ASSOCIATIVE MEMORY DESIGN : FOKKER - PLANCK FORMALISM

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Abstract

A design methodology for associative memory (AM) using Fokker-Planck formalism is proposed. This method allows AM to be designed for noisy conditions by taking into account the noise level to be tolerated by the AM as a design specification. The AM is viewed as a stochastic nonlinear dynamical system governed by a Fokker-Planck Equation (FPE). Using eigenfunction expansion, spectrum of the FPE governing the dynamics of the desired AM is derived. A method to design neural networks having the derived FPE dynamics is also given.

keywords: Associative memory, noise, Fokker-Planck equation.

1 INTRODUCTION

AM has received ~~much~~ attention in the recent years due to its pattern recognition capabilities. AMs are dynamical systems having stable point attractors corresponding to the patterns to be stored [1]. The AM dynamics take initial states corresponding to distorted version of the stored pattern to the stable points thus retrieving the closest correct pattern. While taking into account noise in the patterns, it is more realistic to account for the noise in the dynamics of the AM itself. This means the governing equation of AM dynamics should be stochastic differential in nature rather than ordinary differential equations. Under these situations the AM dynamics can be viewed as overdamped Brownian motion of a ball under gravitation.

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This paper is organised as follows. In section 2 we give a brief review of the eigenfunction expansion method of FPE, which will aid us in explaining our procedure for deriving the FPE spectrum from the target stored patterns. In section 3 we give estimates for the eigenfunctions. The drift coefficient ($A(x)$) of the FPE is derived in section 4. Before concluding we finally present in section 5 a method for designing neural networks having the derived FPE dynamics.

2 Eigenfunction Expansion Method

Equations governing brownian motion, called Langevin equations are of the form [2]

$$\dot{x} = A(x)x + f(t) \quad (1)$$

where $f(t)$ is the white noise with

$$\langle f(t) \rangle = 0 \text{ and } \langle f(t)f(t') \rangle = \sqrt{B(x)}\delta(t-t')$$

The equivalent time homogeneous FPE reads

$$\partial_t P(x, t) = -\partial_x [A(x)P(x, t)] + \frac{1}{2} \partial_x^2 [B(x)P(x, t)] \quad (2)$$

In eigen function expansion method [3] we look for solutions to FPE of the form

$$P(x, t) = P_\lambda(x) e^{-\lambda t} \quad (3)$$

Substituting eq 3 in the FPE given by equation 2 we get the eigenfunction equation

$$-\partial_x [A(x)P_\lambda(x)] + \frac{1}{2} \partial_x^2 [B(x)P_\lambda(x)] = -\lambda P_\lambda(x)$$

Let

$$P(x, t) = P_s(x)Q(x, t) \quad (4)$$

(where $P_s(x)$ is the stationary solution of FPE). It can be seen that $Q(x, t)$ satisfies the kolmogorov Backward equation (KBE)

$$\partial_t Q(x, t) = A(x)\partial_x Q(x, t) + \frac{1}{2} B(x)\partial_x^2 Q(x, t) \quad (5)$$

Solutions for KBE of the form $Q(x, t) = Q_\lambda(x)e^{-\lambda t}$

lead to the equation

$$A(x)\partial_x Q_\lambda(x) + \frac{1}{2}B(x)\partial_x^2 Q_\lambda(x) = -\lambda Q_\lambda(x) \quad (6)$$

$P_\lambda(x)$ and $Q_\lambda(x)$ form a bi-orthogonal system and this allows us to write any solution of FPE in the form

$$P(x, t) = \sum_{\lambda} A_{\lambda} P_{\lambda}(x) e^{-\lambda t} \quad (7)$$

where

$$A_{\lambda} = \int dx Q_{\lambda}(x) P(x, 0) \quad (8)$$

(integrations are assumed to be over appropriate ranges). From equation 7, we can see that

$$P_s(x) = P_o(x) \quad (9)$$

3 Estimate for eigenfunctions

In this section we derive the spectrum of the FPE (the eigenfunctions) of the equation which will have the desired **AM** dynamics. We list some of the characteristics of the FPE dynamics [4] and see how they are reflected in its spectrum.

3.1 Characteristics of FPE Dynamics

1. Under moderate noise conditions the stationary probability distribution (SPDF) will be concentrated near the attractors of the dynamics.
2. If there are M stable attractors then the first M eigenvalues are of the smaller order in magnitude than the higher eigenvalues.
3. There exists two time scales the smaller one(t_r), in which the dynamics relaxes to one of the stable attractor and the larger one(mean transition time t_m), in which the system exhibit transitions from one attractor to another.
4. The first M eigenvalues are of the order of the inverse of the mean transition time.

We will show how one can estimate the eigenfunctions of a 1- dimensional FPE having M stable attractors at points u_i ; $i=1$ to M. Extension to K-dimensional FPE, which is encountered in case of AM having K-neurons is straightforward.

3.2 Estimate for $P_0(x)$

From the point 1 of section 3.1, and from equation 9 we can approximate $P_0(x)$ by a function which is peaked around the attractors and practically zero outside the basin of attractors. A function satisfying these criteria can be written as,

$$P_0(x) = N \sum_{i=1}^M \frac{1}{\sigma_1 \sqrt{2\pi}} \text{EXP}(-\frac{1}{2} \frac{[x - u_i]^2}{\sigma_1^2})$$

where σ_1 determine the strength of the noise and N a normalization constant.

3.3 Estimate for $P_\lambda(x)(\lambda > 0)$

From point 2 of section 3.1 and from equation 7 we can see that to a good degree of approximation, we can truncate the eigenfunction expansion to first M terms because, eigenfunctions $P_\lambda(x)(\lambda > M)$ contribute less towards the dynamics. With this approximation equation 7 reads

$$P(x, t) = \sum_{\lambda=0}^M A_\lambda P_\lambda(x) e^{-\lambda t} \quad (10)$$

we now give estimates for $P_\lambda(x)$ for $\lambda = 1$ to M .

The time dependent solution of FPE can now be written using equations 3, 4, 8 and 10

$$P(x, t) = \sum_{\lambda=0}^M P_s(x) Q_\lambda(x) \left[\int Q_\lambda(x) P(x, 0) dx \right] e^{-\lambda t} \quad (11)$$

We want the dynamics to be such that for any initial state within the basin of attractor, the $P(x, t)$ should converge to SPDF with a relaxation time(t_r).

That is for an initial distribution

$$P(x, 0) = N(u_i, \sigma)$$

the solution $P(x, t)$ of equation 11 for the time scale of the order t_r should be

$$P(x, t) = N(u_i, \sigma \frac{1}{t_r})$$

Here N stands for a normal distribution with mean u_i and variance σ and $\sigma \frac{1}{t_r}$ respectively (the variance σ decides the size of the basin of attractors) . This condition can be imposed on equation 11 to get $Q_\lambda(x)$ and hence $P_\lambda(x)$ by using equation 3 and 4. Point 4 of section 3.1 mentioned above enables us to fix the eigenvalues (λ_1 to λ_M) to be below $1/t_m$ but having distinct values.

4 Estimate for A(x)

Substituting a constant diffusion coefficient D for B(x) in equation 6 we get M equations one for each λ_i (for i = 1 to M)

$$A(x)\partial_x Q_{\lambda_i}(x) + \frac{1}{2}D\partial_x^2 Q_{\lambda_i}(x) = -\lambda_i Q_{\lambda_i}(x); \quad (12)$$

Adding these M equations and simplifying we get

$$A(x) = - \sum_{i=1}^M \left[\frac{\{\lambda_i Q_{\lambda_i}(x) - D/2\partial_x^2 Q_{\lambda_i}(x)\}}{\partial_x Q_{\lambda_i}(x)} \right]$$

Here D can be fixed depending upon the noise level which the AM has to tolerate. So finally we get the drift coefficient A(x) and diffusion coefficient B(x) of the FPE governing the dynamics of the AM which stores the desired patterns.

5 A Neural Network with the Derived Fokker - Planck Dynamics

Here we give an estimate for the weights (W_{ij}) connecting neuron i to neuron j of a network having K neurons exhibiting the above derived FPE dynamics. If A(x) can be identified as a gradient of a potential then a network having the energy function as this potential will have the FPE dynamics. Consider the simplified formula for the energy function of a Hopfield neural network.

$$E = -\frac{1}{2} \left[\sum_{ij} W_{ij} x_i x_j \right]$$

$$dE/dx_i = - \sum_j W_{ij} x_j \text{ for } i = 1 \text{ to } K \quad (13)$$

It can be seen that the derivative of the energy function vanishes at u_i for $i=1$ to M. (i.e. the derivative vanishes in each direction of the K- dimensional state space of the network at M points. Solving equation 13 at u_i for $i= 1$ to M we get KM simultaneous linear equations, but we have ($K^2/2 - K$) unknowns (because the weight matrix is symmetric with zero diagonal entries). Since M (the number of patterns) will be less than K (the number of neurons) we get nonunique solution. This is not a problem because many neural networks can store the same set of patterns, if their weights are bounded within a certain range. Thus any one of the solution satisfying the KM equations can be chosen as the weight matrix.

6 Conclusions

Our method of designing an associative memory differs from other existing approaches in two ways. Existing methods train the network by carving the energy landscape of the network so as to have the bottom of the energy valleys corresponding to the stored patterns. Whereas here we first derive the gradient of the energy function and then look for a network having such a energy function. Next we take into consideration noise as a design criteria itself. Whereas in other methods one can test the network performance under noise only after the design is over. This results in redesigning, in case the performance is not satisfactory. The estimates for the eigenfunctions and for the weights of neural network given in this paper are only approximate. As pointed out already accurate estimates are not mandatory for designing the network to store a given set of patterns.

References

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