

# FINITE DATA PERFORMANCE ANALYSIS OF MVDR BEAMFORMER WITH SPATIAL SMOOTHING

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## ABSTRACT

In this paper, we analyze the finite-data performance of **MVDR** beamformer with spatial smoothing, using first order perturbation theory. In particular, we develop expressions for the mean values of the power gain in any direction of interest, the output power and the norm of the weight-error vector, as a function of the number of snapshots and the number of smoothing steps. We simplify these expressions for a single interference case without smoothing to show explicitly how the **SNR**, spacing of the interference from the desired signal and the correlation between them influence the beamformer performance. Simulations are used to verify the usefulness of the theoretical expressions and the results show an excellent agreement with predicted results.

## 1 Introduction

In a **MVDR** (minimum variance distortionless response) beamformer, the array weights are chosen so as to pass the desired directional (look direction) signal without any distortion while maximally rejecting the interfering signals. The only assumption made is that the desired signal direction is known a priori.

Since the pioneering work of Capon, there has been much activity in the development of optimum arrays for radar, sonar, communication etc. Sensitivity of the adaptive arrays to element errors and to those in the look direction has been extensively studied in the recent past. However, the performance of the optimum beamformer in finite data has received little attention. The only papers (to our knowledge) which address this specific problem are those of Reed et al. [1] and Boroson [2]. Their analysis, however, assumes that the weight vector computed from one set of data operates on an independent set of data. Also, the main thrust of their analysis is directed to the case where the weight vector is estimated from the noise-alone matrix inverse, and the interferences are uncorrelated with the look-direction signal.

In this paper, we analyze the performance of the **MVDR** beamformer in finite data with spatial smoothing. Our analysis uses first order perturbation theory,

originally developed by Wilkinson [3] and first applied in array processing problems by Kaveh and Barabell [4].

## 2 Problem Statement

Consider  $D$  narrowband sources impinging on an uniform linear array consisting of  $M$  elements. We assume that the  $D$  signals and the additive noise are zero mean, stationary, and ergodic complex Gaussian random processes. In the finite data case, given  $L$  snapshots of data, the array covariance matrix is estimated as

$$\hat{R} = \frac{1}{L} \sum_{t=1}^L \mathbf{r}(t)\mathbf{r}^+(t) \quad (2.1)$$

where  $\mathbf{r}(t)$  is array snapshot vector. In spatial smoothing, the total array is divided into  $K$  overlapping subarrays, each of size  $m$  and the smoothed covariance matrix is obtained by averaging over these. Thus the smoothed array covariance matrix is given by

$$\bar{R} = \frac{1}{K} \sum_{i=1}^K \hat{R}_i \quad (2.2)$$

where,  $\hat{R}_i$  is a  $m$  by  $m$  sub-matrix of  $\hat{R}$  with  $\hat{R}(i, i)$  as its first element. Here, all the terms corresponding to the finite data are denoted by  $(\hat{\cdot})$ . We may point out here that when spatial smoothing is not used,  $L$  should at least be equal to  $m$ , since  $\hat{R}$  would otherwise be rank deficient. On the other hand, with spatial smoothing, the minimum number of snapshots required (for  $\bar{R}$  to attain full rank) can be traded with the number of subarrays,  $K$ . For example, for  $L=1$ ,  $\hat{R}$  would attain full rank if  $K \geq m$ .

The weight vector in **MVDR** beamformer with spatial smoothing is given by

$$\bar{\mathbf{w}} = \frac{\bar{R}^{-1} \mathbf{a}_1}{\mathbf{a}_1^+ \bar{R}^{-1} \mathbf{a}_1} \quad (2.3)$$

where  $\mathbf{a}_1$  denotes the desired signal direction vector. The power gain and the output power can then be expressed as

$$\hat{G}_j = \mathbf{g}_j^+ \bar{\mathbf{w}} \quad \bar{P}_{out} = \frac{1}{\mathbf{a}_1^+ \bar{R}^{-1} \mathbf{a}_1} \quad (2.4)$$

where

$$\bar{g}_j = \frac{\mathbf{a}_j^+ \bar{R}^{-1} \mathbf{a}_1}{\mathbf{a}_1^+ \bar{R}^{-1} \mathbf{a}_1} \quad (2.5)$$

When we refer to the asymptotic quantities we drop the symbol hat. Further, when we consider the unsmoothed case, i.e.,  $K=1$ , we drop the symbol over-bar.

To study the deviation of the weight vector from its asymptotic value, w. as a function of  $L$ , we define the weight-error vector norm as

$$\frac{\|\bar{\mathbf{w}} - \bar{\mathbf{w}}\|^2}{\|\bar{\mathbf{w}}\|^2} \quad (2.6)$$

where  $\|\cdot\|$  denotes the Euclidean norm.

The problem is to evaluate the mean values,  $E[\bar{G}_j]$ ,  $E[\hat{P}_{out}]$  and  $E[\|\bar{\mathbf{w}} - \bar{\mathbf{w}}\|^2/\|\bar{\mathbf{w}}\|^2]$ , as a function of the number of snapshots.

### 3 Derivation Of Expressions

The estimated array covariance matrix,  $\hat{\mathbf{R}}$ , is Wishart distributed and can be decomposed as [4,3]

$$\hat{\mathbf{R}} = \mathbf{R} + \epsilon \mathbf{B} \quad (3.1)$$

where  $\mathbf{R}$  is the true or asymptotic covariance matrix,  $\mathbf{B}$  is a zero mean random matrix and  $\epsilon$  is a perturbation constant. Since  $\hat{\mathbf{R}}_i$  is a submatrix of  $\hat{\mathbf{R}}$ , we can write

$$\hat{\mathbf{R}}_i = \mathbf{R}_i + \epsilon \mathbf{B}_i \quad (3.2)$$

where  $\mathbf{B}_i$  is the corresponding submatrix of  $\mathbf{B}$ . It then follows that

$$\bar{\hat{\mathbf{R}}} = \bar{\mathbf{R}} + \epsilon \bar{\mathbf{B}} \quad (3.3)$$

where  $\mathbf{R}$  and  $\mathbf{B}$  are given by

$$\bar{\mathbf{R}} = \frac{1}{K} \sum_{i=1}^K \mathbf{R}_i, \quad \bar{\mathbf{B}} = \frac{1}{K} \sum_{i=1}^K \mathbf{B}_i \quad (3.4)$$

We can express

$$\begin{aligned} \bar{\hat{\mathbf{R}}}^{-1} &= (\bar{\mathbf{R}} + \epsilon \bar{\mathbf{B}})^{-1} \\ &= \left\{ I - \epsilon \bar{\mathbf{R}}^{-1} \bar{\mathbf{B}} + \epsilon^2 (\bar{\mathbf{R}}^{-1} \bar{\mathbf{B}})^2 - \dots \right\} \bar{\mathbf{R}}^{-1} \quad (3.5) \end{aligned}$$

When the number of snapshots is not small,  $\epsilon$  is a small quantity and hence, we can neglect terms of order  $\epsilon^3$  and higher and obtain results accurate to the first order in  $L$ . We thus approximate

$$\bar{\hat{\mathbf{R}}}^{-1} \approx \left\{ I - \epsilon \bar{\mathbf{R}}^{-1} \bar{\mathbf{B}} + \epsilon^2 (\bar{\mathbf{R}}^{-1} \bar{\mathbf{B}})^2 \right\} \bar{\mathbf{R}}^{-1} \quad (3.6)$$

### 3.1 Mean Power Gain

Using (3.6) in (2.4) and the approximation

$$(1 + y)^{-1} \approx 1 - y + y^2, \quad (3.7)$$

and neglecting the terms of order  $\epsilon^3$  and higher, we obtain the following expression for the power gain in a direction with steering vector  $\mathbf{a}_j$ :

$$\begin{aligned} \bar{G}_j &\approx \bar{G}_j + \epsilon (2\bar{G}_j \delta_2 - 2J_1) + \epsilon^2 (|\delta_3|^2 \\ &\quad - 4\delta_2 J_1 + 3\bar{G}_j \delta_2^2 - 2\bar{G}_j \delta_4 + 252) \quad (3.8) \end{aligned}$$

where

$$\begin{aligned} \delta_2 &= \frac{\mathbf{q}_1 + \bar{\mathbf{B}} \mathbf{q}_1}{\delta_1}, & \delta_3 &= \frac{\mathbf{q}_1 + \bar{\mathbf{B}} \mathbf{q}_1}{\delta_1} \\ \delta_4 &= \frac{\mathbf{q}_1 + \bar{\mathbf{B}} \bar{\mathbf{R}}^{-1} \bar{\mathbf{B}} \mathbf{q}_1}{\delta_1}, & \delta_5 &= \frac{\mathbf{q}_1 + \bar{\mathbf{B}} \bar{\mathbf{R}}^{-1} \bar{\mathbf{B}} \mathbf{q}_1}{\delta_1} \\ \delta_1 &= \mathbf{a}_1^+ \bar{\mathbf{R}}^{-1} \mathbf{a}_1, & \mathbf{q}_1 &= \bar{\mathbf{R}}^{-1} \mathbf{a}_1 \\ \mathbf{q}_j &= \bar{\mathbf{R}}^{-1} \mathbf{a}_j, & J_1 &= \text{real}(\bar{g}_j^* \delta_3) \\ J_2 &= \text{real}(\bar{g}_j^* \delta_5), \end{aligned} \quad (3.9)$$

Note that  $\delta_2, \delta_3, \delta_4$  and  $\delta_5$  are scalar random quantities while  $\delta_1$  is deterministic.

For any deterministic vectors  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  of a proper dimension, we can show that (see [5],[6] for more details)

$$E[\alpha_1^+ \bar{\mathbf{B}} \alpha_2] = 0 \quad (3.10)$$

and

$$E[\alpha_1^+ \bar{\mathbf{B}} \alpha_2 \alpha_3^+ \bar{\mathbf{B}} \alpha_4] = \frac{1}{\epsilon^2 K^2 L} \sum_{u=1}^K \sum_{v=1}^K \alpha_1^+ \mathbf{R}_{uv} \alpha_4 \alpha_3^+ \mathbf{R}_{vu} \alpha_2 \quad (3.11)$$

where  $\mathbf{R}_{uv}$  denotes the submatrix of size  $m$  by  $m$  of total array covariance matrix  $\mathbf{R}$  beginning at element  $\mathbf{R}(u,v)$ . Note that the subarray matrix  $\mathbf{R}_i$  is same as  $\mathbf{R}_{ii}$ . Using these identities we can evaluate the expectation of each term in (3.8) and obtain

$$\begin{aligned} E[\bar{G}_j] &\approx \bar{G}_j + \frac{1}{K^2 L \delta_1^2} \sum_{u=1}^K \sum_{v=1}^K \{ \phi_{jj} \phi_{11}^* - 4 \text{real}(\bar{g}_j^* \phi_{j1} \phi_{11}^*) \\ &\quad + 3\bar{G}_j \phi_{11} \phi_{11}^* - 2\bar{G}_j \phi_{11} T_1 \delta_1 + 2 \text{real}(\bar{g}_j^* \phi_{j1} T_1 \delta_1) \} \quad (3.12) \end{aligned}$$

where

$$\phi_{xy} = \mathbf{q}_x^+ \mathbf{R}_{uv} \mathbf{q}_y, \quad T_1 = \text{tr}(\bar{\mathbf{R}}^{-1} \mathbf{R}_{vu}) \quad (3.13)$$

Since  $\mathbf{a}_j$  is arbitrary, (3.12) can be used to find the power gain in any direction and thus obtain the mean power beam pattern of the beamformer for finite data.

### 3.2 Mean Output Power

Following the approach used above, we can show that (see [5],[6] for more details)

$$E[\bar{P}_{out}] \approx \bar{P}_{out} \left\{ 1 + \frac{1}{K^2 L} \sum_{u=1}^K \sum_{v=1}^K \left( \frac{\phi_{11} \phi_{11}^*}{\delta_1^2} - \frac{\phi_{11} T_1}{\delta_1} \right) \right\} \quad (3.14)$$

### 3.3 Mean Weight-Error Vector Norm

We can also show that (see [5],[6])

$$E \left[ \frac{\|\tilde{\mathbf{w}} - \bar{\mathbf{w}}\|^2}{\|\bar{\mathbf{w}}\|^2} \right] \approx \frac{1}{K^2 L} \sum_{u=1}^K \sum_{v=1}^K \left\{ \frac{\phi_{11} T_2}{x_1} + \frac{|\phi_{11}|^2}{\delta_1^2} - \frac{\tau}{\delta_1 x_1} \right\} \quad (3.15)$$

where

$$\begin{aligned} T_2 &= \text{tr}(R_{vu} \bar{R}^{-1} \bar{R}^{-1}) \\ \tau &= 2 \text{real}(\phi_{11} \mathbf{q}_1^+ \bar{R}^{-1} R_{vu} \mathbf{q}_1) \end{aligned} \quad (3.16)$$

Note that the RHS in (3.15) goes to zero as L tends to infinity, which is consistent with the fact that w tends to w asymptotically.

## 4 Unsmoothed Case

In this section, we reduce the above results to the unsmoothed case. It should, however, be noted that even though the above results were derived for the case of a uniform array, the following results are valid for an arbitrary array.

For the unsmoothed case, i.e.,  $K=1$ , (3.12),(3.14) and (3.15) reduce to

$$E[\hat{G}_j] \approx G_j \left(1 - \frac{1}{L}\right) + \frac{Q_j}{L} \quad (4.1)$$

$$E[\hat{P}_{out}] \approx P_{out} \left(1 - \frac{m-1}{L}\right) \quad (4.2)$$

$$E \left[ \frac{\|\hat{\mathbf{w}} - \mathbf{w}\|^2}{\|\mathbf{w}\|^2} \right] \approx \frac{1}{L} \left( \frac{\delta_1 \text{tr}(R^{-1})}{x_1} - 1 \right) \quad (4.3)$$

where

$$Q_j = \frac{\mathbf{a}_j^+ R^{-1} \mathbf{a}_j}{\delta_1} \quad (4.4)$$

Note from (4.2) that the mean value of the output power is within 3 dB of the asymptotic value when the number of snapshots, L, is approximately twice the array size.

We now study the finite data performance of the beamformer in certain special, but pertinent, scenarios. We do this by considering a single interference .

### 4.1 Single Interference Case

For the case of a single interference, we can write

$$\begin{aligned} R &= \sigma_n^2 \left( \mathbf{I} + \frac{\sigma_1^2}{\sigma_n^2} \mathbf{a}_1 \mathbf{a}_1^+ + \frac{\sigma_2^2}{\sigma_n^2} \mathbf{a}_2 \mathbf{a}_2^+ + \frac{\sigma_1 \sigma_2}{\sigma_n^2} \rho \mathbf{a}_1 \mathbf{a}_2^+ \right. \\ &\quad \left. + \frac{\sigma_1 \sigma_2}{\sigma_n^2} \rho^* \mathbf{a}_2 \mathbf{a}_1^+ \right) \end{aligned} \quad (4.5)$$

where  $\sigma_1^2, \sigma_2^2$  and  $\sigma_n^2$  are the powers of the desired signal, interference (jammer) and noise, respectively,  $\rho$  is the coefficient of correlation between the signal and

the jammer, and  $\mathbf{a}_2$  is the jammer direction vector. Applying the matrix inversion lemma repeatedly to (4.5), we can express (4.1) as

$$\begin{aligned} E[\hat{G}_j] &\approx \frac{\psi \psi^*}{\{\sigma_2^2(m^2 - |\beta|^2) + m\sigma_n^2\}^2} \left(1 - \frac{1}{L}\right) \\ &\quad + \frac{\sigma_1^2(m^2 - |\beta|^2) + m\sigma_n^2}{L\{\sigma_2^2(m^2 - |\beta|^2) + m\sigma_n^2\}} \end{aligned} \quad (4.6)$$

where  $\beta = \mathbf{a}_2^+ \mathbf{a}_1$  and

$$\psi = \sigma_1 \sigma_2 \rho^* (|\beta|^2 - m^2) + \sigma_n^2 \beta \quad (4.7)$$

In (4.6), we used the simplified result for the power gain from [7]. We will now simplify (4.6) for some special scenarios.

#### 4.1.1 Low Noise Case ( $\sigma_n^2 \ll \sigma_1^2, \sigma_2^2$ )

Assuming that the interference is not too close to the desired direction, (4.6) can be simplified to

$$E[\hat{G}_j] \approx \frac{\sigma_1^2}{\sigma_2^2} \left( |\rho|^2 + \frac{1 - |\rho|^2}{L} \right) \quad (4.8)$$

Note from (4.8) that when the signals are uncorrelated ( $\rho = 0$ ), the gain varies as  $1/L$  and thus, for a 20 dB suppression of jammer power below the signal power, we require about 100 snapshots. When signals are correlated, (4.8) shows that the degradation in the interference rejection performance due to finite data is overshadowed by that due to correlation. For example, for  $\rho = .5$ , the reduction in the jammer power goes up by a mere 0.1 dB when L is increased from 100 to infinity.

We now consider the mean output power given by (4.2). Assuming once again that the interference is not too close to the desired direction,

$$E[\hat{P}_{out}] \approx \sigma_1^2 (1 - |\rho|^2) \left(1 - \frac{m-1}{L}\right) \quad (4.9)$$

where we used the simplified result for  $P_{out}$  from [7]. Note from (4.9) that in addition to signal cancellation due to correlation, a reduction in output signal power also takes place because of the finite data.

#### 4.1.2 Jammer Close to Look Direction ( $|\beta|^2 \approx m^2$ )

When the jammer is close to the look direction,  $|\beta|$  approaches m and (4.6) simplifies to

$$E[\hat{G}_j] \approx \frac{|\beta|^2}{m^2} \quad (4.10)$$

Thus, the power gain is independent of L and the MVDR beamformer reduces to a conventional beamformer, as in the asymptotic case (see [7]).

## 5 Simulation Results

We conducted simulations to test the usefulness of the expressions derived in the earlier sections. In our simulations, a uniform linear array with a half wavelength inter-element spacing was used, though the expressions for the unsmoothed case are valid for an arbitrary array. In all the scenarios, the desired signal was assumed to arrive from the broadside direction (0 degrees). The simulated result was obtained by averaging over 50 monte carlo runs. The scenario used in the simulation is described in the figures.

First we considered the power gain with spatial smoothing. The results show a good agreement between the theoretical and simulated results. Fig 1(a) shows the mean power gain performance as a function of the number of snapshots for various number of smoothing steps. Here, the improvement with increasing number of smoothing steps is mainly due to progressive reduction in the effective correlation between the impinging signals. Note that for  $K=6$ , when sources are sufficiently decorrelated, we can observe a change in the gain with the number of snapshots. This is consistent with our earlier comments that when the correlation between the desired signal and the interference is high, the degradation in the gain due to correlation is so high that the effect of  $L$  on gain performance is hardly noticed.

Fig. 1(b) shows the effect of smoothing on the gain performance in the presence of uncorrelated interference. Recall that in the asymptotic case, progressive smoothing does not affect the gain when the sources are uncorrelated. But for the finite data case, there is a significant improvement in the gain with increasing number of sub-arrays for any given data size. This improvement in the performance is because of the reduction in the finite data perturbations due to progressive smoothing.

Fig. 2 shows the finite-data behaviour of the mean output power and the mean weight-error vector norm for the case of a fully correlated interference, for two different smoothing steps, i.e.,  $K = 2$  and  $K = 6$ . With more smoothing, the effective correlation between the desired signal and the interference comes down thereby resulting in a reduced signal cancellation. Consequently, the output power goes up, as seen in Fig 2(b). Recall further that progressive smoothing results in progressive reduction in the finite data perturbations, and as a result, the finite data weight vector is nearer to its asymptotic value when  $K$  is larger (see Fig. 2(a)). Because of this, the output power is closer to the asymptotic value in the case with  $K=6$  compared to that with  $K=2$  (see Fig. 2(b)).

## 6 Conclusions

In this paper, we have analyzed the performance of the MVDR beamformer with and without spatial smoothing, using first order perturbation theory.

Our analysis shows that for uncorrelated sources, the performance degradation from the asymptotic value is dependent on the data size, while in the correlated case, the degradation due to finite data is overshadowed by that due to correlation. The results for the spatially smoothed case indicate that in addition to decorrelating the sources, the smoothing can also reduce the perturbation effects caused by finite data. Thus we can compensate for low number of snapshots by using more smoothing steps and hence a larger array. We may point out here that in a practical scenario, it may not be known a priori whether the interferences are correlated or not, and hence, a good thumb rule is to use smoothing. This will result in decorrelation of the sources mainly when correlated interferences are present while it will cause a reduction in the finite-data perturbations when the interferences are uncorrelated.

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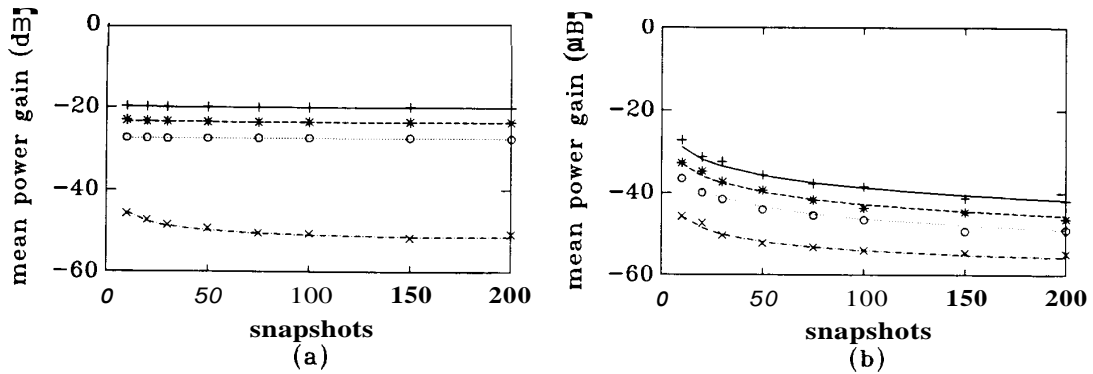


Fig.1 Finite data performance curves with different number of smoothing steps,  $K=1,3,4,6$   
 $(m=4, DOA's=0^\circ, 20^\circ, \text{powers}=0 \text{ dB}, 20 \text{ dB}, \sigma_n^2=0 \text{ dB})$   
 (a) fully correlated interference  $\rho = 1$  (b) uncorrelated interference  $\rho = 0$

$K$	theoretical	simulated
1	—	+
3	- - -	*
4	⋯	o
6	- . -	x

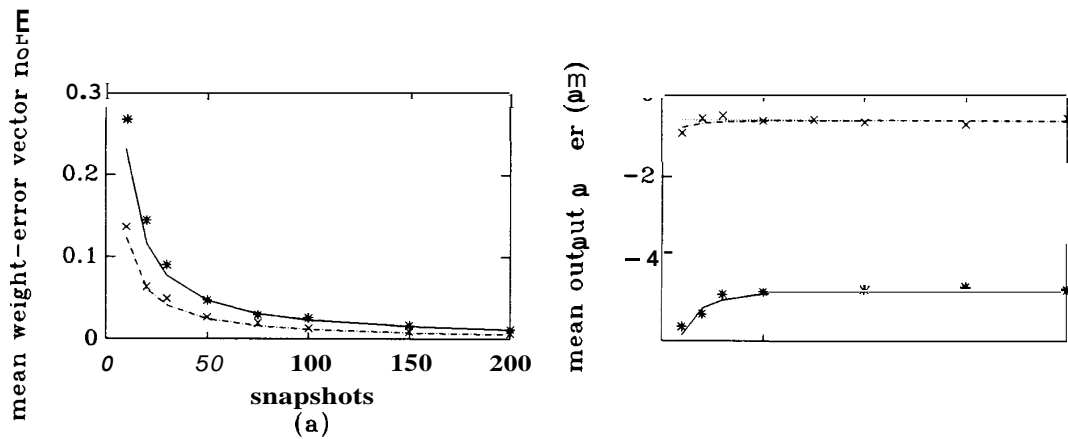


Fig.2 Finite data behaviour of mean weight-error vector norm and mean output power for different number of smoothing steps,  $K = 2$  and  $6$ .  
 $(m = 6, DOA's = 0^\circ, 10^\circ, \text{powers} = 0 \text{ dB}, 20 \text{ dB}, \sigma_n^2 = 0 \text{ dB}, \rho = 1)$

$K$	theoretical	simulated
2	—	*
6	- - -	x

... asymptotic