslant 3$, using the same arguments along the lines as in the case of Q_2 we get that,

$$\mathrm{Ind}(Q_n) \simeq \Sigma(\mathrm{Ind}(M_n)) \vee \Sigma^2(\mathrm{Ind}(M_{n-1})). \tag{10}$$

Corollary 16. *For $n \geqslant 1$, $\mathrm{Ind}(Q_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.8, Section 3.2.8, Theorem 15, induction hypothesis, and Equation (10). □

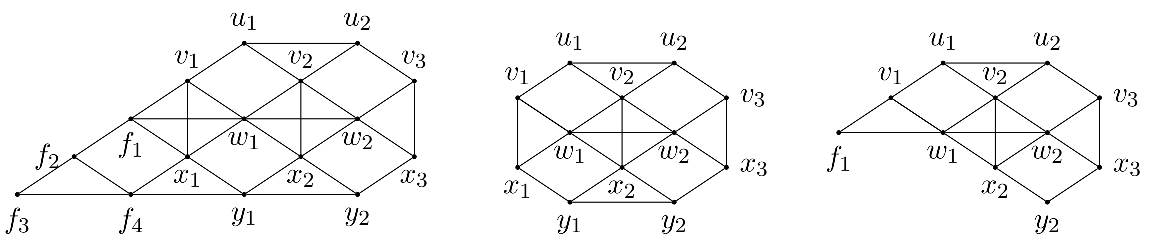
3.3.9 F_n

Observe that f_3 is a simplicial vertex in F_n with $N_{F_n}(f_3) = \{f_2, f_4\}$, and therefore, $\mathrm{Ind}(F_n) \simeq \Sigma(\mathrm{Ind}(F_n - N_{F_n}[f_2])) \vee \Sigma(\mathrm{Ind}(F_n - N_{F_n}[f_4]))$ by Theorem 6. Since $F_n - N_{F_n}[f_2] \simeq G_n$ and $F_n - N_{F_n}[f_4] \simeq H_{n-1}$ (see Figure 3.21), we get that

$$\mathrm{Ind}(F_n) \simeq \Sigma(\mathrm{Ind}(G_n)) \vee \Sigma(\mathrm{Ind}(H_{n-1})). \tag{11}$$

Corollary 17. *For $n \geqslant 1$, $\mathrm{Ind}(F_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.9, Section 3.2.9, Theorem 9, induction hypothesis, and Equation (11). □



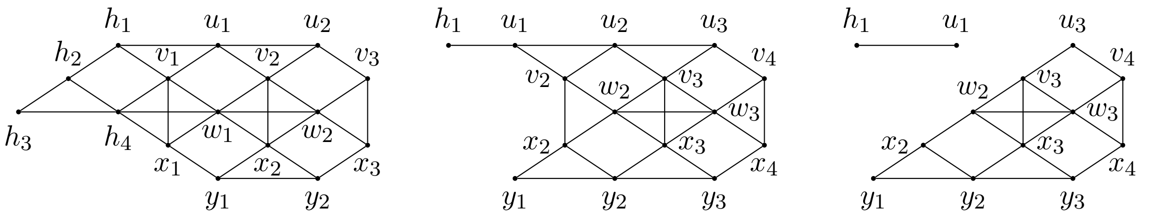
(a) F_3 (b) $F_3 - N_{F_3}[f_2]$ (c) $F_3 - N_{F_3}[f_4]$

Figure 3.21

3.3.10 H_n

Observe that h_3 is a simplicial vertex in H_n with $N_{H_n}(h_3) = \{h_2, h_4\}$ (see Figure 3.22a). Therefore from Theorem 6,

$$\mathrm{Ind}(H_n) \simeq \Sigma(\mathrm{Ind}(H_n - N_{H_n}[h_2])) \vee \Sigma(\mathrm{Ind}(H_n - N_{H_n}[h_4])).$$



(a) H_3 (b) $H_4 - N_{H_4}(h_4)$ (c) $H_4 - N_{H_4}(h_4) - \{u_2, v_2\}$

Figure 3.22

We see that $N_{H_n - N_{H_n}[h_4]}(h_1) \subseteq N_{H_n - N_{H_n}[h_4]}(u_2) \cap N_{H_n - N_{H_n}[h_4]}(v_2)$, therefore Theorem 5 implies that $\text{Ind}(H_n - N_{H_n}[h_4]) \simeq \text{Ind}(H_n - N_{H_n}[h_4] - \{u_2, v_2\})$. Since $H_n - N_{H_n}[h_4] - \{u_2, v_2\} \cong P_2 \sqcup F_{n-2}$ (see Figure 3.22c) and $H_n - N_{H_n}[h_2] \cong G_n$, we get that

$$\text{Ind}(H_n) \simeq \Sigma(\text{Ind}(G_n)) \vee \Sigma^2(\text{Ind}(F_{n-2})). \tag{12}$$

Corollary 18. *For $n \geqslant 1$, $\text{Ind}(H_n)$ is homotopy equivalent to a wedge of spheres.*

Proof. The result follows from Section 3.1.10, Section 3.2.10, Theorem 9, induction hypothesis, and Equation (12). □

4 Dimension of the spheres occurring in the homotopy type

In this section we determine the dimensions of all the spheres occurring in the homotopy type of the independence complexes of all ten classes of graphs defined in Section 3.1. For any $m \geqslant n \geqslant 1$, let $[n, m] = \{a \in \mathbb{Z} : n \leqslant a \leqslant m\}$ and

$$\mathcal{S}^{[n,m]} = \{X : X \simeq \bigvee_{d_1} \mathbb{S}^n \vee \ldots \vee \bigvee_{d_{m+1}} \mathbb{S}^{n+m} \text{ for some } d_1, \ldots, d_{m+1} > 0\}.$$

Theorem 19. *Let $n \geqslant 1$.*

1. *If $n \in [9k, 9k+8]$, then $\text{Ind}(G_n) \in \mathcal{S}^{[n-1,n+k-1]}$.*
2. *If $n \in [9k+8, 9k+16]$, then $\text{Ind}(B_n) \in \mathcal{S}^{[n,n+k+1]}$.*
3. *If $n \in [9k+7, 9k+15]$, then $\text{Ind}(A_n) \in \mathcal{S}^{[n-1,n+k]}$.*
4. *If $n \in [9k+6, 9k+14]$, then $\text{Ind}(D_n) \in \mathcal{S}^{[n-1,n+k]}$.*
5. *If $n \in [9k+4, 9k+12]$, then $\text{Ind}(J_n) \in \mathcal{S}^{[n,n+k+1]}$.*
6. *If $n \in [9k+3, 9k+11]$, then $\text{Ind}(O_n) \in \mathcal{S}^{[n+1,n+k+2]}$.*
7. *If $1 \neq n \in [9k+2, 9k+10]$, then $\text{Ind}(M_n) \in \mathcal{S}^{[n,n+k]}$, and $\text{Ind}(M_1) \simeq \text{pt}$.*
8. *If $1 \neq n \in [9k+2, 9k+10]$, then $\text{Ind}(Q_n) \in \mathcal{S}^{[n+1,n+k+1]}$, and $\text{Ind}(Q_1) \simeq \text{pt}$.*

9. *If* $n \in [9k, 9k+8]$, *then* $\mathrm{Ind}(F_n) \in \mathcal{S}^{[n,n+k]}$.

10. *If* $n \in [9k, 9k+8]$, *then* $\mathrm{Ind}(H_n) \in \mathcal{S}^{[n,n+k]}$.

Proof. The proof is by induction on n. For $n \leqslant 8$, the explicit homotopy types can be computed using the results from Section 3 and are listed in the Table 1.

n	1	2	3	4	5	6	7	8
$\mathrm{Ind}(G_n)$	$\mathbb{S}^0$	$\mathbb{S}^1 \vee \mathbb{S}^1$	$\vee_5 \mathbb{S}^2$	$\vee_9 \mathbb{S}^3$	$\vee_{16} \mathbb{S}^4$	$\vee_{31} \mathbb{S}^5$	$\vee_{55} \mathbb{S}^6$	$\vee_{94} \mathbb{S}^7$
$\mathrm{Ind}(B_n)$	$\vee_2 \mathbb{S}^1$	$\vee_4 \mathbb{S}^2$	$\vee_7 \mathbb{S}^3$	$\vee_{14} \mathbb{S}^4$	$\vee_{26} \mathbb{S}^5$	$\vee_{45} \mathbb{S}^6$	$\vee_{76} \mathbb{S}^7$	$\vee_{136} \mathbb{S}^8 \vee_{16} \mathbb{S}^9$
$\mathrm{Ind}(A_n)$	$\vee_2 \mathbb{S}^0$	$\mathbb{S}^1 \vee \mathbb{S}^1$	$\vee_5 \mathbb{S}^2$	$\vee_{10} \mathbb{S}^3$	$\vee_{14} \mathbb{S}^4$	$\vee_{21} \mathbb{S}^5$	$\vee_{42} \mathbb{S}^6 \vee_{16} \mathbb{S}^7$	$\vee_{70} \mathbb{S}^7 \vee_{22} \mathbb{S}^8$
$\mathrm{Ind}(D_n)$	$\mathbb{S}^0$	$\mathbb{S}^1 \vee \mathbb{S}^1$	$\vee_4 \mathbb{S}^2$	$\vee_6 \mathbb{S}^3$	$\vee_8 \mathbb{S}^4$	$\vee_{16} \mathbb{S}^5 \vee_8 \mathbb{S}^6$	$\vee_{28} \mathbb{S}^6 \vee_{11} \mathbb{S}^7$	$\vee_{44} \mathbb{S}^7 \vee_{20} \mathbb{S}^8$
$\mathrm{Ind}(J_n)$	$\vee_2 \mathbb{S}^1$	$\vee_2 \mathbb{S}^2$	$\vee_4 \mathbb{S}^3$	$\vee_8 \mathbb{S}^4 \vee_8 \mathbb{S}^5$	$\vee_{12} \mathbb{S}^5 \vee_3 \mathbb{S}^6$	$\vee_{16} \mathbb{S}^6 \vee_9 \mathbb{S}^7$	$\vee_{32} \mathbb{S}^7 \vee_{36} \mathbb{S}^8$	$\vee_{56} \mathbb{S}^8 \vee_{66} \mathbb{S}^9$
$\mathrm{Ind}(O_n)$	$\mathbb{S}^2$	$\vee_2 \mathbb{S}^3$	$\vee_4 \mathbb{S}^4 \vee \mathbb{S}^5$	$\vee_6 \mathbb{S}^5 \vee_3 \mathbb{S}^6$	$\vee_8 \mathbb{S}^6 \vee_9 \mathbb{S}^7$	$\vee_{16} \mathbb{S}^7 \vee_{28} \mathbb{S}^8$	$\vee_{28} \mathbb{S}^8 \vee_{55} \mathbb{S}^9$	$\vee_{44} \mathbb{S}^9 \vee_{108} \mathbb{S}^{10}$
$\mathrm{Ind}(M_n)$	pt	$\mathbb{S}^2$	$\vee_3 \mathbb{S}^3$	$\vee_6 \mathbb{S}^4$	$\vee_{14} \mathbb{S}^5$	$\vee_{30} \mathbb{S}^6$	$\vee_{58} \mathbb{S}^7$	$\vee_{93} \mathbb{S}^8$
$\mathrm{Ind}(Q_n)$	pt	$\mathbb{S}^3$	$\vee_3 \mathbb{S}^4$	$\vee_9 \mathbb{S}^5$	$\vee_{20} \mathbb{S}^6$	$\vee_{44} \mathbb{S}^7$	$\vee_{88} \mathbb{S}^8$	$\vee_{151} \mathbb{S}^9$
$\mathrm{Ind}(F_n)$	$\vee_2 \mathbb{S}^1$	$\vee_3 \mathbb{S}^2$	$\vee_8 \mathbb{S}^3$	$\vee_{16} \mathbb{S}^4$	$\vee_{28} \mathbb{S}^5$	$\vee_{35} \mathbb{S}^6$	$\vee_{102} \mathbb{S}^7$	$\vee_{177} \mathbb{S}^8$
$\mathrm{Ind}(H_n)$	$\mathbb{S}^1$	$\vee_3 \mathbb{S}^2$	$\vee_7 \mathbb{S}^3$	$\vee_{12} \mathbb{S}^4$	$\vee_{24} \mathbb{S}^5$	$\vee_{47} \mathbb{S}^6$	$\vee_{83} \mathbb{S}^7$	$\vee_{129} \mathbb{S}^8$

Table 1: Independence complexes of all ten classes of graphs for $n \leqslant 8$

For $n \geqslant 9$, let us assume that the result holds for all $m < n$.

1. From Equation (3), we have the following:

$$\mathrm{Ind}(G_n) \simeq \mathrm{Ind}(B_{n-1}) \vee \Sigma^3(\mathrm{Ind}(A_{n-3})). \tag{13}$$

Let $n \in [9k, 9k+8]$, then $n-1 \in [9(k-1)+8, 9(k-1)+16]$. By induction, $\mathrm{Ind}(B_{n-1}) \in \mathcal{S}^{[n-1,n+k-1]}$. Clearly $n-3 \in [9(k-1)+6, 9(k-1)+14]$. If $n-3 = 9(k-1)+6 = 9(k-2)+15$, then by induction $\mathrm{Ind}(A_{n-3}) \in \mathcal{S}^{[n-4,n-3+k-2]}$ and for $n-3 \in [9(k-1)+7, 9(k-1)+14]$, $\mathrm{Ind}(A_{n-3}) \in \mathcal{S}^{[n-4,n-3+k-1]}$. Hence from Equation (13), $\mathrm{Ind}(G_n) \in \mathcal{S}^{[n-1,n+k-1]}$.

2. From Equation (4), we have the following:

$$\mathrm{Ind}(B_n) \simeq \Sigma(\mathrm{Ind}(G_n)) \vee \Sigma^2(\mathrm{Ind}(A_{n-1})). \tag{14}$$

Let $n \in [9k+8, 9k+16]$. From part (1), we know that for $n = 9k+8$, $\Sigma(\mathrm{Ind}(G_n)) \in \mathcal{S}^{[n,n+k]}$ and for $n \in [9k+9, 9k+16] = [9(k+1), 9(k+1)+7]$, $\Sigma(\mathrm{Ind}(G_n)) \in \mathcal{S}^{[n,n+k+1]}$. Clearly $n \in [9k+8, 9k+16]$ implies $n-1 \in [9k+7, 9k+15]$. By induction, $\mathrm{Ind}(A_{n-1}) \in \mathcal{S}^{[n-2,n+k-1]}$ and therefore $\Sigma^2(\mathrm{Ind}(A_{n-1})) \in \mathcal{S}^{[n,n+k+1]}$. Thus Equation (14) implies that $\mathrm{Ind}(B_n) \in \mathcal{S}^{[n,n+k+1]}$.

3. From Equation (5), we have the following:

$$\mathrm{Ind}(A_n) \simeq \vee_2 \Sigma(\mathrm{Ind}(D_{n-1})) \vee \Sigma^3(\mathrm{Ind}(A_{n-3})). \tag{15}$$

Let $n \in [9k+7, 9k+15]$. If $n-1 \in [9k+6, 9k+14]$, then by induction $\mathrm{Ind}(D_{n-1}) \in \mathcal{S}^{[n-2,n+k-1]}$ and therefore $\Sigma(\mathrm{Ind}(D_{n-1})) \in \mathcal{S}^{[n-1,n+k]}$. Clearly $n \in [9k+7, 9k+15]$

implies that $n-3 \in [9k+4, 9k+12]$. If $n-3 \in [9k+4, 9k+6] = [9(k-1)+13, 9(k-1)+15]$, then by induction $\text{Ind}(A_{n-3}) \in \mathcal{S}^{[n-4,n+k-4]}$ and therefore $\Sigma^3(\text{Ind}(A_{n-3})) \in \mathcal{S}^{[n-1,n+k-1]}$. For $n-3 \in [9k+7, 9k+12]$, $\text{Ind}(A_{n-3}) \in \mathcal{S}^{[n-4,n-3+k]}$ and therefore $\Sigma^3(\text{Ind}(A_{n-3})) \in \mathcal{S}^{[n-1,n+k]}$. The result now follows from Equation (15).

4. From Equation (6), we have the following:

$$\text{Ind}(D_n) \simeq \Sigma(\text{Ind}(D_{n-1})) \vee \Sigma(\text{Ind}(J_{n-2})). \tag{16}$$

Let $n \in [9k+6, 9k+14]$, then $n-1 \in [9k+5, 9k+13]$. If $n-1 = 9k+5 = 9(k-1)+14$, then by induction $\text{Ind}(D_{n-1}) \in \mathcal{S}^{[n-2,n+k-2]}$ and therefore $\Sigma(\text{Ind}(D_{n-1})) \in \mathcal{S}^{[n-1,n+k-1]}$. Otherwise $n-1 \in [9k+7, 9k+14]$ and by induction $\text{Ind}(D_{n-1}) \in \mathcal{S}^{[n-2,n+k-1]}$ and thus $\Sigma(\text{Ind}(D_{n-1})) \in \mathcal{S}^{[n-1,n+k]}$. Also $n \in [9k+6, 9k+14]$ implies $n-2 \in [9k+4, 9k+12]$. By induction $\text{Ind}(J_{n-2})) \in \mathcal{S}^{[n-2,n-2+k+1]}$, thereby implying that $\Sigma(\text{Ind}(J_{n-2})) \in \mathcal{S}^{[n-1,n+k]}$. Result then follows from Equation (16).

5. From Equation (7), we have the following:

$$\text{Ind}(J_n) \simeq \text{Ind}(O_{n-1}) \vee \Sigma^2(\text{Ind}(D_{n-1})). \tag{17}$$

Let $n \in [9k+4, 9k+12]$, then $n-1 \in [9k+3, 9k+11]$. By induction, we have $\text{Ind}(O_{n-1}) \in \mathcal{S}^{[n,n+k+1]}$. Moreover, if $n-1 \in [9k+3, 9k+5] = [9(k-1)+12, 9(k-1)+14]$, then by induction $\Sigma^2(\text{Ind}(D_{n-1})) \in \mathcal{S}^{[n,n+k]}$. For $n-1 \in [9k+6, 9k+11]$, $\Sigma^2(\text{Ind}(D_{n-1})) \in \mathcal{S}^{[n,n+k+1]}$. Therefore the result follows from Equation (17).

6. From Equation (8), we have the following:

$$\text{Ind}(O_n) \simeq \Sigma^2(\text{Ind}(D_n)) \vee \Sigma^2(\text{Ind}(Q_{n-1})). \tag{18}$$

Let $n \in [9k+3, 9k+11]$. If $n \in [9k+3, 9k+5] = [9(k-1)+12, 9(k-1)+14]$, then by induction $\Sigma^2(\text{Ind}(D_n)) \in \mathcal{S}^{[n+1,n+k+1]}$. Further, if $n \in [9k+6, 9k+11]$, then again by induction $\Sigma^2(\text{Ind}(D_n)) \in \mathcal{S}^{[n+1,n+k+2]}$. Observe that $n \in [9k+3, 9k+11]$ implies $n-1 \in [9k+2, 9k+10]$ and therefore by induction we have $\Sigma^2(\text{Ind}(Q_{n-1})) \in \mathcal{S}^{[n+2,n+k+2]}$. The result then follows from Equation (18).

7. From Equation (9), we have the following:

$$\text{Ind}(M_n) \simeq \Sigma(\text{Ind}(M_{n-1})) \vee \Sigma^2(\text{Ind}(F_{n-2})). \tag{19}$$

Let $n \in [9k+2, 9k+10]$, then $n-1 \in [9k+1, 9k+9]$. If $n-1 = 9k+1 = 9(k-1)+10$, then $\Sigma(\text{Ind}(M_{n-1})) \in \mathcal{S}^{[n,n+k-1]}$. For $n-1 \in [9k+2, 9k+9]$, $\Sigma(\text{Ind}(M_{n-1})) \in \mathcal{S}^{[n,n+k]}$ by induction. Clearly $n \in [9k+2, 9k+10]$ implies $n-2 \in [9k, 9k+8]$. Therefore by induction, $\text{Ind}(F_{n-2}) \in \mathcal{S}^{[n-2,n+k-2]}$ and hence $\Sigma^2(\text{Ind}(F_{n-2})) \in \mathcal{S}^{[n,n+k]}$. Equation (19) then implies that for $n \in [9k+2, 9k+10]$, $\text{Ind}(M_n) \in \mathcal{S}^{[n,n+k]}$.

8. From Equation (10), we have the following:

$$\text{Ind}(Q_n) \simeq \Sigma(\text{Ind}(M_n)) \vee \Sigma^2(\text{Ind}(M_{n-1})). \tag{20}$$

Let $n \in [9k+2, 9k+10]$, then $\Sigma(\mathrm{Ind}(M_n)) \in \mathcal{S}^{[n+1,n+k+1]}$. Clearly $n-1 \in [9k+1, 9k+9]$. If $n-1 = 9k+1 = 9(k-1)+10$, then $\mathrm{Ind}(M_{n-1})) \in \mathcal{S}^{[n-1,n+k-2]}$ thereby implying that $\Sigma^2(\mathrm{Ind}(M_{n-1})) \in \mathcal{S}^{[n+1,n+k]}$. For $n-1 \in [9k+2, 9k+9]$, $\Sigma^2(\mathrm{Ind}(M_{n-1})) \in \mathcal{S}^{[n+1,n+k+1]}$ by induction. The result now follows from Equation (20).

9. From Equation (11), we have the following:

$$\mathrm{Ind}(F_n) \simeq \Sigma(\mathrm{Ind}(G_n)) \vee \Sigma(\mathrm{Ind}(H_{n-1})). \tag{21}$$

Let $n \in [9k, 9k+8]$. From part (1), we have $\mathrm{Ind}(G_n) \in \mathcal{S}^{[n-1,n+k-1]}$ and therefore $\Sigma(\mathrm{Ind}(G_n)) \in \mathcal{S}^{[n,n+k]}$. Clearly $n-1 \in [9k-1, 9k+7]$. If $n-1 = 9k-1 = 9(k-1)+8$, then by induction $\mathrm{Ind}(H_{n-1}) \in \mathcal{S}^{[n-1,n+k-2]}$ and therefore $\Sigma(\mathrm{Ind}(H_{n-1})) \in \mathcal{S}^{[n,n+k-1]}$. For $n-1 \in [9k, 9k+7]$, $\mathrm{Ind}(H_{n-1}) \in \mathcal{S}^{[n-1,n+k-1]}$ and hence $\Sigma(\mathrm{Ind}(H_{n-1})) \in \mathcal{S}^{[n,n+k]}$ by induction. The result now follows from Equation (21).

10. From Equation (12), we have the following:

$$\mathrm{Ind}(H_n) \simeq \Sigma(\mathrm{Ind}(G_n)) \vee \Sigma^2(\mathrm{Ind}(F_{n-2})). \tag{22}$$

Let $n \in [9k, 9k+8]$. From part (1), we have $\mathrm{Ind}(G_n) \in \mathcal{S}^{[n-1,n+k-1]}$ and therefore $\Sigma(\mathrm{Ind}(G_n)) \in \mathcal{S}^{[n,n+k]}$. Clearly $n-2 \in [9k-2, 9k+6]$. If $n-2 \in [9k-2, 9k-1] = [9(k-1)+7, 9(k-1)+8]$, then by induction $\Sigma^2(\mathrm{Ind}(F_{n-2})) \in \mathcal{S}^{[n,n+k-1]}$. Further, if $n-2 \in [9k, 9k+6]$, then again from induction $\Sigma^2(\mathrm{Ind}(F_{n-2})) \in \mathcal{S}^{[n,n+k]}$. The result then follows from Equation (22). □

5 Future directions

From Theorem 19, we see that the number 9 plays an important role in determining the dimensions of spheres that occur in the homotopy type of $M(\Gamma_{3,n})$. It would be interesting to see if there is any relation between the number or the dimension of spheres in the homotopy type of $M(\Gamma_{3,n})$ and the combinatorial description of $\Gamma_{3,n}$. Another interesting enumerative problem is to calculate the Betti numbers of $M(\Gamma_{3,n})$. More precisely,

Question 20. Can we determine the closed form formula for the homotopy type of $M(\Gamma_{3,n})$?

Based on the main result of this article, our computer-based computations for various general grid graphs, and [9], we propose the following.

Conjecture 21. The complex $M(\Gamma_{m,n})$ is homotopy equivalent to a wedge of spheres for any grid graph $\Gamma_{m,n}$.

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