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Numerical radius and a notion of smoothness in the space of bounded linear operators ☆

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Dedicated to Rafael Nadal, the Tennis maestro at the Rolland Garros and everywhere else, on the occasion of his ever inspiring 13-th French Open title

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ABSTRACT

We observe that the classical notion of numerical radius gives rise to a notion of smoothness in the space of bounded linear operators on certain Banach spaces, whenever the numerical radius is a norm. We characterize Birkhoff-James orthogonality in the space of bounded linear operators on a finite-dimensional Banach space, endowed with the numerical radius norm. Some examples are also discussed to illustrate the geometric differences between the numerical radius norm and the usual operator norm, from the viewpoint of operator smoothness.

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1. Introduction

The main objective of the present article is to study smoothness in the space of bounded linear operators on a Banach space, induced by the numerical radius. The study of smoothness in the space of bounded linear operators on a Banach space, *with respect to the usual operator norm*, is a classical area of research in geometry of Banach spaces [1,6–8,12,15–18,24,25]. The space of bounded linear operators on a Banach space, endowed with the numerical radius norm, need not be isometrically isomorphic to the space of bounded linear operators on the same Banach space, endowed with the usual operator norm, in general. Therefore, it is expected to have differences in geometric structures in the space of bounded linear operators on a Banach space, equipped with these two different norms. The current work explores the said differences from the point of view of smoothness.

The symbol $\mathbb{X}$ signifies a Banach space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Unless otherwise mentioned, we work with both real and complex Banach spaces. Given any $\lambda \in \mathbb{C}$, let $\mathfrak{Re}\ \lambda$ denote the real part of λ. For any subset D of $\mathbb{F}$, let $\mathsf{CO}(\mathsf{D})$ denote the convex hull of D. It is immediate that if D is a compact subset of $\mathbb{F}$ then $\mathsf{CO}(\mathsf{D})$ is also a compact subset of $\mathbb{F}$.

Let $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ denote the closed unit ball and the unit sphere of $\mathbb{X}$, respectively. Let $ext(B_{\mathbb{X}})$ denote the collection of all extreme points of the closed unit ball $B_{\mathbb{X}}$. We denote the zero vector of any vector space by θ, other than the scalar field $\mathbb{F}$. Let $\mathbb{L}(\mathbb{X})$ ($\mathbb{K}(\mathbb{X})$) denote the collection of all bounded (compact) linear operators on $\mathbb{X}$ endowed with the usual operator norm. We use the symbol M_T to denote the norm attainment set of a bounded linear operator $T \in \mathbb{L}(\mathbb{X})$, i.e., $\mathsf{M}_T := \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. Given any $x, y \in \mathbb{X}$, we say that x is Birkhoff-James orthogonal [2] to y, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all scalars $\lambda \in \mathbb{F}$. Let $\mathbb{X}^*$ denote the topological dual of $\mathbb{X}$. The collection of all support functionals at a non-zero $x \in \mathbb{X}$ is denoted by $J(x)$ and is defined by:

$$J(x) := \{x^* \in S_{\mathbb{X}^*} : x^*(x) = \|x\|\}.$$

By the Hahn-Banach Theorem, the collection $J(x)$ is non-empty. It is well known [9,10] that $x \perp_B y$ if and only if there exists $x^* \in J(x)$ such that $x^*(y) = 0$. The element x is said to be a smooth point in $\mathbb{X}$, if $J(x)$ is singleton. Equivalently, x is a smooth point in $\mathbb{X}$ if and only if for any $y_1, y_2 \in \mathbb{X}$ with $x \perp_B y_1$ and $x \perp_B y_2$ imply that $x \perp_B (y_1 + y_2)$, i.e., Birkhoff-James orthogonality is right additive at x. The space $\mathbb{X}$ is called smooth if every non-zero element of $\mathbb{X}$ is smooth.

Given any $T \in \mathbb{L}(\mathbb{X})$, the numerical range of T is defined by:

$$\mathsf{W}(T) := \{x^*(Tx) : (x, x^*) \in \mathsf{J}\}, \text{ where } \mathsf{J} := \{(x, x^*) \in \mathbb{X} \times \mathbb{X}^* : x \in S_{\mathbb{X}}, x^* \in J(x)\}.$$

The numerical radius of the linear operator T is defined by:

$$\|T\|_w = \sup\{|\lambda| : \lambda \in \mathsf{W}(T)\}.$$

It is well known that whenever $\mathbb{F} = \mathbb{C}$, the numerical radius defines a norm on $\mathbb{L}(\mathbb{X})$. However, $\|\cdot\|_w$ need not be a norm on $\mathbb{L}(\mathbb{X})$, if $\mathbb{F} = \mathbb{R}$. Throughout the text, we will only consider those Banach spaces for which $\|\cdot\|_w$ defines a norm on $\mathbb{L}(\mathbb{X})$. The space of bounded linear operators on $\mathbb{X}$ endowed with the numerical radius norm is denoted by $(\mathbb{L}(\mathbb{X}))_w$. For a detailed study on numerical range of operators and their possible applications, we refer the readers to [4,5,11,13,14].

Birkhoff-James orthogonality is an important tool in the study of smoothness of elements in a given Banach space. Indeed, using the Birkhoff-James orthogonality of operators [19], a complete characterization of smoothness in $\mathbb{K}(\mathbb{X})$ was provided in [15], for a real reflexive Banach space $\mathbb{X}$. The geometry of $\mathbb{L}(\mathbb{X})$ is heavily dependent on the norm attainment sets of its members [3,20–22]. In particular, the norm attainment set of a linear operator T plays a pivotal role in determining the smoothness of T in $\mathbb{L}(\mathbb{X})$. A characterization of smoothness without any restriction on M_T was provided in [24]. The above studies motivate us to explore the concept of *numerical radius smoothness* in $(\mathbb{L}(\mathbb{X}))_w$.

Definition 1.1. Let $\mathbb{X}$ be a Banach space and let $T, A \in (\mathbb{L}(\mathbb{X}))_w$. We say that T is numerical radius Birkhoff-James orthogonal to A, written as $T \perp_B^w A$ if

$$\|T + \lambda A\|_w \geq \|T\|_w \qquad \forall\, \lambda \in \mathbb{F}.$$

Definition 1.2. Let $\mathbb{X}$ be a Banach space and let $T \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero. We say that T is *nu-smooth* (the abbreviated form of numerical radius smooth), if

$$T \perp_B^w A,\ T \perp_B^w B \text{ imply that } T \perp_B^w (A + B) \qquad \forall\, A, B \in (\mathbb{L}(\mathbb{X}))_w.$$

The concept of extreme points is closely related to the numerical radius attainment problem, especially in the finite-dimensional case. In general, $ext(B_{\mathbb{X}})$ can be empty for a given Banach space $\mathbb{X}$. As a result, $(ext(B_{\mathbb{X}}) \times ext(B_{\mathbb{X}^*})) \bigcap \mathsf{J}$ can also be empty. Obviously, this is not true if $\mathbb{X}$ is finite-dimensional. In that case, given any $T \in (\mathbb{L}(\mathbb{X}))_w$, we have the following useful formula:

$$\|T\|_w = \sup\left\{|x^*(Tx)| : (x, x^*) \in (ext(B_{\mathbb{X}}) \times ext(B_{\mathbb{X}^*})) \bigcap \mathsf{J}\right\}. \tag{1.1}$$

The above formulation is particularly advantageous whenever $\mathbb{X}$ is a finite-dimensional real polyhedral Banach space, i.e., $ext(B_{\mathbb{X}})$ is finite. Note that a finite-dimensional real Banach space $\mathbb{X}$ is polyhedral if and only if $\mathbb{X}^*$ is polyhedral, and a member x^* of $\mathbb{X}^*$ is an extreme point of $B_{\mathbb{X}^*}$ if and only if x^* is the unique supporting functional corresponding to a facet of $B_{\mathbb{X}}$ [23, Lemma 2.1]. In this context, it is worth mentioning that the numerical radius defines norm on $\mathbb{L}(\mathbb{X})$, whenever $\mathbb{X}$ is a finite-dimensional real polyhedral Banach space. Indeed, if $\mathbb{X}$ is a finite-dimensional real Banach space so that the numerical radius is not a norm on $\mathbb{L}(\mathbb{X})$, then there exists a non-zero operator $T \in \mathbb{L}(\mathbb{X})$ with $\|T\|_w = 0$. Using Lemma 2.3 of [14] one gets that there are infinitely many onto isometries on $\mathbb{X}$ which is not compatible with $\mathbb{X}$ being a finite-dimensional real polyhedral Banach space.

For any non-zero element $T \in (\mathbb{L}(\mathbb{X}))_w$, let $\mathsf{M}_{\mathsf{W}(T)}$ denote the numerical radius attainment set of T, i.e.,

$$\mathsf{M}_{\mathsf{W}(T)} := \{(x, x^*) \in \mathsf{J} : x^*(Tx) = \sigma,\ |\sigma| = \|T\|_w\}.$$

Note that $\mathsf{M}_{\mathsf{W}(T)}$ is non-empty, whenever $\mathbb{X}$ is finite-dimensional. The collection of support functionals at T is defined by:

$$J_{\mathsf{W}}(T) := \{f\ : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F} : f \text{ is linear}, \|f\| = 1, f(T) = \|T\|_w\}.$$

It follows from the James characterization ([10, Theorem 2.1]) that $T \perp_B^w A$, for some $A \in (\mathbb{L}(\mathbb{X}))_w$ if and only if there exists $f \in J_{\mathsf{W}}(T)$ such that $A \in \ker f$. Also, for any non-zero $T \in (\mathbb{L}(\mathbb{X}))_w$, T is nu-smooth if and only if $J_{\mathsf{W}}(T)$ is singleton.

In the next Section, we acquire a characterization of smoothness in $(\mathbb{L}(\mathbb{X}))_w$, both in the finite-dimensional and the infinite-dimensional cases. In due course of our development, we also obtain a necessary and sufficient condition for Birkhoff-James orthogonality in $(\mathbb{L}(\mathbb{X}))_w$ whenever $\mathbb{X}$ is finite-dimensional. Some examples have been discussed to show that smoothness in $\mathbb{L}(\mathbb{X})$ and smoothness in $(\mathbb{L}(\mathbb{X}))_w$ are not equivalent.

2. Smoothness induced by the numerical radius

We devote this section to study nu-smoothness of bounded linear operators, which is the integral theme of the present article. We start by characterizing nu-smoothness of a bounded linear operator on any Banach space $\mathbb{X}$.

Theorem 2.1. *Let $\mathbb{X}$ be a Banach space and let $T \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero. Then the following conditions are equivalent:*
(i) T is nu-smooth.
(ii) $T \perp_B^w A$ for $A \in (\mathbb{L}(\mathbb{X}))_w$ implies that for any sequence $((x_n, x_n^*)) \subseteq \mathsf{J}$ with the property that

$$lim\ x_n^*(Tx_n) \to \sigma, \quad |\sigma| = \|T\|_w, \tag{2.1}$$

every sub-sequential limit of the sequence $(x_n^*(Ax_n))$ is zero.

Proof. $(i) \implies (ii)$: Suppose on the contrary that there exists a sequence $((x_n, x_n^*)) \subseteq \mathsf{J}$ satisfying (2.1) such that

$$\lim x_{n_k}^*(Ax_{n_k}) = r \neq 0,$$

for some sub-sequence $\left(x_{n_k}^*\left(Ax_{n_k}\right)\right)$ of $(x_n^*(Ax_n))$. Let $B \in (\mathbb{L}(\mathbb{X}))_w$ be defined by $B = T - \dfrac{\sigma}{r}A$. Then we obtain

$$\lim x_{n_k}^*\left(Bx_{n_k}\right) = \lim x_{n_k}^*\left(\left(T - \frac{\sigma}{r}A\right)x_{n_k}\right) = \lim x_{n_k}^*\left(Tx_{n_k}\right) - \frac{\sigma}{r}\lim x_{n_k}^*\left(Ax_{n_k}\right) = 0.$$

This leads us to conclude that

$$\|T + \lambda B\|_w \geq \lim \left|x_{n_k}^*(T + \lambda B)(x_{n_k})\right| = \lim \left|x_{n_k}^*(Tx_{n_k}) + \lambda x_{n_k}^*(Bx_{n_k})\right| = \|T\|_w,$$

for all scalars λ. In other words, $T \perp_B^w B$. Since $\perp_B^w$ is homogeneous and T is nu-smooth, we get that $T \perp_B^w \left(\dfrac{\sigma}{r}A + B\right) = T$, which is a contradiction.

$(ii) \implies (i)$: Suppose that $T \perp_B^w A$ and $T \perp_B^w B$ for some non-zero $A, B \in (\mathbb{L}(\mathbb{X}))_w$. Consider any sequence $((x_n, x_n^*)) \subseteq \mathsf{J}$ that satisfies the condition (2.1). It now follows from the hypothesis of the theorem that we can find monotonically increasing sequence of natural numbers, say (n_k), such that

$$\lim x_{n_k}^*(Ax_{n_k}) = \lim x_{n_k}^*(Bx_{n_k}) = 0.$$

Therefore, $\lim x_{n_k}^*\left((A + B)(x_{n_k})\right) = 0$. Now, for every scalar λ, we have that

$$\|T + \lambda(A + B)\|_w \geq \lim \left|x_{n_k}^*\left(T + \lambda(A + B)\right)(x_{n_k})\right| = \|T\|_w.$$

In other words, $T \perp_B^w (A + B)$. Thus, T is nu-smooth and the proof follows. $\square$

An interesting query on this context is whether the above characterization takes any special form if $\mathbb{X}$ is finite-dimensional. An extra advantage in assuming $\mathbb{X}$ to be finite-dimensional is that we now have $\mathsf{M}_{\mathsf{W}(T)} \neq \emptyset$ for any $T \in (\mathbb{L}(\mathbb{X}))_w$. Therefore, we can expect $\mathsf{M}_{\mathsf{W}(T)}$ to play an important role in determining the nu-smoothness of T. To explore the said connection, we first prove a lemma that is particularly helpful in our further developments.

Lemma 2.2. *Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T, A \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero. The set D defined by*

$$\mathsf{D} := \left\{\overline{x^*(Tx)}x^*(Ax) : (x, x^*) \in \mathsf{M}_{\mathsf{W}(T)}\right\}, \tag{2.2}$$

is a compact subset of $\mathbb{F}$.

Proof. It is trivial to see that D is bounded. Therefore, to show that D is compact, it is sufficient to show that D is closed. Assume that (μ_n) is a sequence in D with $\mu_n \to \mu_0$. Obviously, for each n

$$\mu_n = \overline{x_n^*(Tx_n)}x_n^*(Ax_n), \text{ where } (x_n, x_n^*) \in \mathsf{M}_{\mathsf{W}(T)}.$$

Passing through a suitable sub-sequence if necessary, we may assume that $x_n \to x_0$ and $x_n^* \to x_0^*$, as $n \to \infty$, where $x_0 \in S_{\mathbb{X}}$ and $x_0^* \in S_{\mathbb{X}^*}$. Observe that

$$\begin{aligned}|x_n^*(Tx_n) - x_0^*(Tx_0)| &= |x_n^*(Tx_n) - x_n^*(Tx_0) + x_n^*(Tx_0) - x_0^*(Tx_0)| \\ &\leq |x_n^*(Tx_n - Tx_0)| + |(x_n^* - x_0^*)(Tx_0)| \\ &\leq \|Tx_n - Tx_0\| + \|x_n^* - x_0^*\|\|Tx_0\|.\end{aligned}$$

Since T is continuous, $Tx_n \to Tx_0$, as $n \to \infty$. Thus, $x_n^*(Tx_n) \to x_0^*(Tx_0)$, as $n \to \infty$. Evidently, $x_0^*(Tx_0) = \sigma$ for some $\sigma \in \mathbb{F}$ with $|\sigma| = \|T\|_w$. Similar argument shows that $x_n^*(Ax_n) \to x_0^*(Ax_0)$ and $x_n^*(x_n) \to x_0^*(x_0) = 1$. This proves that

$$(x_0, x_0^*) \in \mathsf{M}_{\mathsf{W}(T)} \text{ and } \lim \overline{x_n^*(Tx_n)}x_n^*(Ax_n) = \mu_0 = \overline{x_0^*(Tx_0)}x_0^*(Ax_0).$$

Consequently, $\mu_0 \in \mathsf{D}$. Thus, D is closed and this completes the proof of the lemma. $\square$

The following theorem completely characterizes Birkhoff-James orthogonality in $(\mathbb{L}(\mathbb{X}))_w$ for a finite-dimensional Banach space $\mathbb{X}$.

Theorem 2.3. *Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T, A \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero. Then the following conditions are equivalent:*
(i) $T \perp_B^w A$.
(ii) $0 \in \mathsf{CO}(\mathsf{D})$, where D is the subset of $\mathbb{F}$ defined by (2.2).

Proof. $(i) \implies (ii)$: Since numerical radius Birkhoff-James orthogonality is homogeneous, without loss of generality, we may assume that $\|A\|_w = 1$. Suppose on the contrary that $0 \notin \mathsf{CO}(\mathsf{D})$. Since $\mathsf{CO}(\mathsf{D})$ is a compact convex subset of $\mathbb{F}$ (Lemma 2.2), rotating $\mathsf{CO}(\mathsf{D})$ suitably if necessary, we may and do assume that $\mathfrak{Re}\ d > 0$ for all $d \in \mathsf{CO}(\mathsf{D})$. Moreover, due to the compactness of D, we can find $r \in (0, \frac{1}{2})$ such that $\mathfrak{Re}\ d > r$ for all $d \in \mathsf{D}$. In other words,

$$\mathfrak{Re}\ \overline{x^*(Tx)}x^*(Ax) > r \qquad \forall\ (x, x^*) \in \mathsf{M}_{\mathsf{W}(T)}. \tag{2.3}$$

Next, we define

$$\mathsf{G} := \left\{(x, x^*) \in \mathsf{J} : \mathfrak{Re}\ \overline{x^*(Tx)}x^*(Ax) \leq \frac{r}{2}\right\}.$$

We claim that

$$\sup\{|x^*(Tx)| : (x, x^*) \in \mathsf{G}\} < \|T\|_w - 2\varepsilon \qquad \text{for some } \varepsilon \in (0, \frac{1}{2}).$$

It follows from (2.3) and the definition of G that $\mathsf{G} \cap \mathsf{M}_{\mathsf{W}(T)} = \emptyset$. Suppose that $((x_n, x_n^*)) \subseteq \mathsf{G}$ with $\lim |x_n^*(Tx_n)| = \|T\|_w$. Without loss of generality, we may assume that $x_n \to x_0$ and $x_n^* \to x_0^*$, as $n \to \infty$, where $x_0 \in S_{\mathbb{X}}$ and $x_0^* \in S_{\mathbb{X}^*}$. Applying the similar techniques as in the proof of Lemma 2.2, it can be shown that

$$(1)\ x_n^*(Tx_n) \to x_0^*(Tx_0) \quad (2)\ x_n^*(Ax_n) \to x_0^*(Ax_0) \quad (3)\ x_0^*(x_0) = 1.$$

Since $|x_0^*(Tx_0)| = \|T\|_w$, we have that $(x_0, x_0^*) \in \mathsf{M}_{\mathsf{W}(T)}$. Therefore,

$$\mathfrak{Re}\ \overline{x_n^*(Tx_n)} x_n^*(Ax_n) \to \mathfrak{Re}\ \overline{x_0^*(Tx_0)} x_0^*(Ax_0) > r \qquad \text{(using (2.3))}.$$

However, this is a contradiction, as $((x_n, x_n^*)) \subseteq \mathsf{G}$. Thus, $\sup\{|x^*(Tx)| : (x, x^*) \in \mathsf{G}\} < \|T\|_w - 2\varepsilon$ for some $\varepsilon \in (0, \frac{1}{2})$.

Choose $0 < \lambda < \min\{\varepsilon, r\}$. Now, for any $(x, x^*) \in \mathsf{G}$

$$\begin{aligned} |x^*(Tx - \lambda Ax)| &\leq |x^*(Tx)| + |\lambda x^*(Ax)| \\ &< \|T\|_w - 2\varepsilon + \lambda \\ &< \|T\|_w - \varepsilon. \end{aligned}$$

Also, for any $(x, x^*) \in \mathsf{J} \setminus \mathsf{G}$

$$\begin{aligned} |x^*(Tx - \lambda Ax)|^2 &= x^*(Tx - \lambda Ax)\, \overline{x^*(Tx - \lambda Ax)} \\ &\leq \|T\|_w^2 + \lambda^2 - 2\lambda \mathfrak{Re}\ \overline{x^*(Tx)} x^*(Ax) \\ &\leq \|T\|_w^2 + \lambda^2 - \lambda r. \end{aligned}$$

Since $\lambda^2 - \lambda r < 0$, we get

$$\|T - \lambda A\|_w = \sup\{|x^*(Tx - \lambda Ax)| : (x, x^*) \in \mathsf{J}\} < \|T\|_w.$$

This is a contradiction to the fact that $T \perp_B^w A$. Therefore, $0 \in \mathsf{CO}(\mathsf{D})$, as desired.

$(ii) \implies (i)$: Since $0 \in \mathsf{CO}(\mathsf{D})$, applying Carathéodory Theorem we can find $t_j \in [0, 1]$ and $(x_j, x_j^*) \in \mathsf{M}_{\mathsf{W}(T)}$, $j = 1, 2, 3$; such that

$$\sum_{j=1}^{3} t_j = 1 \text{ and } \sum_{j=1}^{3} t_j \overline{x_j^*(Tx_j)} x_j^*(Ax_j) = 0. \tag{2.4}$$

Let $\rho : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F}$ be defined by

$$\rho(B) = \frac{1}{\|T\|_w} \sum_{j=1}^{3} t_j \overline{x_j^*(Tx_j)} x_j^*(Bx_j) \qquad \forall\ B \in (\mathbb{L}(\mathbb{X}))_w\,.$$

Clearly, $\rho(A) = 0$. Also, note that for any $B \in (\mathbb{L}(\mathbb{X}))_w$,

$$|\rho(B)| = \left| \frac{1}{\|T\|_w} \sum_{j=1}^{3} t_j \overline{x_j^*(Tx_j)} x_j^*(Bx_j) \right| \leq \frac{1}{\|T\|_w} \sum_{j=1}^{3} t_j |\overline{x_j^*(Tx_j)}| |x_j^*(Bx_j)| \leq \|B\|_w,$$

and

$$\rho(T) = \frac{1}{\|T\|_w} \sum_{j=1}^{3} t_j \overline{x_j^*(Tx_j)} x_j^*(Tx_j) = \frac{1}{\|T\|_w} \sum_{j=1}^{3} t_j \|T\|_w^2 = \|T\|_w.$$

This shows that $\rho \in J_{\mathsf{W}}(T)$. Therefore, $T \perp_B^w A$ and the proof follows. $\square$

Whenever $\mathsf{M}_{\mathsf{W}(T)} = \{(\mu x_0, \overline{\mu} x_0^*) : |\mu| = 1, (x_0, x_0^*) \in \mathsf{J}\}$, for some fixed $x_0 \in S_{\mathbb{X}}$ and $x_0^* \in S_{\mathbb{X}^*}$, we have the following corollary:

Corollary 2.4. *Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T, A \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero with $\mathsf{M}_{\mathsf{W}(T)} = \{(\mu x_0, \overline{\mu} x_0^*) : |\mu| = 1, (x_0, x_0^*) \in \mathsf{J}\}$. Then $T \perp_B^w A$ if and only if $x_0^*(Ax_0) = 0$.*

Proof. It follows from Theorem 2.3 that

$$T \perp_B^w A \;\Leftrightarrow\; 0 \in \mathsf{CO}\left(\left\{\overline{\overline{\mu} x_0^*(T\mu x_0)} \overline{\mu} x_0^*(A\mu x_0) : |\mu| = 1, (x_0, x_0^*) \in \mathsf{J}\right\}\right).$$

Clearly,

$$\overline{\overline{\mu} x_0^*(T\mu x_0)} \overline{\mu} x_0^*(A\mu x_0) = \overline{x_0^*(Tx_0)} x_0^*(Ax_0).$$

As a result, $T \perp_B^w A$ if and only if $x_0^*(Ax_0) = 0$. This completes the proof. $\square$

Finally, we characterize nu-smoothness in $(\mathbb{L}(\mathbb{X}))_w$, for a finite-dimensional Banach space $\mathbb{X}$.

Theorem 2.5. *Let $\mathbb{X}$ be a finite-dimensional complex Banach space and let $T \in (\mathbb{L}(\mathbb{X}))_w$ be non-zero. Then the following conditions are equivalent:*
(i) T is nu-smooth.
(ii) $\mathsf{M}_{\mathsf{W}(T)} = \{(\mu x_0, \overline{\mu} x_0^*) : |\mu| = 1, (x_0, x_0^*) \in \mathsf{J}\}$.
In case $\mathbb{X}$ is a finite-dimensional real Banach space, then for any non-zero $T \in (\mathbb{L}(\mathbb{X}))_w$, the condition (i) is equivalent to:

(iii) $\mathsf{M}_{\mathsf{W}(T)} = \{(ax_0, ax_0^*) : a \in \{-1, 1\}, (x_0, x_0^*) \in \mathsf{J}\}$.

Proof. We only prove $(i) \iff (ii)$ and note that the proof of $(i) \iff (iii)$ can be completed in a similar manner.

$(i) \implies (ii)$: Suppose on the contrary that there exists $(y_0, y_0^*) \in \mathsf{M}_{\mathsf{W}(T)}$ such that $(y_0, y_0^*) \neq (\mu x_0, \overline{\mu} x_0^*)$ for any unimodular constant μ. We now consider the following two cases:

Case I: Let $y_0 = \sigma_0 x_0$ for some unimodular constant σ_0. We claim that $\ker x_0^* \neq \ker y_0^*$. Indeed, if $\ker x_0^* = \ker y_0^*$, then $y_0^* = \alpha_0 x_0^*$ for some unimodular scalar α_0. Since $y_0^*(\sigma_0 x_0) = 1$, we get $\alpha_0 \sigma_0 = 1$, which is true if and only if $\alpha_0 = \overline{\sigma_0}$. However, this proves that $y_0^* = \overline{\sigma_0} x_0^*$, which is a contradiction, since $(y_0, y_0^*) \neq (\mu x_0, \overline{\mu} x_0^*)$ for any unimodular constant μ. Therefore, $\ker x_0^* \neq \ker y_0^*$, as we have claimed.

Next, we consider $x_1, x_2 \in \mathbb{X}$ such that $x_1 \in \ker x_0^* \setminus \ker y_0^*$ and $x_2 \in \ker y_0^* \setminus \ker x_0^*$. Observe that for any $z \in \mathbb{X}$, there exist a unique scalar α_z and a unique vector $h_z \in \ker x_0^*$ such that

$$z = \alpha_z x_0 + h_z.$$

Now, we define $A_1, A_2 : \mathbb{X} \to \mathbb{X}$ by

$$A_1(z) = \alpha_z x_1 \text{ and } A_2(z) = \alpha_z x_2 \qquad \forall\, z \in \mathbb{X}.$$

Clearly, $A_1, A_2 \in (\mathbb{L}(\mathbb{X}))_w$. Note that $A_1(x_0) = x_1$ and $A_2(y_0) = \sigma_0 A_2(x_0) = \sigma_0 x_2$. Since (x_0, x_0^*), $(y_0, y_0^*) \in \mathsf{M}_{\mathsf{W}(T)}$, we have that

$$x_0^*(Tx_0) = \sigma_1 \|T\|_w \text{ and } y_0^*(Ty_0) = \sigma_2 \|T\|_w,$$

for some unimodular scalars σ_1, σ_2. We define $\rho, \tau : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F}$ by

$$\rho(B) = x_0^*(Bx_0) \text{ and } \tau(B) = y_0^*(By_0) \qquad \forall\, B \in (\mathbb{L}(\mathbb{X}))_w\,.$$

Observe that ρ and τ are linear and the linear functional $\overline{\sigma_1}\rho : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F}$ satisfies the following:

$$\begin{aligned}
&(i)\ |\overline{\sigma_1}\rho(B)| = |\rho(B)| \leq \|B\|_w \qquad \forall\, B \in (\mathbb{L}(\mathbb{X}))_w\,,\\
&(ii)\ \overline{\sigma_1}\rho(T) = \|T\|_w,\\
&(iii)\ \overline{\sigma_1}\rho(A_1) = \overline{\sigma_1} x_0^*(A_1 x_0) = \overline{\sigma_1} x_0^*(x_1) = 0,\\
&(iv)\ \overline{\sigma_1}\rho(A_2) = \overline{\sigma_1} x_0^*(A_2 x_0) = \overline{\sigma_1} x_0^*(x_2) \neq 0.
\end{aligned}$$

Therefore, we get $\overline{\sigma_1}\rho \in J_{\mathsf{W}}(T)$, $A_1 \in \ker \overline{\sigma_1}\rho$ and $A_2 \notin \ker \overline{\sigma_1}\rho$. Similar arguments show that $\overline{\sigma_2}\tau \in J_{\mathsf{W}}(T)$, $A_2 \in \ker \overline{\sigma_2}\tau$ and $A_1 \notin \ker \overline{\sigma_2}\tau$. Thus, $\overline{\sigma_1}\rho$, $\overline{\sigma_2}\tau$ are distinct members of $J_{\mathsf{W}}(T)$. As a result, T is not nu-smooth, which is a contradiction.

Case II: Let $y_0 \neq \sigma x_0$ for any unimodular scalar σ. Let $z_0 \in \mathbb{X}$ be such that

$$z_0 = x_0^*(y_0) x_0 - y_0. \tag{2.5}$$

Evidently, z_0 is non-zero, as otherwise, $x_0^*(y_0)x_0 = y_0$ and $|x_0^*(y_0)| = 1$. Observe that $z_0 \in \ker x_0^*$. Consider any $z_0^* \in J(z_0)$. Evidently, for any $z \in \mathbb{X}$ there exist unique scalars α_z, β_z and $h_z \in \ker x_0^* \cap \ker z_0^*$ such that

$$z = \alpha_z x_0 + \beta_z z_0 + h_z. \tag{2.6}$$

Plugging the expression of z_0 (see (2.5)) into (2.6), we get

$$z = (\alpha_z + \beta_z x_0^*(y_0))x_0 + (-\beta_z)y_0 + h_z.$$

As a result, for every $z \in \mathbb{X}$, there exist $\gamma_z, \zeta_z \in \mathbb{F}$ and $h_z \in \ker x_0^* \cap \ker z_0^*$ such that

$$z = \gamma_z x_0 + \zeta_z y_0 + h_z.$$

Now, we define $T_1, T_2 : \mathbb{X} \to \mathbb{X}$ by

$$T_1(z) = \gamma_z x_0 \text{ and } T_2(z) = \zeta_z y_0 \qquad \forall\ z \in \mathbb{X}.$$

Clearly, $T_1, T_2 \in (\mathbb{L}(\mathbb{X}))_w$. Note that $T_1(y_0) = T_2(x_0) = \theta$. Moreover, since $(x_0, x_0^*), (y_0, y_0^*)$ are contained in $\mathsf{M}_{\mathsf{W}(T)}$, we have that

$$x_0^*(Tx_0) = \sigma_1\|T\|_w \text{ and } y_0^*(Ty_0) = \sigma_2\|T\|_w,$$

for some unimodular scalars σ_1, σ_2. We define $\psi, \eta : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F}$ by

$$\psi(B) = x_0^*(Bx_0) \text{ and } \eta(B) = y_0^*(By_0) \qquad \forall\ B \in (\mathbb{L}(\mathbb{X}))_w.$$

Consider the linear functionals $\overline{\sigma_1}\psi, \overline{\sigma_2}\eta : (\mathbb{L}(\mathbb{X}))_w \to \mathbb{F}$. Then using analogous techniques as in Case I, we can show the following:

$$(i)\ \overline{\sigma_1}\psi, \overline{\sigma_2}\eta \in J_{\mathsf{W}}(T),$$
$$(ii)\ T_2 \in \ker \overline{\sigma_1}\psi,\ T_1 \notin \ker \overline{\sigma_1}\psi,$$
$$(iii)\ T_1 \in \ker \overline{\sigma_2}\eta,\ T_2 \notin \ker \overline{\sigma_2}\eta.$$

Thus, $\overline{\sigma_1}\psi$, $\overline{\sigma_2}\eta$ are distinct members of $J_{\mathsf{W}}(T)$. As a result, T is not nu-smooth, which is a contradiction.

$(ii) \implies (i)$: Suppose that $T \perp_B^w U_1$ and $T \perp_B^w U_2$, for some non-zero $U_1, U_2 \in (\mathbb{L}(\mathbb{X}))_w$. Then it follows from Corollary 2.4 that $x_0^*(U_1x_0) = 0$ and $x_0^*(U_2x_0) = 0$. Therefore, we have that $x_0^*((U_1 + U_2)x_0) = 0$. As a result, $T \perp_B^w (U_1 + U_2)$. This proves that T is nu-smooth and thereby establishes the theorem completely. □

Equipped with the above characterization, we are now in a position to explore the geometrical dissimilarities between $\mathbb{L}(\mathbb{X})$ and $(\mathbb{L}(\mathbb{X}))_w$ from the perspective of smoothness. Applying Theorem 2.5, it is possible to construct linear operators between finite-dimensional polyhedral Banach spaces which are (not) nu-smooth. This further illustrates that the concepts of classical smoothness and nu-smoothness are indeed different.

To serve our purpose, we state the following result, which characterizes smoothness in the space of compact linear operators on a real reflexive Banach space endowed with the usual operator norm. Given a Banach space $\mathbb{X}$ and a normed linear space $\mathbb{Y}$, let $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ denote the space of all compact linear operators from $\mathbb{X}$ to $\mathbb{Y}$.

Theorem 2.6. *[15, Theorem 4.1 and Theorem 4.2] Let $\mathbb{X}$ be a real reflexive Banach space and let $\mathbb{Y}$ be a real normed space. Then $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$ is smooth if and only if T attains norm at a unique (upto scalar multiplication) vector x_0 (say) of $S_{\mathbb{X}}$ and Tx_0 is a smooth point.*

Let us end this article with the following two explicit illustrative examples:

Example 2.7. Let $\mathbb{Z} = \mathbb{X} \oplus_\infty \mathbb{R}$, where $\mathbb{X}$ is a two-dimensional real Banach space whose unit sphere is given by:

$$S_{\mathbb{X}} := \left\{ (x, y) \in \mathbb{R}^2 : \frac{\sqrt{3}|y| + |x| + \left| \frac{|y|}{\sqrt{3}} - |x| \right|}{2} = 1 \right\}. \tag{2.7}$$

It is not difficult to see that $B_{\mathbb{X}}$ is a regular hexagon in $\mathbb{R}^2$ and $B_{\mathbb{Z}}$ is a hexagonal prism in $\mathbb{R}^3$. Also, $ext(B_{\mathbb{Z}}) = \{\pm x_1, \pm x_2, \pm x_3, \pm x_4, \pm x_5, \pm x_6\}$, where $x_1 = (1, 0, 1)$, $x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$, $x_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$, $x_4 = (-1, 0, 1)$, $x_5 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$, $x_6 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$. A pictorial description of $B_{\mathbb{Z}}$ can be seen in Fig. 1.

Let $g : \mathbb{Z} \to \mathbb{R}$ be defined by

$$g(x, y, z) = \frac{x + \sqrt{3}y - z}{3} \qquad \forall\ (x, y, z) \in \mathbb{Z}.$$

A simple computation reveals that $|g(x_i)| < 1$ for all $i \in \{1, 2, 3, 4, 6\}$ and $|g(x_5)| = 1$. Thus, $\|g\| = 1$ and $\mathsf{M}_g = \{\pm x_5\}$. Now, define $T : \mathbb{Z} \to \mathbb{Z}$ by

$$T(x, y, z) = g(x, y, z)u \qquad \forall\ (x, y, z) \in \mathbb{Z},$$

where $u = (-1, 0, 0)$. Our aim is to show that T is not smooth with respect to the usual operator norm but T is smooth with respect to the numerical radius norm.

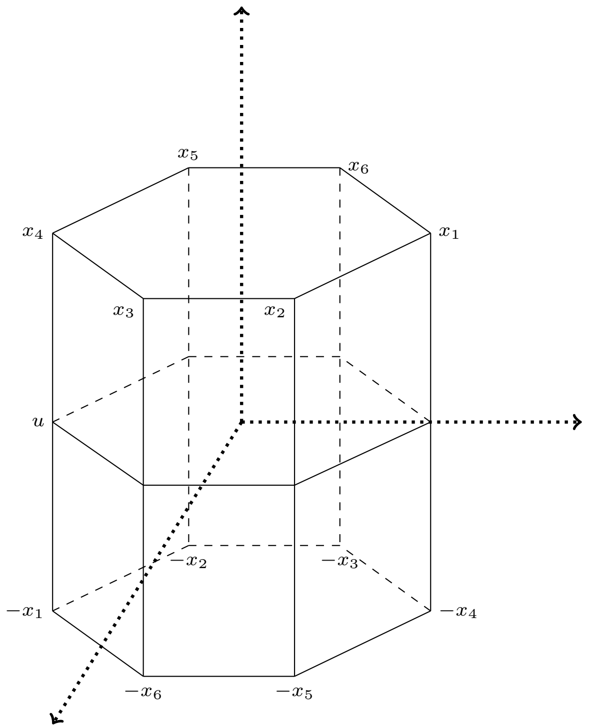

Fig. 1. Closed unit ball of $\mathbb{X} \oplus_\infty \mathbb{R}$.

Given any $(x, y, z) \in S_{\mathbb{Z}} \setminus \{\pm x_5\}$,

$$\begin{aligned} \|T(x, y, z)\| &= \|g(x, y, z)u\| \\ &= |g(x, y, z)|\|u\| \\ &= |g(x, y, z)| \\ &< 1. \end{aligned}$$

On the other hand, $\|T(x_5)\| = 1$. Therefore, $\|T\| = 1$ and $\mathsf{M}_T = \{\pm x_5\}$. Note that u is a non-smooth point and $T(x_5) = -u$. Therefore, it follows from Theorem 2.6 that T is not smooth with respect to the usual operator norm.

Next, let

$$\Lambda := \{\pm(x_5, h) : h \in J(x_5)\}.$$

Observe that for any $(v, f) \in \mathsf{J} \setminus \Lambda$,

$$|f(Tv)| = |f(g(v)u)| \leq \|f\|\|g(v)u\| = |g(v)|\|u\| < 1.$$

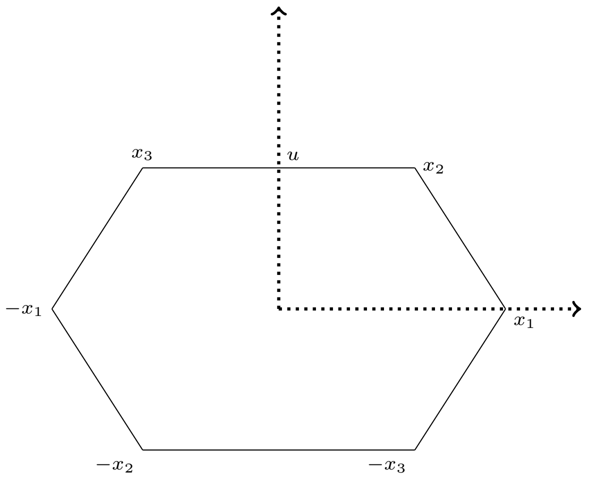

Fig. 2. Closed unit ball of $\mathbb{X}$.

Let

$$f_1(x,y,z) = z, f_2(x,y,z) = -\frac{2}{\sqrt{3}}y, f_3(x,y,z) = -x - \frac{1}{\sqrt{3}}y \qquad \forall\ (x,y,z) \in \mathbb{Z}.$$

Note that f_1, f_2 and f_3 are support functionals of $B_{\mathbb{Z}}$ at x_5 and contained in $ext(B_{\mathbb{Z}^*})$. Consequently,

$$J(x_5) = \{\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 : \lambda_1, \lambda_2, \lambda_3 \geq 0,\ \lambda_1 + \lambda_2 + \lambda_3 = 1\}.$$

Evidently, $Tx_5 \in \ker f_1 \cap \ker f_2$ and $f_3(Tx_5) = 1$. Therefore, for any $(x_5, h) \in \Lambda$,

$$|h(Tx_5)| \leq 1,$$

and the equality holds for $h = f_3$. This shows that $\|T\|_w = 1$, and $\mathsf{M}_{\mathsf{W}(T)} = \{(x_5, f_3), (-x_5, -f_3)\}$. Consequently, T is nu-smooth by Theorem 2.5.

Example 2.8. Let $\mathbb{X}$ be a two-dimensional Banach space whose unit ball is the regular hexagon in $\mathbb{R}^2$, defined in Example 2.7. Clearly, $ext(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$, where $x_1 = (1,0)$, $x_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $x_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. A pictorial description of $B_{\mathbb{X}}$ can be seen from Fig. 2.

Evidently, $ext(B_{\mathbb{X}^*}) = \{\pm f_1, \pm f_2, \pm f_3\}$, where

$$f_1(x,y) = x - \frac{1}{\sqrt{3}}y, f_2(x,y) = x + \frac{1}{\sqrt{3}}y, f_3(x,y) = \frac{2}{\sqrt{3}}y \qquad \forall\ (x,y) \in \mathbb{X}.$$

Let $g : \mathbb{X} \to \mathbb{R}$ be defined by

$$g(x,y) = x \qquad \forall\ (x,y) \in \mathbb{X}.$$

Define $T : \mathbb{X} \to \mathbb{X}$ by

$$T(x, y) = g(x, y)u \qquad \forall\ (x, y) \in \mathbb{X},$$

where $u = (0, \frac{\sqrt{3}}{2})$. Our aim is to show that T is not nu-smooth but T is smooth with respect to the usual operator norm.

Clearly, $\mathsf{M}_T = \mathsf{M}_g = \{\pm(1, 0)\}$ and $T(1, 0) = u = (0, \frac{\sqrt{3}}{2})$ is a smooth point of $S_{\mathbb{X}}$. Therefore, it follows from Theorem 2.6 that T is smooth with respect to the usual operator norm. Observe that $J(x_1) \cap ext(B_{\mathbb{X}^*}) = \{f_1, f_2\}$, $J(x_2) \cap ext(B_{\mathbb{X}^*}) = \{f_2, f_3\}$ and $J(x_3) \cap ext(B_{\mathbb{X}^*}) = \{f_3, -f_1\}$. Now,

$$\begin{aligned}
f_1(Tx_1) &= f_1(g(x_1)u) = f_1(u) = -\frac{1}{2}, \\
f_2(Tx_1) &= f_2(g(x_1)u) = f_2(u) = \frac{1}{2}; \\
f_2(Tx_2) &= f_2(g(x_2)u) = \frac{1}{2} f_2(u) = \frac{1}{4}, \\
f_3(Tx_2) &= f_3(g(x_2)u) = \frac{1}{2} f_3(u) = \frac{1}{2}; \\
f_3(Tx_3) &= f_3(g(x_3)u) = -\frac{1}{2} f_3(u) = -\frac{1}{2}, \\
-f_1(Tx_3) &= -f_1(g(x_3)u) = \frac{1}{2} f_1(u) = \frac{1}{4}.
\end{aligned}$$

Thus, it follows from (1.1) that $\|T\|_w = \frac{1}{2}$ and $\{(x_1, f_1), (-x_1, -f_1), (x_1, f_2), (-x_1, -f_2)\}$ is a subset of $\mathsf{M}_{\mathsf{W}(T)}$. Consequently, T is not nu-smooth by Theorem 2.5.

Declaration of competing interest

The authors would like to declare that there is no competing interest.

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