Platonic solids, Archimedean solids and semi-equivelar maps on the sphere

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ABSTRACT

A map X on a surface is called vertex-transitive if the automorphism group of X acts transitively on the set of vertices of X. A map is called semi-equivelar if the cyclic arrangement of faces around each vertex is same. In general, semi-equivelar maps on a surface form a bigger class than vertex-transitive maps. There are semi-equivelar maps on the torus, the Klein bottle and other surfaces which are not vertex-transitive.

It is known that the boundaries of Platonic solids, Archimedean solids, regular prisms and anti-prisms are vertex-transitive maps on $\mathbb{S}^2$. Here we show that there is exactly one semi-equivelar map on $\mathbb{S}^2$ which is not vertex-transitive. As a consequence, we show that all the semi-equivelar maps on $\mathbb{RP}^2$ are vertex-transitive. Moreover, every semi-equivelar map on $\mathbb{S}^2$ can be geometrized, i.e., every semi-equivelar map on $\mathbb{S}^2$ is isomorphic to a semi-regular tiling of $\mathbb{S}^2$. In the course of the proof of our main result, we present a combinatorial characterisation in terms of an inequality of all the types of semi-equivelar maps on $\mathbb{S}^2$. Here we present combinatorial proofs of all the results.

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1. Introduction

By a map we mean a polyhedral map on a surface. So, a face of a map is an n-gon for some $n \geq 3$ and two intersecting faces intersect either on a vertex or on an edge. A map on a surface is also called a *topological tiling* of the surface. If all the faces of a map are triangles then the map is called *simplicial*. Maps on the sphere and the torus are called *spherical* and *toroidal* maps respectively. A map X is said to be *vertex-transitive* if the automorphism group Aut(X) of X acts transitively on the set $V(X)$ of vertices of X. In [14], Lutz found all the (77 in numbers) vertex-transitive simplicial maps with at most 15 vertices.

For a vertex u in a map X, the faces containing u form a cycle (called the *face-cycle* at u) C_u in the dual graph of X. So, C_u is of the form $(F_{1,1}\text{-}\cdots\text{-}F_{1,n_1})\text{-}\cdots\text{-}(F_{k,1}\text{-}\cdots\text{-}F_{k,n_k})\text{-}F_{1,1}$, where $F_{i,j}$ is a p_i-gon for $1 \leq j \leq n_i$ and $p_i \neq p_{i+1}$ for $1 \leq i \leq k$ (addition in the suffix is modulo k). A map X is called *semi-equivelar* if the cyclic arrangement of faces around each vertex is same. More precisely, there exist integers $p_1, \ldots, p_k \geq 3$ and $n_1, \ldots, n_k \geq 1$, $p_i \neq p_{i+1}$, such that C_u is of the form as above for all $u \in V(X)$. In such a case, X is called a *semi-equivelar map of vertex-type* $[p_1^{n_1}, \ldots, p_k^{n_k}]$ (or, a *map of type* $[p_1^{n_1}, \ldots, p_k^{n_k}]$). (We identify a cyclic tuple $[p_1^{n_1}, p_2^{n_2}, \ldots, p_k^{n_k}]$ with $[p_k^{n_k}, \ldots, p_2^{n_2}, p_1^{n_1}]$ and with $[p_2^{n_2}, \ldots, p_k^{n_k}, p_1^{n_1}]$.) Clearly, vertex-transitive maps are semi-equivelar.

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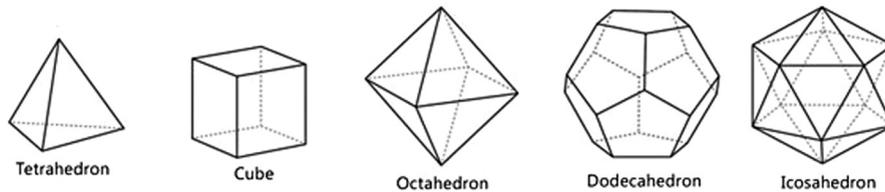
Fig. 1. Platonic Solids (from [23]).

A *semi-regular tiling* of a surface S of constant curvature (*e.g.*, the round sphere, the Euclidean plane or the hyperbolic plane) is a semi-equivelar map on S in which each face is a regular polygon and each edge is a geodesic. An *Archimedean tiling* of the plane $\mathbb{R}^2$ is a semi-regular tiling of the Euclidean plane. There are eleven types of semi-equivelar toroidal maps and all these are quotients of Archimedean tilings of the plane ([6], [7]). Among these 11 types, 4 types of maps are always vertex-transitive and there are infinitely many such examples in each type ([1], [6]). For each of the other seven types, there exists a semi-equivelar toroidal map which is not vertex-transitive ([6]). Although, there are vertex-transitive maps of each of these seven types also ([1], [16], [19]). Similar results are known for the Klein bottle ([1], [6], [17]). If the Euler characteristic $\chi(M)$ of a surface M is negative then the number of semi-equivelar maps on M is finite and at most $-84\chi(M)$ ([1]). Nine examples of non-vertex-transitive semi-equivelar maps on the surface of Euler characteristic -1 are known ([20]). There are exactly three non vertex-transitive semi-equivelar simplicial maps on the double torus ([9]).

It follows from the results in [5] that there exist semi-regular tilings of the hyperbolic plane of infinitely many different vertex-types. It is also shown that there exists a unique semi-regular tiling of the hyperbolic plane of vertex-type $[p^q]$ for each pair (p, q) of positive integers satisfying $1/p + 1/q < 1/2$. Moreover, these tilings are vertex-transitive. It follows from the results in [11] and [15] that there exists a semi-equivelar map on the plane of vertex-type $[3^1, p^3]$ for each $p \geq 5$ odd and there does not exist any vertex-transitive map on the plane of vertex-type $[3^1, q^3]$, for each $q \equiv \pm 1 \pmod 6$.

All vertex-transitive spherical maps are known. These are the boundaries of Platonic solids, Archimedean solids and two infinite families ([1], [12]). Other than these, there exists a semi-equivelar spherical map, namely, the boundary of the pseudorhombicuboctahedron ([10], [24]). It is known that quotients of ten centrally symmetric vertex-transitive spherical maps (namely, the boundaries of icosahedron, dodecahedron and eight Archimedean solids) are all the vertex-transitive maps on the real projective plane $\mathbb{RP}^2$ ([1]). Here we show that these are also all the semi-equivelar maps on $\mathbb{RP}^2$. We prove

Theorem 1.1. *Let X be a semi-equivelar spherical map. Then, up to isomorphism, X is the boundary of a Platonic solid, an Archimedean solid, a regular prism, an antiprism or the pseudorhombicuboctahedron.*

Theorem 1.2. *If Y is a semi-equivelar map on $\mathbb{RP}^2$ then the vertex-type of Y is $[5^3]$, $[3^5]$, $[4^1, 6^2]$, $[3^1, 5^1, 3^1, 5^1]$, $[3^1, 4^3]$, $[4^1, 6^1, 8^1]$, $[3^1, 4^1, 5^1, 4^1]$, $[4^1, 6^1, 10^1]$, $[3^1, 10^2]$ or $[5^1, 6^2]$. Moreover, in each case, there exists a unique semi-equivelar map on $\mathbb{RP}^2$.*

Corollary 1.3. *The boundary of the pseudorhombicuboctahedron is not vertex-transitive and all the other semi-equivelar spherical maps are vertex-transitive.*

As consequences we get

Corollary 1.4. (*a*) *Each semi-equivelar spherical map is isomorphic to a semi-regular tiling of $\mathbb{S}^2$. (b) Each semi-equivelar map on $\mathbb{RP}^2$ is isomorphic to a semi-regular tiling of $\mathbb{RP}^2$.*

Corollary 1.5. *All the semi-equivelar maps on $\mathbb{RP}^2$ are vertex-transitive.*

2. Examples of twenty two face-regular polyhedra

Here we present twenty two classically known 3-polytopes (Figs. 1, 2 and 3).

If P is one of the first nineteen of above twenty two polytopes then (i) all the vertices of P are points on a sphere whose centre is same as the centre of P, (ii) each 2-face of P is a regular polygon, and (iii) the cyclic arrangement of faces around each vertex is same (equivalently, the boundary complex ∂P of P is a semi-equivelar map). Without loss of generality, we assume that (iv) the vertices of P are points on the unit 2-sphere $\mathbb{S}^2$ with centre $(0, 0, 0)$. We say a 3-polytope is *face-regular* if it satisfies (ii).

For $n \geq 3$, let P_n be the polytope whose vertex-set is

$$\left\{ (1 + \sin^2 \tfrac{\pi}{n})^{-\frac{1}{2}} \left(\cos \frac{2m\pi}{n}, \sin \frac{2m\pi}{n}, \pm \sin \frac{\pi}{n} \right) : 0 \leq m \leq n - 1 \right\}.$$
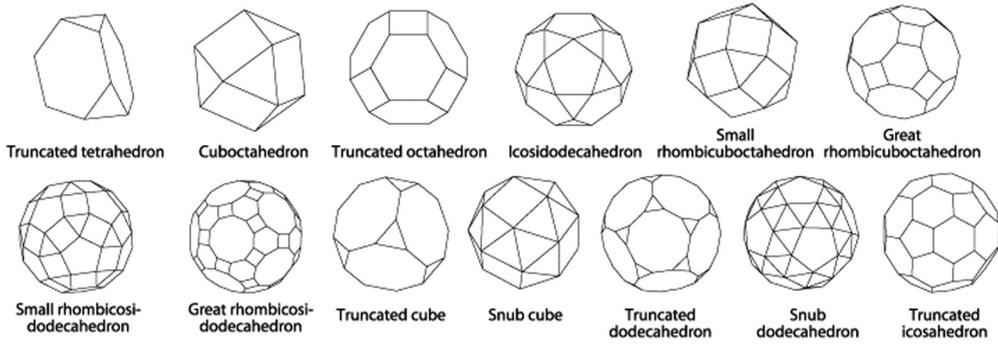
Fig. 2. Archimedean Solids (from [23]).
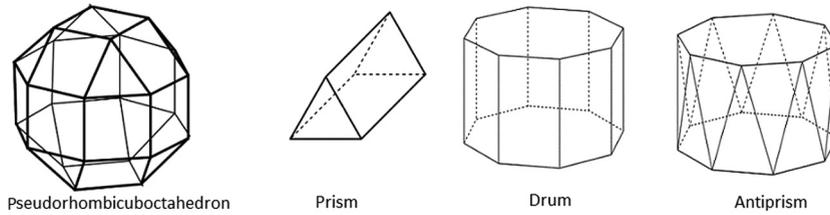
Fig. 3. Pseudorhombicuboctahedron, Prisms P_3 & P_8, Antiprism Q_8 (from [10], [22]).

The polytope P_4 is a cube and, for $n \neq 4$, the boundary of P_n consists of n squares and two regular n-gons. This polytope P_n is called a $2n$-vertex *regular prism* or *drum* or *ladder*. We have chosen the vertices of P_n on the unit sphere $\mathbb{S}^2$ and coordinates are such that P_n satisfies (i), ..., (iv) above.

For $n \geq 3$, let Q_n be the polytope whose vertex-set is

$$\left\{ (\sin^2 \frac{\pi}{n} + \cos^2 \frac{\pi}{2n})^{-\frac{1}{2}} \left(\cos \frac{(2m+1)\pi}{n}, \sin \frac{(2m+1)\pi}{n}, (\sin^2 \frac{\pi}{n} - \sin^2 \frac{\pi}{2n})^{\frac{1}{2}} \right), \right.$$
$$\left. (\sin^2 \frac{\pi}{n} + \cos^2 \frac{\pi}{2n})^{-\frac{1}{2}} \left(\cos \frac{2m\pi}{n}, \sin \frac{2m\pi}{n}, -(\sin^2 \frac{\pi}{n} - \sin^2 \frac{\pi}{2n})^{\frac{1}{2}} \right) : 0 \leq m \leq n-1 \right\}.$$

The polytope Q_3 is an octahedron and, for $n \geq 4$, the boundary of Q_n consists of $2n$ equilateral triangles and two regular n-gons. This polytope Q_n is called a $2n$-vertex *antiprism*. Moreover, with the chosen coordinates of vertices, Q_n satisfies (i), ..., (iv) above.

A *truncation* is an operation that cuts polytope vertices, creating a new face in place of each vertex. A *rectification* or *complete-truncation* is the process of truncating a polytope by marking the midpoints of all its edges, and cutting off its vertices at those points. Thus, a rectification truncates edges to points. For more on truncation see [3, Chapter 8]. In Section 4, we present some combinatorial versions of truncation and rectification of polytopes. Proposition 2.1 gives some relations among the five Platonic solids and the thirteen Archimedean solids in terms of truncation and rectification.

Proposition 2.1 *(Coxeter).* The truncation of tetrahedron (respectively, cube, octahedron, dodecahedron, icosahedron, cuboctahedron and icosidodecahedron) gives the truncated tetrahedron (respectively, truncated cube, truncated octahedron, truncated dodecahedron, truncated icosahedron, great rhombicuboctahedron and great rhombicosidodecahedron). The rectification of cube (respectively, dodecahedron, icosidodecahedron and cuboctahedron) gives the cuboctahedron (respectively, icosidodecahedron, small rhombicosidodecahedron and small rhombicuboctahedron).

Remark 2.2. We all know that a 3-polytope is called *regular* if its automorphism group acts transitively on the set of flags. But, there are confusions about the names 'semi-regular polyhedra', 'uniform polyhedra' and 'Archimedean polyhedra' ([10]). Agreeing with most of the authors in the literature, we write the following. A 3-polytope is *semi-regular* (or *uniform*) if its automorphism group acts transitively on the set of vertices. A 3-polytope is an *Archimedean polyhedron* if it satisfies (ii) and (iii) above ([10,13]). So, a semi-regular polyhedron is an Archimedean polyhedron and the pseudorhombicuboctahedron is an Archimedean polyhedron which is not a semi-regular polyhedron. Observe that all the twenty two polyhedra presented above are Archimedean polyhedra.

3. Classification of vertex-types of semi-equivelar maps on $\mathbb{S}^2$

In this section, we present a combinatorial characterisation of all the types of semi-equivelar maps on $\mathbb{S}^2$ in terms of an inequality. We need this in Section 4 to prove Theorem 1.1.

Let $F_1\text{-}\cdots\text{-}F_m\text{-}F_1$ be the face-cycle of a vertex u in a map. Then $F_i \cap F_j$ is either u or an edge through u. Thus, the face F_i must be of the form $u_{i+1}\text{-}u\text{-}u_i\text{-}P_i\text{-}u_{i+1}$, where $P_i = \emptyset$ or a path and $P_i \cap P_j = \emptyset$ for $i \neq j$. Here addition in the suffix is modulo m. So, $u_1\text{-}P_1\text{-}u_2\text{-}\cdots\text{-}u_m\text{-}P_m\text{-}u_1$ is a cycle and said to be the *link-cycle* of u. For a simplicial complex, $P_i = \emptyset$ for all i, and the link-cycle of a vertex is the link of that vertex.

A face in a map of the form $u_1\text{-}u_2\text{-}\cdots\text{-}u_n\text{-}u_1$ is also denoted by $u_1 u_2 \cdots u_n$. The faces with 3, 4, …, 10 vertices are called *triangle*, *square*, …, *decagon* respectively.

If X is the boundary of a Platonic solid, an Archimedean solid, the pseudorhombicuboctahedron, a regular prism or an antiprism then the vertex-type of X is one of the cycle tuples of the following set.

$$\mathcal{A} := \Big\{ [3^3], [3^4], [4^3], [3^5], [5^3], [3^4, 5^1], [3^4, 4^1], [3^1, 5^1, 3^1, 5^1], [3^1, 4^1, 3^1, 4^1],$$
$$[3^1, 4^1, 5^1, 4^1], [3^1, 4^3], [5^1, 6^2], [4^1, 6^1, 8^1], [4^1, 6^1, 10^1], [4^1, 6^2],$$
$$[3^1, 6^2], [3^1, 8^2], [3^1, 10^2], [3^1, 4^2]\} \cup \{[4^2, r^1], [3^3, s^1], r \geq 5, s \geq 4\} \Big\}. \tag{1}$$

If $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \in \mathcal{A}$ then one readily checks that $\sum_{i=1}^{\ell} \frac{n_i(p_i - 2)}{p_i} < 2$. Here we prove the following converse.

Theorem 3.1. *Let X be a semi-equivelar map of vertex-type $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$ on a 2-manifold. If $\sum_{i=1}^{\ell} \frac{n_i(p_i - 2)}{p_i} < 2$ then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \in \mathcal{A}$.*

We need the following technical lemma of [8] to prove Theorem 3.1.

Lemma 3.2 (Datta & Maity). *If $[p_1^{n_1}, \ldots, p_k^{n_k}]$ satisfies any of the following three properties then $[p_1^{n_1}, \ldots, p_k^{n_k}]$ cannot be the vertex-type of any semi-equivelar map on a surface.*

1. *There exists i such that $n_i = 2$, p_i is odd and $p_j \neq p_i$ for all $j \neq i$.*
2. *There exists i such that $n_i = 1$, p_i is odd, $p_j \neq p_i$ for all $j \neq i$ and $p_{i-1} \neq p_{i+1}$. (Here, addition in the subscripts are modulo k.)*
3. *$[p_1^{n_1}, \ldots, p_k^{n_k}]$ is of the form $[p^1, q^m, p^1, r^n]$, where p, q, r are distinct and p is odd.*

Proof of Theorem 3.1. Let d be the degree of each vertex in X. Consider the k-tuple $(q_1^{m_1}, \ldots, q_k^{m_k})$, where $3 \leq q_1 < \cdots < q_k$ and, for each $i = 1, \ldots, k$, $q_i = p_j$ for some j, $m_i = \sum_{p_j = q_i} n_j$. So, $\sum_{i=1}^{k} m_i = \sum_{j=1}^{\ell} n_j = d$ and $\sum_{i=1}^{k} \frac{m_i}{q_i} = \sum_{j=1}^{\ell} \frac{n_j}{p_j}$. Thus,

$$2 > \sum_{j=1}^{\ell} \frac{n_j(p_j - 2)}{p_j} = \sum_{j=1}^{\ell} n_j - 2\sum_{j=1}^{\ell} \frac{n_j}{p_j} = \sum_{i=1}^{k} m_i - 2\sum_{i=1}^{k} \frac{m_i}{q_i} = d - 2\sum_{i=1}^{k} \frac{m_i}{q_i}. \tag{2}$$

So, $d - 2 < 2\sum_{i=1}^{k} \frac{m_i}{q_i} \leq 2\sum_{i=1}^{k} \frac{m_i}{3} \leq \frac{2d}{3}$. This implies that $3d - 6 < 2d$ and hence $d < 6$. Therefore, $d = 3, 4$ or 5.

Case 1: First assume $d = 5$. If $q_1 \geq 4$ then $\frac{m_1}{q_1} + \cdots + \frac{m_k}{q_k} \leq \frac{d}{q_1} \leq \frac{5}{4}$. Therefore, by (2), $2 > d - 2\sum_{i=1}^{k} \frac{m_i}{q_i} \geq 5 - \frac{10}{4} = \frac{10}{4}$, a contradiction. So, $q_1 = 3$. If $m_1 \leq 3$ then $3 = d - 2 < 2(\frac{m_1}{q_1} + \cdots + \frac{m_k}{q_k}) \leq 2(\frac{m_1}{q_1} + \frac{d - m_1}{q_2}) \leq 2(\frac{m_1}{3} + \frac{5 - m_1}{4}) = \frac{15 + m_1}{6} \leq \frac{15 + 3}{6} = 3$, a contradiction. So, $m_1 \geq 4$. Since $m_1 \leq d = 5$, it follows that $m_1 = 4$ or 5.

1.1: Let $m_1 = 5$. Then, $d = m_1$ and $k = 1$. So, $(q_1^{m_1}, q_2^{m_2}, \ldots, q_k^{m_k}) = (3^5)$ and hence $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^5]$.

1.2: Let $m_1 = 4$. Then $m_2 = 1$. Therefore, $3 = d - 2 < 2\sum_{i=1}^{k} \frac{m_i}{q_i} = 2(\frac{m_1}{q_1} + \frac{m_2}{q_2}) = 2(\frac{4}{3} + \frac{1}{q_2})$. This implies that $1/q_2 > 3/2 - 4/3 = 1/6$ and hence $q_2 < 6$. Since $q_2 > q_1 = 3$, $q_2 = 4$ or 5.

If $q_2 = 5$, then $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^4, 5^1)$ and hence $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^4, 5^1]$. Similarly, if $q_2 = 4$ then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^4, 4^1]$.

Case 2: Now, assume $d = 4$. Then, $1 = \frac{d}{2} - 1 < \sum_{i=1}^{k} \frac{m_i}{q_i} \leq \frac{d}{q_1} = \frac{4}{q_1}$. So, $q_1 < 4$ and hence $q_1 = 3$.

2.1: If $m_1 = d = 4$, then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^4]$.

2.2: If $m_1 = 3$, then $m_2 = 1$. So, $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^3, q_2^1)$. This implies that $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^3, s^1]$ for some $s \geq 4$.

2.3: If $m_1 = 2$, then $1 = \frac{d}{2} - 1 < \sum_{i=1}^{k} \frac{m_i}{q_i} = \frac{2}{3} + \frac{m_2}{q_2} + \frac{m_3}{q_3} \leq \frac{2}{3} + \frac{2}{q_2}$. So, $\frac{2}{q_2} > \frac{1}{3}$ and hence $q_2 < 6$. Thus, $q_2 = 5$ or 4.

2.3.1: If $q_2 = 5$, then $1 = \frac{d}{2} - 1 < \frac{2}{3} + \frac{m_2}{5} + \frac{m_3}{q_3}$ and hence $\frac{m_2}{5} + \frac{m_3}{q_3} > \frac{1}{3}$, where $m_2 + m_3 = d - m_1 = 2$ and $m_2 \geq 1$. These imply that $q_3 \leq 7$. If $q_3 = 7$ then $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^2, 5^1, 7^1)$. This implies that $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^2, 5^1, 7^1]$ or $[3^1, 5^1, 3^1, 7^1]$. But $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^2, 5^1, 7^1]$ is not possible by Lemma 3.2 (i) and $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 5^1, 3^1, 7^1]$ is not possible by Lemma 3.2 (iii). So, $q_3 \neq 7$. If $q_3 = 6$, then $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^2, 5^1, 6^1)$. Again, by Lemma 3.2 (i) and (iii), $[3^2, 5^1, 6^1]$ and $[3^1, 5^1, 3^1, 6^1]$ are not vertex-types of any maps. So, $q_3 \neq 6$. Since $q_3 > q_2 = 5$, it follows that $m_2 = 2$ (and $q_2 = 5$). Then $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^2, 5^2)$. By Lemma 3.2 (i), $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \neq [3^2, 5^2]$. Therefore, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 5^1, 3^1, 5^1]$.

2.3.2: If $q_2 = 4$, then $1 = \frac{d}{2} - 1 < \frac{2}{3} + \frac{m_2}{4} + \frac{m_3}{q_3}$ and hence $\frac{m_2}{4} + \frac{m_3}{q_3} > \frac{1}{3}$. If $m_2 = 1$ then $m_3 = 1$. So, $\frac{1}{4} + \frac{1}{q_3} > \frac{1}{3}$ and hence $4 < q_3 < 12$. Therefore, $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^2, 4^1, q_3^1)$ and hence $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^2, 4^1, q_3^1]$, where $q_3 > 4$. But this is not possible by Lemma 3.2 (i). So, $m_2 \neq 1$ and hence $m_2 = 2$. Then $(q_1^{m_1}, \ldots, q_k^{m_k}) = (3^2, 4^2)$. By Lemma 3.2 (i), $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \neq [3^2, 4^2]$. Therefore, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 4^1, 3^1, 4^1]$.

2.4: Let $m_1 = 1$. Then, $1 = \frac{d}{2} - 1 < \frac{1}{3} + \frac{m_2}{q_2} + \frac{m_3}{q_3} + \frac{m_4}{q_4}$, where $m_2 + m_3 + m_4 = 3$ and $4 \leq q_2 < q_3 < q_4$. These imply that $q_2 = 4$. If $m_2 = 1$ then $1 < \frac{1}{3} + \frac{1}{4} + \frac{m_3}{q_3} + \frac{m_4}{q_4} \leq \frac{7}{12} + \frac{2}{q_3}$. So, $\frac{2}{q_3} > \frac{5}{12}$ and hence $q_3 \leq 4 = q_2$, a contradiction. Thus, $m_2 \geq 2$ and hence $m_2 = 2$ or 3.

If $m_2 = 2$, then $m_3 = 1$. So, $1 = \frac{d}{2} - 1 < \frac{1}{3} + \frac{2}{4} + \frac{1}{q_3}$ and hence $\frac{1}{q_3} > 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$. Therefore, $q_3 < 6$ and hence $q_3 = 5$. Then, $(q_1^{m_1}, q_2^{m_2}, \ldots, q_k^{m_k}) = (3^1, 4^2, 5^1)$. By Lemma 3.2 (ii), $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \neq [3^1, 4^2, 5^1]$. Therefore, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 4^1, 5^1, 4^1]$.

If $m_2 = 3$, then $(q_1^{m_1}, q_2^{m_2}, \ldots, q_k^{m_k}) = (3^1, 4^3)$ and hence $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 4^3]$.

Case 3: Finally, assume $d = 3$. Then, $\frac{1}{2} = \frac{d}{2} - 1 < \frac{m_1}{q_1} + \frac{m_2}{q_2} + \frac{m_3}{q_3}$, where $m_1 + m_2 + m_3 = 3$ and $3 \leq q_2 < q_3 < q_4$. This implies that $q_1 < 6$ and hence $q_1 = 3, 4$ or 5.

3.1: Let $q_1 = 5$. Now, $m_1 = 1, 2$ or 3. If $m_1 = 2$ then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [5^2, q_2^1]$, where $q_2 > 5$. This is not possible by Lemma 3.2 (i). So, $m_1 = 1$ or 3.

3.1.1: If $m_1 = 1$, then $\frac{1}{2} < \frac{1}{5} + \frac{m_2}{q_2} + \frac{m_3}{q_3}$. So, $\frac{m_2}{q_2} + \frac{m_3}{q_3} > \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$, where $m_2 + m_3 = 2$ and $5 = q_1 < q_2 < q_3$. These imply that $q_2 = 6$. If $m_2 = 1$ then $m_3 = 3 - m_1 - m_2 = 1$ and hence $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [5^1, 6^1, q_3^1]$, where $q_3 \geq 7$. But, this is not possible by Lemma 3.2 (ii). Thus, $m_2 = 2$. Then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [5^1, 6^2]$.

3.1.2: If $m_1 = 3$, then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [5^3]$.

3.2: Let $q_1 = 4$. Since $d = 3$, $(m_1, \ldots, m_k) = (1, 1, 1), (1, 2), (2, 1)$ or (3).

3.2.1: If $(m_1, \ldots, m_k) = (1, 1, 1)$, then $\frac{1}{2} < \frac{1}{4} + \frac{1}{q_2} + \frac{1}{q_3}$. So, $\frac{1}{q_2} + \frac{1}{q_3} > \frac{1}{4}$. Since $q_2 < q_3$, it follows that $q_2 < 8$. If $q_2 = 5$ then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^1, 5^1, q_3^1]$, $q_3 > 5$. This is not possible by Lemma 3.2 (ii). So, $q_2 \neq 5$. Similarly, $q_2 \neq 7$. Thus, $q_2 = 6$. Then $\frac{1}{q_3} > \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ and hence $q_3 < 12$. Then, by the same argument, $q_3 \neq 9, 11$. So, $q_3 = 8$ or 10. Therefore, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^1, 6^1, 8^1]$ or $[4^1, 6^1, 10^1]$.

3.2.2: If $(m_1, \ldots, m_k) = (1, 2)$, then $\frac{1}{2} < \frac{1}{4} + \frac{2}{q_2}$ and hence $4 = q_1 < q_2 < 8$. Thus, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^1, q_2^2]$, $5 \leq q_2 \leq 7$. By Lemma 3.2 (i), $q_2 \neq 5$ or 7. So, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^1, 6^2]$.

3.2.3: If $(m_1, \ldots, m_k) = (2, 1)$, then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^2, q_2^1]$ for some $q_2 \geq 5$.

3.2.4: If $(m_1, \ldots, m_k) = (3)$, then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^3]$.

3.3: Let $q_1 = 3$. By Lemma 3.2, $(m_1, \ldots, m_k) = (3)$ or $(1, 2)$.

3.3.1: If $(m_1, \ldots, m_k) = (3)$, then $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [q_1^3] = [3^3]$.

3.3.2: If $(m_1, \ldots, m_k) = (1, 2)$, then $\frac{1}{2} < \frac{1}{3} + \frac{2}{q_2}$. So, $q_2 < 12$. Again, by Lemma 3.2 (i), q_2 is not odd. So, $q_2 = 4, 6, 8$ or 10. Therefore, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^1, 4^2], [3^1, 6^2], [3^1, 8^2]$ or $[3^1, 10^2]$. This completes the proof. □

As a consequence we prove

Corollary 3.3. *Suppose there exists an n-vertex semi-equivelar spherical map of vertex-type* $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$. *Then* $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (4, [3^3]), (6, [3^4]), (8, [4^3]), (12, [3^5]), (20, [5^3]),$ $(60, [3^4, 5^1]), (24, [3^4, 4^1]), (30, [3^1, 5^1, 3^1, 5^1]), (12, [3^1, 4^1, 3^1, 4^1]),$ $(60, [3^1, 4^1, 5^1, 4^1]), (24, [3^1, 4^3]), (60, [5^1, 6^2]), (48, [4^1, 6^1, 8^1]), (120, [4^1, 6^1, 10^1]), (24, [4^1, 6^2]), (12, [3^1, 6^2]), (24, [3^1, 8^2]),$ $(60, [3^1, 10^2]), (6, [3^1, 4^2]), (2r, [4^2, r^1])$ *for some* $r \geq 5$ *or* $(2s, [3^3, s^1])$ *for some* $s \geq 4$.

Proof. Let X be an n-vertex semi-equivelar map on $\mathbb{S}^2$ of vertex-type $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$. Let f_1, f_2 be the number of edges and faces of X respectively. Let d be the degree of each vertex. So, $f_1 = (nd)/2$. Consider the k-tuple $(q_1^{m_1}, \ldots, q_k^{m_k})$, where $3 \leq q_1 < \cdots < q_k$ and, for each $i = 1, \ldots, k$, $q_i = p_j$ for some j, $m_i = \sum_{p_j = q_i} n_j$. So, $\sum_i m_i = \sum_j n_j = d$ and $\sum_i \frac{m_i}{q_i} = \sum_j \frac{n_j}{p_j}$. A two-way counting the number of ordered pairs (F, v), where F is a q_i-gons in X and v is a vertex of F, we get: (the

number of q_i-gons) $\times q_i = n \times m_i$. This implies that $f_2 = n \times \sum_{i=1}^{k} \frac{m_i}{q_i} = n \times \sum_{j=1}^{\ell} \frac{n_j}{p_j}$. Since the Euler characteristic of $\mathbb{S}^2$ is 2, we get

$$2 = n - f_1 + f_2 = n \times (1 - \frac{1}{2} \sum_{j=1}^{\ell} n_j + \sum_{j=1}^{\ell} \frac{n_j}{p_j}) = \frac{n}{2} \times (2 - \sum_{j=1}^{\ell} \frac{n_j(p_j - 2)}{p_j}). \tag{3}$$

Thus,

$$n = 4 \left(2 - \sum_{j=1}^{\ell} \frac{n_j(p_j - 2)}{p_j} \right)^{-1}. \tag{4}$$

From (3), we get $\sum_{j=1}^{\ell} \frac{n_j(p_j-2)}{p_j} = 2 - \frac{4}{n} < 2$. Therefore, by Theorem 3.1, $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] \in \mathcal{A}$. The result now follows from (4) and the set $\mathcal{A}$ given in (1). (For example, if $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [3^4, 5^1]$ then $n = 4(2 - (\frac{4(3-2)}{3} + \frac{1(5-2)}{5}))^{-1} = 60$. So, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (60, [3^4, 5^1])$.) $\square$

4. Proofs of main results

In this section, we present proofs of Theorems 1.1, 1.2 and Corollaries 1.3, 1.4, 1.5

Lemma 4.1. *Let K be a semi-equivelar spherical map. If the number of vertices and the vertex-type of K are same as those of the boundary ∂P of a Platonic solid P then $K \cong \partial P$.*

Proof. If P is the tetrahedron, then K is a 4-vertex triangulation of $\mathbb{S}^2$ and it is trivial to see that K is unique up to isomorphism.

If P is the octahedron, then K is a 6-vertex triangulation of $\mathbb{S}^2$ and degree of each vertex is 4. It is easy to see that K is unique up to an isomorphism (cf. [4]). This also implies that the dual map of K which has 8 vertices and is of vertex-type $[4^3]$ is unique up to an isomorphism. Hence an 8-vertex map of vertex-type $[4^3]$ on $\mathbb{S}^2$ is isomorphic to the boundary of the cube.

If P is the icosahedron, then K is a 12-vertex triangulation of $\mathbb{S}^2$ and degree of each vertex is 5. We know that it is unique up to an isomorphism (cf. [18, Table 8], [21, Lemma 1]). This also implies that the dual map of K which has 20 vertices and is of vertex-type $[5^3]$ is unique up to an isomorphism. Hence a 20-vertex map of vertex-type $[5^3]$ on $\mathbb{S}^2$ is isomorphic to the boundary of the dodecahedron. $\square$

We need Lemmas 4.2, 4.4 and 4.6 to prove Lemma 4.8.

Lemma 4.2. *Let X be a semi-equivelar spherical map of vertex-type $[p^1, q^2]$, where $q \geq 6$. If α, β are two p-gonal faces of X then there exists at most one edge from α to β.*

Proof. Since the degree of each vertex is 3, for each p-gonal face α of X and $u \in \alpha$, there exists a unique edge of the form uv where $v \notin \alpha$ and v is in another p-gon. Consider the graph G whose nodes are p-gonal faces of X. Two such nodes α, β form a link in G if there exists an edge uv in X, where $u \in \alpha$ and $v \in \beta$. It is sufficient to show that G is simple.

Suppose X has n vertices. Then, by Corollary 3.3, $(n, [p^1, q^2]) = (12, [3^1, 6^2]), (24, [3^1, 8^2]), (60, [3^1, 10^2]), (24, [4^1, 6^2])$ or $(60, [5^1, 6^2])$. Since X is a polyhedral map, G has no loop. Suppose there is a pair of links between two nodes α and β of G. Then there exist $u, v \in \alpha$ and $x, y \in \beta$ such that $e = ux$, $f = vy$ are edges in X. Suppose uv is an edge (of X). Let γ be the q-gonal face containing uv. Since the degree of each vertex is 3, it follows that ux & vy are edges of γ and hence xy is a diagonal of γ. This is not possible since $x, y \in \beta$. So, uv is not an edge. Similarly, xy is a non-edge. Hence $p > 3$.

Let $p = 5$. Then $n = 60$ and $q = 6$. Since two p-gons in X are disjoint, it follows that G has 12 nodes. By above uv, xy are non-edges, and there is no more edges between α & β. Then α, β and the edges e, f subdivide $\mathbb{S}^2$ into two disks C, D (and interiors of α & β).

Claim. Number of pentagons inside D is at least 6.

Let $\mathcal{P}$ be the set of pentagons inside D. The boundary of D contains u, v, x, y and m (say) vertices, where $2 \leq m \leq 4$. For each of the m vertices, w, there exists an edge wz and a pentagon γ containing z. Then γ is inside D. So, $\mathcal{P} \neq \emptyset$. Let the number of pentagons in $\mathcal{P}$ be ℓ. Then, the number of links between ℓ nodes is $(5\ell - m)/2$. Since the number of links between any two nodes is at most 2, $\frac{5\ell - m}{2} \leq 2\binom{\ell}{2}$. Thus, $2\ell(\ell - 1) \geq 5\ell - m \geq 5\ell - 4$ and hence $2\ell^2 - 7\ell + 4 \geq 0$. This implies that $\ell \geq 3$. First consider the case where the induced subgraph $G[\mathcal{P}]$ is simple. Then, $\frac{5\ell - m}{2} \leq \binom{\ell}{2}$. Thus, $\ell(\ell - 1) \geq 5\ell - m \geq 5\ell - 4$ and hence $\ell^2 - 6\ell + 4 \geq 0$. This implies that $\ell \geq 6$ and we are done.

Now, assume that $G[\mathcal{P}]$ is not simple. Suppose there is a pair of links between two nodes α_1 and β_1 of $G[\mathcal{P}]$. Then, as before, there exist edges $e_1 = u_1 x_1$, $f_1 = v_1 y_1$, where $u_1, v_1 \in \alpha_1$, $x_1, y_1 \in \beta_1$, such that α_1, β_1, e_1 and f_1 subdivide $\mathbb{S}^2$ into two disks C_1, D_1 (and interiors of α_1 & β_1). Then α, β are inside one of C_1, D_1. Assume, without loss of generality, that α, β are inside C_1. Then D_1 is inside D. Let $\mathcal{P}_1$ be the set of pentagons inside D_1 and $\ell_1 = \#(\mathcal{P}_1)$. Again, if $G[\mathcal{P}_1]$ is simple then, by the same arguments as above, $\ell_1 \geq 6$. This implies that the number of pentagons inside D is at least 8 and we are done. So, assume that $G[\mathcal{P}_1]$ is not simple. Then, repeating the same arguments, we get a pair of links between two nodes α_2 & β_2 in $G[\mathcal{P}_1]$ and a disk D_2 bounded by α_2, β_2 and two edges of the form $e_2 = u_2 x_2$, $f_2 = v_2 y_2$, where $u_2, v_2 \in \alpha_2$, $x_2, y_2 \in \beta_2$, inside D_1. Let ℓ_2 be the number of pentagons inside D_2. Then, as before, $\ell_2 \geq 3$ and hence the number of pentagons inside D is at least $\#(\{\alpha_1, \beta_1, \alpha_2, \beta_2\}) + 3 = 7$. This proves the claim.

Similarly, the number of pentagons inside C is at least 6. Therefore, the number of pentagons in X is $\geq 6 + 6 + 2 = 14$, a contradiction. Thus, $p \neq 5$. By similar arguments $p \neq 4$. This is a contradiction. So, G is a simple graph. This completes the proof. □

We now present some combinatorial versions of truncation and rectification of polytopes.

Definition 4.3. Let P be a 3-polytope and TP be the truncation of P. Let $X \cong \partial P$ and $V(X) = \{u_1, \ldots, u_n\}$. Without loss of generality, we identify X with ∂P. Consider a new set (of nodes) $V := \{v_{ij} : u_i u_j \text{ is an edge of } X\}$. So, if $v_{ij} \in V$ then $i \neq j$ and v_{ji} is also in V. Let $E := \{v_{ij} v_{ji} : v_{ij} \in V\} \sqcup \{v_{ij} v_{ik} : u_j, u_k \text{ are in a face containing } u_i, 1 \leq i \leq n\}$. Then (V, E) is a graph on X. Clearly, from the construction, $(V, E) \cong$ the edge graph of TP. Thus, (V, E) gives a map $T(X)$ on $\mathbb{S}^2$. This map $T(X)$ is said to be the truncation of X.

From Definition 4.3 & the (geometric) construction of truncation of polytopes we get

Lemma 4.4. *Let X, $T(X)$ and TP be as in Definition 4.3. Then, $T(X)$ is isomorphic to the boundary of TP. Moreover, if X is semi-equivelar of vertex-type $[q^p]$ (resp., $[p^1, q^1, p^1, q^1]$), then $T(X)$ is also semi-equivelar and of vertex-type $[p^1, (2q)^2]$ (resp., $[4^1, (2p)^1, (2q)^1]$).*

Proof. Let $P, V(X), V, E$ be as in Definition 4.3. Then, from the definition of truncated polytope and the construction in Definition 4.3, $T(X) \cong \partial(TP)$.

Let X be semi-equivelar of $[q^p]$. From the construction in Definition 4.3, the set of faces of $T(X)$ is $\{\tilde{\alpha} = v_{i_1 i_2}\text{-}v_{i_2 i_1}\text{-}v_{i_2 i_3}\text{-}v_{i_3 i_2}\text{-}v_{i_3 i_4}\text{-}\cdots\text{-}v_{i_q i_1}\text{-}v_{i_1 i_q}\text{-}v_{i_1 i_2} : \alpha = u_{i_1}\text{-}u_{i_2}\text{-}\cdots\text{-}u_{i_q}\text{-}u_{i_1} \text{ is a face of } X\} \sqcup \{\tilde{u}_i = v_{i j_1}\text{-}v_{i j_2}\text{-}\cdots\text{-}v_{i j_q}\text{-}v_{i j_1} : u_{j_1}\text{-}P_1\text{-}u_{j_2}\text{-}P_2\text{-}u_{j_3}\text{-}\cdots\text{-}P_q\text{-}u_{j_1} \text{ is the link-cycle of } u_i \in V(X)\}$. Thus, the faces incident to the vertex v_{ij} are the p-gonal face $\tilde{u}_i$ and the two $2q$-gonal faces $\tilde{\alpha}$ & $\tilde{\beta}$ where α & β are faces in X containing the edge $u_i u_j$. Observe that the face-cycle of v_{ij} is $\tilde{u}_i\text{-}\tilde{\alpha}\text{-}\tilde{\beta}\text{-}\tilde{u}_i$. Thus, $T(X)$ is semi-equivelar and the vertex-type is $[p^1, (2q)^2]$.

Let the vertex-type of X be $[p^1, q^1, p^1, q^1]$. From Definition 4.3, the set of faces of $T(X)$ is $\{\tilde{\alpha} := v_{i_1 i_2}\text{-}v_{i_2 i_1}\text{-}v_{i_2 i_3}\text{-}v_{i_3 i_2}\text{-}v_{i_3 i_4}\text{-}\cdots\text{-}v_{i_r i_1}\text{-}v_{i_1 i_r}\text{-}v_{i_1 i_2} : \alpha = u_{i_1}\text{-}\cdots\text{-}u_{i_r}\text{-}u_{i_1} \text{ is a } r\text{-gonal face of } X, \ r = p, q\} \sqcup \{\tilde{u}_i := v_{i t_1}\text{-}v_{i t_2}\text{-}v_{i t_3}\text{-}v_{i t_4}\text{-}v_{i t_1} : u_{t_1}\text{-}P_1\text{-}u_{t_2}\text{-}P_2\text{-}u_{t_3}\text{-}P_3\text{-}u_{t_4}\text{-}P_4\text{-}u_{t_1} \text{ is the link-cycle of } u_i \in V(X)\}$. Thus, the faces incident to the vertex v_{ij} are the square $\tilde{u}_i$, the $2p$-gonal face $\tilde{\alpha}$ and the $2q$-gonal face $\tilde{\beta}$ where α is a p-gonal face and β is a q-gonal face in X containing the edge $u_i u_j$. Observe that the face-cycle of v_{ij} is $\tilde{u}_i\text{-}\tilde{\alpha}\text{-}\tilde{\beta}\text{-}\tilde{u}_i$. Thus, $T(X)$ is semi-equivelar and the vertex-type is $[4^1, (2p)^1, (2q)^1]$. □

Definition 4.5. Let P be a polytope and RP be the rectification of P. Let $X \cong \partial P$ and $V(X) = \{u_1, \ldots, u_n\}$. Without loss of generality, we identify X with ∂P. Consider the graph (V, E), where V is the edge set $E(X)$ of X and $E := \{ef : e, f \text{ are two adjacent edges in a face of } X\}$. Then (V, E) is a graph on X. From the definition of rectification, it follows that $(V, E) \cong$ the edge graph of RP. Thus, (V, E) gives a map say $R(X)$ on $\mathbb{S}^2$, which is said to be the rectification of X.

From Definition 4.5 & the (geometric) construction of rectification of polytopes we get

Lemma 4.6. *Let X, $R(X)$ and RP be as in Definition 4.5. Then, $R(X)$ is isomorphic to the boundary of RP. Moreover, if X is semi-equivelar of vertex-type $[q^p]$ (respectively, $[p^1, q^1, p^1, q^1]$), then $R(X)$ is also semi-equivelar and of vertex-type $[p^1, q^1, p^1, q^1]$ (respectively, $[4^1, p^1, 4^1, q^1]$).*

Proof. Let $P, V(X), V, E$ be as in Definition 4.5. Then, from the definition of rectified polytope and the construction in Definition 4.5, $R(X) \cong \partial(RP)$.

Let X be semi-equivelar of vertex-type $[q^p]$. From the construction in Definition 4.5, the set of faces of $R(X)$ is $\{\tilde{\alpha} := e_1\text{-}e_2\text{-}\cdots\text{-}e_q\text{-}e_1 : \alpha = (e_1 \cap e_2)\text{-}\cdots\text{-}(e_{q-1} \cap e_q)\text{-}(e_q \cap e_1)\text{-}(e_1 \cap e_2) \text{ is a face of } X\} \sqcup \{\tilde{u}_i := \epsilon_1\text{-}\cdots\text{-}\epsilon_p\text{-}\epsilon_1 : u_{j_1}\text{-}P_1\text{-}u_{j_2}\text{-}P_2\text{-}u_{j_3}\text{-}\cdots\text{-}P_p\text{-}u_{j_1} \text{ is the link-cycle of } u_i \in V(X) \text{ and } \epsilon_\ell = u_i u_{j_\ell}, 1 \leq \ell \leq p\}$. The faces incident to the vertex $e = u_i u_j$ are the two squares $\tilde{u}_i, \tilde{u}_j$ and the two q-gonal faces $\tilde{\alpha}, \tilde{\beta}$ where α, β are faces in X containing the edge $u_i u_j$. Observe that the face-cycle of e ($= u_i u_j = \alpha \cap \beta$) is $\tilde{u}_i\text{-}\tilde{\alpha}\text{-}\tilde{u}_j\text{-}\tilde{\beta}\text{-}\tilde{u}_i$. Thus, $R(X)$ is semi-equivelar and the vertex-type is $[p^1, q^1, p^1, q^1]$.
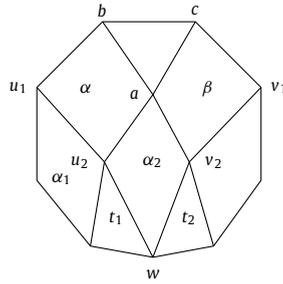
Fig. 4. Part of the map X when $(s_2, s_3, s_4) = (6, 12, 0)$.
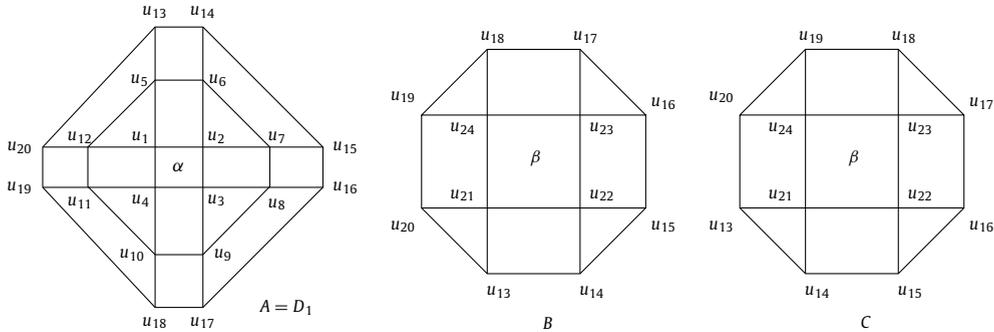
Fig. 5. A is a part of X with 13 squares. B and C are complements of A in X.

Let X be semi-equivelar of vertex-type $[p^1, q^1, p^1, q^1]$. From Definition 4.5, the set of faces of $R(X)$ is $\{\tilde{\alpha} := e_1\text{-}\cdots\text{-}e_r\text{-}e_1 \colon \alpha = (e_1 \cap e_2)\text{--}\cdots\text{--}(e_{r-1} \cap e_r)\text{-}(e_r \cap e_1)\text{-}(e_1 \cap e_2)$ is a r-gonal face of X, $r = p, q\} \sqcup \{\tilde{u}_i = e_1\text{-}e_2\text{-}e_3\text{-}e_4\text{-}e_1 \colon u_{j_1}\text{-}P_1\text{-} u_{j_2}\text{-}P_2\text{-}u_{j_3}\text{-}P_3\text{-}u_{j_4}\text{-}P_4\text{-}u_{j_1}$ is the link-cycle of $u_i \in V(X)$ and $e_\ell = u_i u_{j_\ell}$, $1 \le \ell \le 4\}$. The faces incident to the vertex $e = u_i u_j$ are the two squares $\tilde{u}_i$ & $\tilde{u}_j$, the p-gonal face $\tilde{\alpha}$ and the q-gonal face $\tilde{\beta}$, where α is a p-gonal face and β is a q-gonal face in X containing the edge $u_i u_j$. Observe that the face-cycle of e is $\tilde{u}_i$-$\tilde{\alpha}$-$\tilde{u}_j$-$\tilde{\beta}$-$\tilde{u}_i$. Thus, $R(X)$ is semi-equivelar and the vertex-type is $[4^1, p^1, 4^1, q^1]$. $\square$

Lemma 4.7. *(a) If X is a 24-vertex semi-equivelar spherical map of vertex-type $[3^1, 4^3]$ then X is isomorphic to either the boundary of small rhombicuboctahedron or the boundary of the pseudorhombicuboctahedron. (b) The boundaries of the small rhombicuboctahedron and the pseudorhombicuboctahedron are non-isomorphic.*

Proof. Since X is a 24-vertex map of vertex-type $[3^1, 4^3]$, it follows that X has 8 triangles and 18 squares. Since each vertex of X is in one triangle, no two triangles intersect. Thus, a square meets at most two triangles on edges and hence meets at least two other squares on edges. For $2 \le i \le 4$, a square α in X is said to be of type i if α intersects i other squares on edges. Let s_i be the number of squares in X of type i. Consider the set $A := \{(\beta, t) \colon \beta$ a square of X, t a triangle of X, $\beta \cap t$ is an edge$\}$. Two way counting the cardinality of A gives $2 \times s_2 + 1 \times s_3 + 0 \times s_4 = 8 \times 3$. So, $2s_2 + s_3 = 24$. Since X has 18 squares, $s_2 + s_3 + s_4 = 18$. These give $(s_2, s_3, s_4) = (6, 12, 0), (7, 10, 1), (8, 8, 2), (9, 6, 3), (10, 4, 4), (11, 2, 5)$ or $(12, 0, 6)$.

Claim. There exists a square in X of type 4.

If the claim is not true, then $(s_2, s_3, s_4) = (6, 12, 0)$. Thus, there are 12 squares of type 3. Each of these 12 type 3 squares meets one triangle on an edge. Since there are 8 triangles, it follows that there is a triangle abc which meets two type 3 squares on edges. Assume, with out loss of generality, that $\alpha := abu_1u_2$ and $\beta := acv_1v_2$ are type 3 squares (see Fig. 4). Since abu_1u_2 is type 3, u_1u_2 is in 2 squares, say in α & α_1. Similarly au_2 is in two squares. Clearly, they are α & $\alpha_2 := au_2wv_2$ for some vertex w. Then $\alpha, \alpha_1, \alpha_2$ are three squares incident to u_2 and hence the 4th face incident to u_2 is a triangle, say t_1, containing u_2w. Similarly, there exists a triangle t_2 containing v_2w. Since $\alpha_2 = au_2wv_2$ is a face, it follows that $u_2 \ne v_2$ and u_2v_2 is a non-edge. These imply that $t_1 \ne t_2$ are two triangles containing w, a contradiction to the fact that each vertex is in one triangle. This proves the claim.

Let $\alpha = u_1u_2u_3u_4$ be a type 4 square. Let the four squares which intersect α on edges be $u_1u_2u_6u_5, u_2u_3u_8u_7, u_3u_4u_{10}u_9$ & $u_1u_4u_{11}u_{12}$ (see Fig. 5). Then $u_2u_6u_7, u_3u_8u_9, u_4u_{10}u_{11}$ and $u_1u_5u_{12}$ must be triangles. If $u_i = u_j$ for $5 \le i \ne j \le 12$ then u_i is in two triangles, a contradiction. Thus, $u_1, \dots, u_{12}$ are distinct vertices. For $5 \le i \le 12$, u_i is in two more squares. This gives us 8 squares of the form $F_1 = u_5u_6u_{14}u_{13}$, $F_2 = u_6u_7u_{15}u_{14}$, $F_3 = u_7u_8u_{16}u_{15}$, $F_4 = u_8u_9u_{17}u_{16}$, $F_5 = u_9u_{10}u_{18}u_{17}$, $F_6 = u_{10}u_{11}u_{19}u_{18}$, $F_7 = u_{11}u_{12}u_{20}u_{19}$, $F_8 = u_{12}u_5u_{13}u_{20}$. Suppose $F_i = F_j$ for some $i \ne j$. Assume, without loss of generality, that $F_1 = F_j$ for some $j \ne 1$. Clearly, $j \ne 2, 3, 7, 8$. If $F_1 = F_4$ then, since $u_5, \dots, u_{12}$ are distinct, $u_5 = u_{16}$ or u_{17}. In either case, the number of edges containing u_5 would be more than 4, a contradiction. Thus, $j \ne 4$. By similar arguments, $j \ne 5, 6$. Thus, $F_1, \dots, F_8$ are distinct.

If $u_k = u_\ell$ for $13 \le k \ne \ell \le 20$, then u_k would be in 4 squares, a contradiction. This implies that $u_1, \ldots, u_{20}$ are distinct vertices and we have a disk D_1 with these 20 vertices, 13 squares and 4 triangles. Then the remaining 5 squares and 4 triangles give a disk D_2 such that $\partial D_1 = \partial D_2$. Since each vertex on ∂D_1 is in 2 squares of D_1, it follows that each vertex on ∂D_2 is in one triangle and one square of D_2. Therefore, the 4 remaining vertices of X are inside D_2 and form a square, say $\beta := u_{21}u_{22}u_{23}u_{24}$. Then 4 remaining squares in D_2 have one edge common with ∂D_2 and one each with β. Since $\partial D_1 = \partial D_2$ and each internal vertex is in 3 squares, it follows that, up to isomorphism, D_2 is B or C. If $D_2 = B$ then X is isomorphic to small rhombicuboctahedron and if $D_2 = C$ then X is isomorphic to the boundary of the pseudorhombicuboctahedron. This proves part (a).

Observe that $(s_2, s_3, s_4) = (12, 0, 6)$ for the small rhombicuboctahedron and $(s_2, s_3, s_4) = (8, 8, 2)$ for the pseudorhombicuboctahedron. Thus, their boundaries are non-isomorphic. This proves part (b). $\square$

Lemma 4.8. *Let K be a semi-equivelar spherical map. Suppose the vertex-type of K is not $[3^1, 4^3]$, $[3^4, 4^1]$ or $[3^4, 5^1]$. If the number of vertices and vertex-type of K are same as those of the boundary ∂P of an Archimedean solid P then $K \cong \partial P$.*

Proof. Let K be an n-vertex map of vertex-type $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$. Then, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (12, [3^1, 6^2])$, $(12, [3^1, 4^1, 3^1, 4^1])$, $(24, [3^1, 8^2])$, $(30, [3^1, 5^1, 3^1, 5^1])$, $(60, [3^1, 4^1, 5^1, 4^1])$, $(60, [5^1, 6^2])$, $(48, [4^1, 6^1, 8^1])$, $(120, [4^1, 6^1, 10^1])$, $(24, [4^1, 6^2])$ or $(60, [3^1, 10^2])$.

If $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (60, [5^1, 6^2])$, then K has twelve pentagons which partition the vertex set. Consider the graph G whose nodes are the pentagons of K. Two nodes are joined by a link in G if the corresponding faces are joined by an edge of K. (Observe that this G is same as in the proof of Lemma 4.2.) By the proof of Lemma 4.2, G is a simple 5-regular graph and can be drawn on $\mathbb{S}^2$. This gives a map $[K]$ whose faces are triangles. (Each hexagon of K gives a 3-cycle.) Since $[K]$ has 12 vertices, by Lemma 4.1, $[K]$ is the boundary of the icosahedron. By Lemma 4.4, $T([K])$ is the boundary of the truncated icosahedron. Clearly, the operations $X \mapsto [X]$ and $Y \mapsto T(Y)$ are inverse operations. Thus, K is isomorphic to $T([K])$. Therefore, K is isomorphic to the boundary of the truncated icosahedron. Similarly, by Lemmas 4.1, 4.2 & 4.4, the maps of vertex-types $[3^1, 6^2]$, $[3^1, 8^2]$, $[3^1, 10^2]$ and $[4^1, 6^2]$ are isomorphic to the boundaries of the truncated tetrahedron, the truncated cube, the truncated dodecahedron and the truncated octahedron respectively.

If $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (12, [3^1, 4^1, 3^1, 4^1])$, then K consists of 8 triangles and 6 squares. Consider the graph G_1 whose nodes are the triangles of K. Two nodes z_i, z_j are joined by a link $z_i z_j$ in G_1 if $z_i \cap z_j \ne \emptyset$. Since the intersecting triangles in K meet on a vertex and through each vertex there are exactly two triangles, it follows that G_1 is a simple 3-regular graph. Since K is a map on $\mathbb{S}^2$, G_1 can be drawn on $\mathbb{S}^2$. Thus, G_1 gives a map $\overline{K}$ on $\mathbb{S}^2$ with 8 vertices. The four triangles intersecting a square of K form a face of $\overline{K}$. So, $\overline{K}$ is of vertex-type $[4^3]$. Thus, $\overline{K}$ is the boundary of the cube. Hence, by Lemma 4.6 and Proposition 2.1, $R(\overline{K})$ is the boundary of cuboctahedron. Clearly, the operations $X \mapsto \overline{X}$ & $Y \mapsto R(Y)$ are inverse operations. Thus, K is isomorphic to $R(\overline{K})$. So, K is isomorphic to the boundary of cuboctahedron. Similarly, a 30-vertex map of vertex-type $[3^1, 5^1, 3^1, 5^1]$ is isomorphic to the boundary of icosidodecahedron.

If $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (60, [3^1, 4^1, 5^1, 4^1])$, then K consists of 20 triangles, 30 squares and 12 pentagons. Consider the graph G_2 whose nodes are the squares of K. Two nodes x_i, x_j are joined by a link $x_i x_j$ in G_2 if $x_i \cap x_j \ne \emptyset$. Since the intersecting squares in K meet on a vertex and through each vertex there are exactly two squares, it follows that G_2 is a simple 4-regular graph. Again, G_2 can be drawn on $\mathbb{S}^2$, and hence G_2 gives a 30-vertex map $\overline{K}$ on $\mathbb{S}^2$. The three squares intersecting a triangle and the four squares intersecting a pentagon form faces of $\overline{K}$. So, $\overline{K}$ is of vertex-type $[3^1, 5^1, 3^1, 5^1]$. Thus, by the observation in the previous paragraph, $\overline{K}$ is the boundary of the icosidodecahedron. Hence, by Lemma 4.6 and Proposition 2.1, $R(\overline{K})$ is the boundary of small rhombicosidodecahedron. Clearly, the operations $X \mapsto \overline{X}$ & $Y \mapsto R(Y)$ are inverse operations. Thus, K is isomorphic to $R(\overline{K})$. Therefore, K is isomorphic to the boundary of small rhombicosidodecahedron.

Finally, if $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (120, [4^1, 6^1, 10^1])$, then K consists of 30 squares, 20 hexagons and 12 decagons. Consider the graph G_3 whose nodes are the squares of K. Two nodes y_i, y_j are joined by a link $y_i y_j$ in G_3 if there exists a hexagon F in K such that $y_i \cap F \ne \emptyset$ & $y_j \cap F \ne \emptyset$. Since K is a polyhedral map, the adjacent squares of each hexagon in K are distinct. This implies that G_3 is a simple 4-regular graph and can be drawn on $\mathbb{S}^2$. This gives a 30-vertex map $\overline{K}$ on $\mathbb{S}^2$. The three squares intersecting a hexagon and the five squares intersecting a decagon form faces of $\overline{K}$. So, $\overline{K}$ is of vertex-type $[3^1, 5^1, 3^1, 5^1]$. Thus, (as above) $\overline{K}$ is the boundary of the icosidodecahedron. Hence by Lemma 4.4 and Proposition 2.1, $R(\overline{K})$ is the boundary of great rhombicosidodecahedron. Clearly, the operations $X \mapsto [X]$ & $Y \mapsto T(Y)$ are inverse operations. Thus, K is isomorphic to $T([K])$. Therefore, K is isomorphic to the boundary of great rhombicosidodecahedron. Similarly, a 48-vertex map of vertex-type $[4^1, 6^1, 8^1]$ is isomorphic to the boundary of great rhombicuboctahedron. $\square$

Lemma 4.9. *A semi-equivelar spherical map of vertex-type $[3^4, 4^1]$ with 24 vertices (respectively, $[3^4, 5^1]$ with 60 vertices) is unique.*

Proof. Let X be a 24-vertex map of vertex-type $[3^4, 4^1]$. So, X has $\frac{24 \times 4}{3} = 32$ triangles and $\frac{24}{6} = 6$ squares. Let us call an edge blue if it is the intersection of two triangles and red if it is the intersection of one triangle and one square. So, each vertex is in two red edges and (hence) in three blue edges. We say a blue edge uv is *deep blue* if three triangles containing u (respectively, v) lie on one side of uv and one lies on the other side of uv. (Clearly, if uvx and uvy are triangles then uv is deep blue if and only if one of ux, uy (respectively, vx, vy) is red. Since two red edges are disjoint, it follows that an edge uv is deep blue if and only if there exist vertices x, y such that ux, vy are red and uy, vx are blue.)
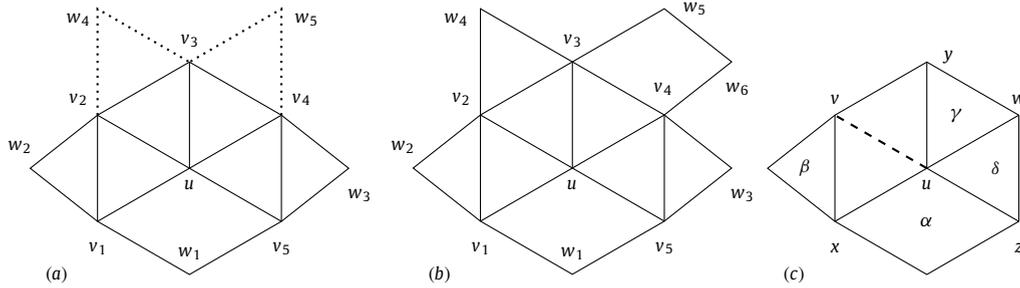
Fig. 6. (a) and (b): Parts of X as in the proof of above claim. (c): Part of $\widetilde{X}$.

Claim. For each vertex u, there exists a unique deep blue edge through u.

Let the faces containing u be uv_1v_2, uv_2v_3, uv_3v_4, uv_4v_5, $uv_5w_1v_1$. So, uv_3 is not deep blue. Since v_1u and v_1w_1 are red edges, v_1v_2 is blue. Let $v_1v_2w_2$ be a triangle containing v_1v_2. Since v_3 is in a unique square, v_2v_3 and v_3v_4 cannot be simultaneously red. Suppose both are blue. Let the face containing v_2v_3 (respectively, v_3v_4) and not containing u be $v_2v_3w_4$ (respectively, $v_3v_4w_5$) (see Fig. 6 (a)). Then the square containing v_2 must contain v_2w_4 and the square containing v_3 must contain v_3w_4. Thus, w_4 is in two squares. This is not possible since vertex-type is $[3^4, 4^1]$. So, exactly one of v_2v_3, v_3v_4 is red. If v_3v_4 is red and v_2v_3 is blue (see Fig. 6 (b)) then uv_4 is deep blue and uv_2 is not deep blue. In the other case, uv_2 is deep blue and uv_4 is not deep blue. This proves the claim.

It follows by the claim that the deep blue edges form a perfect matching in the edge graph of X. Let $\widetilde{X}$ be the map obtained from X by removing all the 12 deep blue edges. (This makes 24 triangles to 12 squares.) Then $\widetilde{X}$ is a 24-vertex map with $32 - 24 = 8$ triangles and $6 + 12 = 18$ squares. Moreover, the degree of each vertex in $\widetilde{X}$ is $5 - 1 = 4$. Since yw and wz cannot both be deep blue (see Fig. 6 (c)). Thus, one of γ, δ is a triangle in $\widetilde{X}$. So, u is in at least one triangle in $\widetilde{X}$. Since there are 8 triangles and 24 vertices, it follows that each vertex is in exactly one triangle and hence in $4 - 1 = 3$ squares. Thus, $\widetilde{X}$ is a semi-equivelar map of vertex-type $[3^1, 4^3]$.

Let $xuyv$ is a new square in $\widetilde{X}$, where uv is a deep blue edge in X. Assume that xuv is the unique triangle in X containing u on one side of uv. Then ux is a red edge and is in a square α (see Fig. 6 (c)). This implies that the other three faces (other than xuv & α) incident to x are triangles in X. Thus, vx is blue and hence vx is in a triangle β of $\widetilde{X}$. Similarly uy is in a triangle γ in $\widetilde{X}$. These imply that the new square $xuyv$ in $\widetilde{X}$ is of type 2 (as in the proof of Lemma 4.7). Therefore, $\widetilde{X}$ has at least 12 type 2 squares and hence by the proof of Lemma 4.7, $\widetilde{X}$ is the boundary of small rhombicuboctahedron.

On the other hand, if Y is the boundary of small rhombicuboctahedron then, as in the proof of Lemma 4.7, Y has twelve type 2 squares. We can divide each of these type 2 squares into two triangles by adding one diagonal in two ways. It is not difficult to see that if we chose one diagonal in one type 2 squares then there exists exactly one diagonal for each of the other 11 type 2 squares so that these 12 new diagonals form a perfect matching. And we get a map $\widehat{Y}$ of vertex-type $[3^4, 4^1]$ in which these 12 new edges are the deep blue edges. If we choose other set of 12 diagonals then we get another map of vertex-type $[3^4, 4^1]$ which is isomorphic to $\widehat{Y}$ under a reflection of Y. Thus, $\widehat{Y}$ is unique up to an isomorphism. Again, it is easy to see that the operations $X \mapsto \widetilde{X}$ and $Y \mapsto \widehat{Y}$ are inverse of each other. Since 24-vertex map of vertex-type $[3^1, 4^3]$ on $\mathbb{S}^2$ with twelve type 2 squares is unique, it follows that 24-vertex map of vertex-type $[3^4, 4^1]$ on $\mathbb{S}^2$ is unique up to isomorphism.

Let X be a 60-vertex map of vertex-type $[3^4, 5^1]$. As before we have 30 deep blue edges and by removing, we get (by similar arguments) a 60-vertex map $\widehat{X}$ of vertex-type $[3^1, 4^1, 5^1, 4^1]$. Hence, by Lemma 4.8, $\widehat{X}$ is isomorphic to the boundary of small rhombicosidodecahedron. On the other hand, if Y is the boundary of small rhombicosidodecahedron then Y has 30 squares. We can divide each of 30 squares into two triangles by adding one diagonal in two ways. Again, it is not difficult to see that if we chose one diagonal in one square then there exists exactly one diagonal for each of the other 29 squares so that these 30 new diagonals form a perfect matching. And we get a map $\widetilde{Y}$ of vertex-type $[3^4, 5^1]$ in which these 30 new edges are the deep blue edges. If we choose other set of 30 diagonals then we get another map of vertex-type $[3^4, 5^1]$ which is isomorphic to $\widetilde{Y}$ under a reflection of Y. Thus, $\widetilde{Y}$ is unique up to an isomorphism. Again, it is easy to see that the operations $X \mapsto \widehat{X}$ and $Y \mapsto \widetilde{Y}$ are inverse of each other. Since, by Lemma 4.8, 60-vertex map of vertex-type $[3^1, 4^1, 5^1, 4^1]$ on $\mathbb{S}^2$ is unique, it follows that 60-vertex map of vertex-type $[3^4, 5^1]$ on $\mathbb{S}^2$ is unique up to isomorphism.   □

Lemma 4.10. *Let K be a semi-equivelar spherical map. If the vertex-type of K is $[4^2, n^1]$ for some $n \geq 3$ then K is isomorphic to the boundary of the $2n$-vertex regular prism P_n.*

If the vertex-type of K is $[3^3, n^1]$ for some $n \geq 4$ then K is isomorphic to the boundary of the $2n$-vertex antiprism Q_n.

Proof. Let the vertex-type of K be $[4^2, n^1]$. Let $u_1 \in V(K)$ and the link-cycle of u_1 be u_2-u_3-$\cdots$-u_n-b_n-b_1-b_2-u_2 where faces $u_1u_2 \ldots u_n$, $u_nu_1b_1b_n$, $u_1u_2b_2b_1$ are incident at u_1. Then, the link-cycle of u_2 would be u_3-u_4-$\cdots$-u_n-u_1-b_1-b_2-b_3-u_3 for some vertex b_3. Clearly, $b_3 \notin \{b_1, b_2, u_1, \ldots, u_n\}$. If $b_3 = b_n$ then the edges $b_3u_3, b_3b_2, b_nu_n, b_nb_1$ are incident with b_3. This implies that $\deg(b_3) \geq 4$, a contradiction. So, $b_3 \neq b_n$. Continuing this way, we get the link-cycle of u_j is
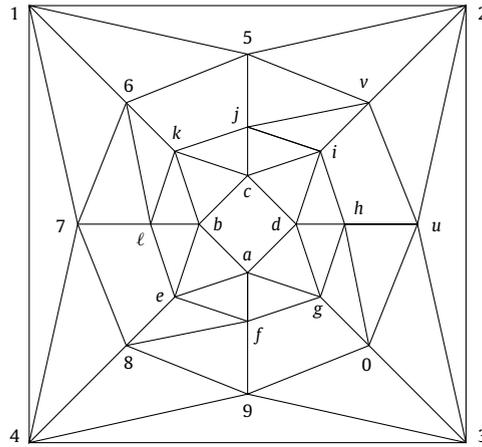
Fig. 7. Projection of the boundary of the snub cube on the square 1234

u_{j+1}-u_{j+2}-$\cdots$-u_n-u_1-$\cdots$-u_{j-1}-b_{j-1}-b_j-b_{j+1}-u_{j+1} for $j \geq 3$ where $b_t \neq b_\ell$ and $u_i \neq u_s$ for $\ell \neq t, i \neq s, 1 \leq \ell, t, i, s \leq n$. Hence, the faces of K are $u_1 u_2 \ldots u_n, b_1 b_2 \ldots b_n, u_n u_1 b_1 b_n, u_i u_{i+1} b_{i+1} b_i$ $(1 \leq i \leq n-1)$. Then, K is isomorphic to the boundary of the prism P_n.

Let the vertex-type of K be $[3^3, n^1]$. Let $u_1 \in V(K)$ and link-cycle of u_1 be u_2-u_3-$\cdots$-u_n -b_1-b_2-u_2, where faces $u_1 u_2 \ldots u_n, b_1 u_1 u_n, b_1 b_2 u_1, u_1 u_2 b_2$ are incident with u_1. Then, the link-cycle of u_2 would be u_3-$\cdots$-u_n-u_1-b_2-b_3-u_3 for some vertex b_3. Then $b_3 \notin \{u_1, u_2, \ldots, u_n, b_2\}$. Also, b_3, b_1 belong in link-cycle of b_2. So, $b_3 \neq b_1$. Hence, let the link-cycle of u_3 be u_4-u_5-$\cdots$-u_n-u_1-u_2- b_3-b_4-u_4 for some vertex b_4. As before, $b_4 \notin \{u_1, u_2, \ldots, u_n, b_3, b_2\}$. If $b_4 = b_1$ then b_4 is in the edges $b_4 u_4, b_4 u_3, b_4 b_3, b_4 u_n, b_4 u_1, b_4 b_2$. This implies that $\deg(b_4) \geq 6$, a contradiction. So, $b_4 \neq b_1$. Continuing this way, we get link-cycle of $u_j = u_{j+1}$-u_{j+2}-$\cdots$- u_n-u_1-$\cdots$-u_{j-1}-b_j-b_{j+1}-u_{j+1} where $b_t \neq b_\ell$ and $u_i \neq u_s$ for $\ell \neq t, i \neq s, 1 \leq \ell, t, i, s \leq n$. Clearly, the cycle b_1-$\cdots$-b_n-b_1 is the third face containing b_j for each j. So, the faces of K are $u_1 u_2 \ldots u_n, b_1 b_2 \ldots b_n, u_n b_n b_1, u_1 b_1 u_n, u_i b_i u_{i+1}, u_i b_{i+1} u_{i+1}$ $(1 \leq i \leq n-1)$. Then, K is isomorphic to the boundary of the antiprism Q_n. □

Proof of Theorem 1.1. The result follows from Corollary 3.3 and Lemmas 4.1, 4.7, 4.8, 4.9, 4.10. □

Definition 4.11. An *involution* of a map is an automorphism of order 2. An involution ρ of a map X is called an *antipodal automorphism* (or *antipodal symmetry*) if no vertex, edge or face of X is fixed by ρ. A map X is called *centrally symmetric* if X has an antipodal symmetry.

We know that the antipodal map of the snub cube (respectively, of the snub dodecahedron) is not an automorphism of the boundary of the snub cube (respectively, of the snub dodecahedron) ([2]). We need the following stronger result for the proof of Theorem 1.2.

Lemma 4.12. *If X is the boundary of the snub cube or the boundary of the snub dodecahedron then X does not have any antipodal symmetry.*

Proof. Let X be the boundary of the snub cube. Assume that X is as in Fig. 7. Suppose there exists an antipodal automorphism ρ of X. Since no edge is fixed by ρ, $x\rho(x)$ is a non-edge for each vertex x. Also, ρ sends a square to another square. If $\rho(1234) = 56kj$ then $\rho(1) = j$ or k. If $\rho(1) = j$ then $\rho(1\text{-}5\text{-}j)$ must be j-5-1 and hence $\rho(5) = 5$, a contradiction. So, $\rho(1) \neq j$. Similarly, $\rho(1) \neq k$. Thus, $\rho(1234) \neq 56kj$. Similarly, $\rho(1234) \neq 78el, 90gf, uvih$. So, $\rho(1234) = abcd$. If $\rho(1) = c$ then $\rho(\{2, 4\}) = \{b, d\}$ and hence $\rho(\{5, 7\}) = \{k, i\}$. This implies that $\rho(6) = j$ and hence $\rho(56kj) = 56kj$, a contradiction. So, $\rho(1) \neq c$. If $\rho(1) = b$ then, by the same argument, $\rho(6) = \ell$ and hence $\rho(6\ell) = 6\ell$, a contradiction. So, $\rho(1) \neq b$. Similarly, $\rho(2) \neq c, d$. So, $\rho((1, 2)) = (a, b)$ or (d, a). In the first case, $\rho(5) = e$ and hence $\rho(56kj) = 78e\ell$. Then $\rho(156) = axy$, where xy is an edge of $78e\ell$ and hence $\rho(156)$ is not a triangle, a contradiction. In the second case, $\rho(5) = g$ and hence $\rho(56kj) = gf90$. Then, by the same argument, $\rho(156)$ is not a triangle, a contradiction. This shows that $\rho(1234) \neq abcd$. Thus, there does not exist any antipodal automorphism of the boundary of the snub cube. By similar arguments, one can show that there does not exist any antipodal automorphism of the boundary of the snub dodecahedron. □

Lemma 4.13. *Let X be the boundary of the pseudorhombicuboctahedron. Then (a) X is not a vertex-transitive map, and (b) X does not have antipodal symmetry.*

Proof. As before, for $2 \leq i \leq 4$, let's call a square *type i* if it intersects i other squares on edges. Then there are two type 4 squares, eight type 3 squares and eight type 2 squares. Let u be a vertex of a type 4 square (there are 8 such vertices) and v be a vertex which is not in any type 4 square (there are 16 such vertices). If φ is an automorphism, then the image of a type 4 square under φ is again a type 4 square. Therefore, $\varphi(u)$ cannot be v. Thus, X is not a vertex-transitive map. This proves part (a).

Consider X as $A \cup C$ as in Fig. 5. If possible, let ρ be an antipodal automorphism of X. Then ρ sends all the eight type 3 squares to themselves. And ρ interchanges two type 4 squares. These imply that $\rho(\{u_5, \ldots, u_{12}\}) = \{u_{13}, \ldots, u_{20}\}$. Since the edge $u_5 u_6$ is in the intersection of two squares, $\rho(u_5 u_6)$ is $u_{14} u_{15}$, $u_{16} u_{17}$, $u_{18} u_{19}$ or $u_{20} u_{13}$. Assume $\rho(u_5 u_6) = u_{16} u_{17}$. Since ρ is an antipodal automorphism, this implies that

$$\rho = (u_5, u_{16})(u_6, u_{17})(u_7, u_{18})(u_8, u_{19})(u_9, u_{20})(u_{10}, u_{13})(u_{11}, u_{14})(u_{12}, u_{15}) \cdots.$$

Then $\rho(u_5 u_6 u_{14} u_{13}) = u_{16} u_{17} u_{11} u_{10}$. This is a contradiction since $u_5 u_6 u_{14} u_{13}$ is a face but $u_{16} u_{17} u_{11} u_{10}$ is not a face of X. Similarly, we get contradictions in other three cases. This proves part (b). $\square$

Proof of Corollary 1.3. From Lemma 4.13 (a), we know that the boundary of the pseudorhombicuboctahedron is not a vertex-transitive map.

From the results in [1], it follows that all the maps on $\mathbb{S}^2$ other than the boundary of the pseudorhombicuboctahedron mentioned in Theorem 1.1 are vertex-transitive. The result now follows by Theorem 1.1. $\square$

Proof of Theorem 1.2. Let the number of vertices in Y be n and the vertex-type of Y be $[p_1^{n_1}, p_2^{n_2}, \ldots, p_\ell^{n_\ell}]$. Since $\mathbb{S}^2$ is a 2-fold cover of $\mathbb{RP}^2$, by pulling back, we get a $2n$-vertex map of vertex-type $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$ on $\mathbb{S}^2$. Hence, by Corollary 3.3, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (2, [3^3])$, $(3, [3^4])$, $(4, [4^3])$, $(6, [3^5])$, $(10, [5^3])$, $(30, [3^4, 5^1])$, $(12, [3^4, 4^1])$, $(15, [3^1, 5^1, 3^1, 5^1])$, $(6, [3^1, 4^1, 3^1, 4^1])$, $(30, [3^1, 4^1, 5^1, 4^1])$, $(12, [3^1, 4^3])$, $(30, [5^1, 6^2])$, $(24, [4^1, 6^1, 8^1])$, $(12, [4^1, 6^2])$, $(60, [4^1, 6^1, 10^1])$, $(6, [3^1, 6^2])$, $(12, [3^1, 8^2])$, $(30, [3^1, 10^2])$, $(3, [3^1, 4^2])$, $(r, [4^2, r^1])$ for some $r \geq 5$ or $(s, [3^3, s^1])$ for some $s \geq 4$.

Clearly, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (2, [3^3])$, $(3, [3^4])$, $(4, [4^3])$, $(6, [3^1, 4^1, 3^1, 4^1])$, $(12, [3^1, 8^2])$, $(6, [3^1, 6^2])$ or $(3, [3^1, 4^2])$ is not possible.

Now, assume $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (12, [3^4, 4^1])$. Then the 2-fold cover $\widetilde{Y}$ of Y is a 24-vertex map of vertex-type $[3^4, 4^1]$ on $\mathbb{S}^2$ with an antipodal symmetry. From Lemma 4.8, $\widetilde{Y}$ is the boundary of snub cube. This is not possible by Lemma 4.12. Thus, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) \neq (12, [3^4, 4^1])$. Similarly, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) \neq (30, [3^4, 5^1])$.

If $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}] = [4^2, n^1]$ for some $n \geq 5$ then Y has a unique face α with n vertices and hence contains all the vertices. Thus, all the vertices of a square β are in α. Then intersection of α and β cannot be a vertex or an edge, a contradiction. Thus, $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) \neq (n, [4^2, n^1])$ for any $n \geq 5$. Similarly $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) \neq (n, [3^3, n^1])$ for any $n \geq 4$. This proves the first part.

Let $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (6, [3^5])$. Then Y is a 6-vertex triangulation of $\mathbb{RP}^2$. It is known (cf. [4]) that there exists unique such triangulation.

Let $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (12, [3^1, 4^3])$ then the 2-fold cover $\widetilde{Y}$ of Y is a 24-vertex map of vertex-type $[3^1, 4^3]$ on $\mathbb{S}^2$ with an antipodal symmetry. Therefore, by Lemma 4.13, $\widetilde{Y}$ is isomorphic to the boundary of small rhombicuboctahedron. Hence Y is unique.

If $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (10, [5^3])$ then the 2-fold cover $\widetilde{Y}$ of Y is a 20-vertex map of vertex-type $[5^3]$ on $\mathbb{S}^2$. Hence, by Lemma 4.1, $\widetilde{Y}$ is the boundary of the dodecahedron. By identifying antipodal vertices of $\widetilde{Y}$ we get Y. Since $\widetilde{Y}$ is unique, Y is unique. Similarly, for each $(n, [p_1^{n_1}, \ldots, p_\ell^{n_\ell}]) = (24, [4^1, 6^1, 8^1])$, $(12, [4^1, 6^2])$, $(30, [5^1, 6^2])$, $(30, [3^1, 10^2])$, $(30, [3^1, 4^1, 5^1, 4^1])$, $(15, [3^1, 5^1, 3^1, 5^1])$ or $(60, [4^1, 6^1, 10^1])$, there is a unique n-vertex map of vertex-type $[p_1^{n_1}, \ldots, p_\ell^{n_\ell}]$ on $\mathbb{RP}^2$. This proves the last part. $\square$

Proof of Corollary 1.4. If P is one of the polytopes mentioned in Section 2 then (i) the centre of P is $(0, 0, 0)$, (ii) all the vertices of P are points on the unit 2-sphere with centre $(0, 0, 0)$, (iii) each 2-face of P is a regular polygon, and (iv) lengths of all the edges of P are same. Let ∂P denote the boundary complex of P. Let $\eta : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \to \mathbb{S}^2$ be the mapping given by $\eta(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}(x_1, x_2, x_3)$. Then $\eta(\partial P)$ is a map on $\mathbb{S}^2$. Since P satisfies the properties (i) - (iv), this map is a semi-regular tiling of $\mathbb{S}^2$. (For example, the image of each edge of P under η is a great circle arc on $\mathbb{S}^2$ and hence a geodesic.) Part (a) now follows from Theorem 1.1.

Since $\mathbb{S}^2$ is the universal cover of $\mathbb{RP}^2$, part (b) follows from part (a). (More explicitly, let Y be a semi-equivelar map on $\mathbb{RP}^2$. By pulling back to the universal cover, we get a semi-equivelar map $\widetilde{Y}$ (of same vertex-type) on $\mathbb{S}^2$. By part (a), up to an isomorphism, $\widetilde{Y}$ is a semi-regular tiling of $\mathbb{S}^2$ and Y is obtained from $\widetilde{Y}$ as quotient by an order 2 automorphism of $\widetilde{Y}$. This implies that, up to an isomorphism, Y is a semi-regular tiling on $\mathbb{RP}^2$.) $\square$

Proof of Corollary 1.5. Since all the maps on $\mathbb{RP}^2$ mentioned in Theorem 1.2 are vertex-transitive ([1]), the result follows by Theorem 1.2. $\square$

Declaration of competing interest

Acknowledgements

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