


Axial Anomaly in $SU(N)$ Yang-Mills Matrix Models

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The $SU(N)$ Yang-Mills matrix model admits self-dual and anti-self-dual instantons. When coupled to N_f flavors of massless quarks, the Euclidean Dirac equation in an instanton background has n_+ positive and n_- negative chirality zero modes. The vacua of the gauge theory are N -dimensional representations of $SU(2)$, and the (anti-) self-dual instantons tunnel between two commuting representations, the initial one composed of $r_0^{(1)}$ irreps and the final one with $r_0^{(2)}$ irreps. We show that the index $(n_+ - n_-)$ in such a background is equal to a new instanton charge $\mathcal{T}_{\text{new}} = \pm[r_0^{(2)} - r_0^{(1)}]$. Thus $\mathcal{T}_{\text{new}} = (n_+ - n_-)$ is the matrix model version of the Atiyah-Singer index theorem. Further, we show that the path integral measure is not invariant under a chiral rotation, and relate the noninvariance of the measure to the index of the Dirac operator. Axial symmetry is broken anomalously, with the residual symmetry being a finite group. For N_f fundamental fermions, this residual symmetry is \mathbb{Z}_{2N_f} , whereas for adjoint quarks it is \mathbb{Z}_{4N_f} .

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Introduction.—A symmetry of a classical theory that cannot be implemented in its quantum counterpart is said to be broken anomalously. A well-known example of such an anomaly occurs in gauge theories with massless Dirac fermions Ψ . The classical action is invariant under the vector rotation $\Psi \rightarrow e^{i\alpha\gamma^5}\Psi$ as well as the axial rotation $\Psi \rightarrow e^{i\alpha\gamma^5}\Psi$, but the latter is anomalously broken [1,2]. This is the *axial anomaly*, which manifests in the nonconservation of the axial current $j_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi$ in the quantum theory. Rather than $\partial_\mu j_A^\mu = 0$, we have $\partial_\mu j_A^\mu \sim e^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$, where $F_{\mu\nu}$ is the gauge curvature.

As argued by Fujikawa [3,4], the origin of the axial anomaly may be understood by examining the Euclidean path integral $\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{-S_E^F}$, where $S_E^F = \int d^4x i\bar{\Psi}\mathcal{D}\Psi$ is the fermionic action and \mathcal{D} is the gauge-covariant Dirac operator. Expanding Ψ and $\bar{\Psi}$ in the basis of eigenfunctions of \mathcal{D} shows that although S_E is invariant under axial rotations, the measure $\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$ is not. The change in the measure is related to the index of the Dirac operator, which in turn is related to the instanton number $\int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ via the Atiyah-Singer index theorem.

In this Letter we will demonstrate a surprising result: the quantum $SU(N)$ Yang-Mills matrix model coupled to N_f

massless quarks also exhibits the axial anomaly. In quantum mechanical situations, anomalies appear when the symmetry operation does not preserve the domain of the Hamiltonian [5–7]. For instance, the Hamiltonian of a free particle on a circle has a family of self-adjoint domains $\mathcal{D}_\varphi = \{f(x) \in L^2[-\pi, \pi] : f(x + \pi) = e^{i\varphi} f(x - \pi)\}$. For $\varphi \neq n\pi$, parity transformation $\mathcal{P} : x \rightarrow -x$, a classical symmetry, changes the domain $\mathcal{D}_\varphi \rightarrow \mathcal{D}_{-\varphi}$, resulting in the anomalous breaking of \mathcal{P} [6]. Another example, where the classical scaling symmetry $S : r \rightarrow \mu r, p \rightarrow \mu^{-1} p$ is anomalously broken has been discussed in careful detail by Jackiw [7]. Anomalies have also been reported in discrete lattice formulations of Kähler-Dirac fermions [8].

Such anomalies have been shown to play an important role in the possible phases of θ -QCD (quantum chromodynamics) [9,10]. Our demonstration of the axial anomaly closely follows the Euclidean approach: we show that there exist zero modes of the Euclidean Dirac operator in the background of (anti-) self-dual instanton gauge configurations, and the index of \mathcal{D} is nonzero.

In the matrix model, the instantons tunnel between the classical vacua $A_i = L_i$, where L_i 's are the generators of $SU(2)$ in any N -dimensional representation, with r_0 irreps (irreducible representations). The (anti-) self-dual instantons tunnelling between two distinct classical vacua say $L_i^{(1)}$ with $r_0^{(1)}$ irreps and $L_i^{(2)}$ with $r_0^{(2)}$ irreps can be associated with a new charge $\mathcal{T}_{\text{new}} \equiv [r_0^{(2)} - r_0^{(1)}]$. As we will demonstrate, the index of \mathcal{D} is equal to \mathcal{T}_{new} , giving us the matrix model version of the Atiyah-Singer index theorem. Finally, we relate the index to the noninvariance

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of the path-integral measure, which gives the anomaly. For the fermions in the fundamental representation of $SU(N)$ (i.e., quarks), there remains a residual \mathbb{Z}_{2N_f} symmetry, while for adjoint fermions the residual symmetry is \mathbb{Z}_{4N_f} . We emphasize that our result holds for any color $N \geq 2$.

The surprise here is that the $SU(N)$ matrix model is vastly different from $SU(N)$ gauge field theory. In the field theory, distinct instantons are labeled by the instanton charge $\pi_3[SU(N)] = \mathbb{Z}$, whereas in the corresponding matrix model, they are labeled by a finite set of integers. But even though the number of distinct instantons is finite, it is enough to disturb the balance between left- and right-handed fermion zero modes, leading to the anomalous breaking of axial symmetry.

The matrix model discussed here was first presented in [11–13], and has been shown to be an excellent candidate for an effective low-energy approximation of $SU(N)$ Yang-Mills theory on $S^3 \times \mathbb{R}$. In particular, numerical investigations of its spectrum gave excellent estimates for the low-lying glueball and hadron masses [14,15]. However, to serve as a correct low-energy approximation of Yang-Mills theory, this quantum-mechanical model must also exhibit the axial anomaly. Our present work provides this important conceptual support.

The matrix model—The dynamical degrees of freedom of the model are elements of \mathcal{M}_N , the set of all $3 \times (N^2 - 1)$ -dimensional real matrices M_{ia} , or equivalently $A_i = M_{ia}T_a$, T_a being the generators of $SU(N)$ in the fundamental representation. Gauge transformations act on A_i via the adjoint action $A_i \rightarrow hA_ih^{-1}$, $h \in SU(N)$. The configuration space is the base space of principle bundle $\text{Ad } SU(N) \rightarrow \mathcal{M}_N \rightarrow \mathcal{M}_N/\text{Ad } SU(N)$. This fiber bundle being twisted [16,17], lies at the heart of the Gribov problem in Yang-Mills theory.

To describe the dynamics, we need to define a gauge-covariant time derivative of A_i . For that, we introduce a set of time-dependent real functions, conveniently named as M_{0a} , and the matrix $A_0 \equiv M_{0a}T_a$. A_0 is the parallel transporter needed to define the covariant derivative along the temporal direction and under a gauge transformation $h(t) \in SU(N)$, $A_0 \rightarrow A'_0 = hA_0h^{-1} - \dot{h}h^{-1}$. Then, the gauge-covariant time derivative of A_i is defined as

$$D_t A_i = \dot{A}_i - i[A_0, A_i]. \quad (1)$$

which transforms as $D_t A_i \rightarrow h(D_t A_i)h^{-1}$.

The curvature F_{ij} is obtained by the pullback of the Maurer-Cartan equation of $SU(N)$ to S^3 : $F_{ij} = -\epsilon_{ijk}R^{-1}A_k - i[A_i, A_j]$, where R is the radius of the S^3 (for a derivation of the matrix model, see the Supplemental Material [18]). The chromoelectric field $E_i \equiv D_t A_i$ and the chromomagnetic field $B_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}$ are given by

$$E_i = \dot{A}_i - i[A_0, A_i], \quad B_i = -\frac{1}{R}A_i - \frac{i}{2}\epsilon_{ijk}[A_j, A_k]. \quad (2)$$

The Yang-Mills Lagrangian is

$$L_{\text{YM}} = \frac{R^3}{g^2} [\text{Tr}(D_t A_i)^2 - V(A)], \quad V(A) \equiv \text{Tr} B_i B_i. \quad (3)$$

The quarks Ψ are Grassmann-valued matrices that depend only on time, and transform in the fundamental representation of $SU(N)$ and in the $\text{spin-}\frac{1}{2}$ representation of spatial rotations. The Lagrangian with minimally coupled massless quarks is $L = L_{\text{YM}} + L_F$ where $L_F = R^3 \bar{\Psi} [i\gamma^0 D_t + \gamma^i A_i - (3/2R)\gamma^5 \gamma^0] \Psi$ is the gauge-covariant Dirac Lagrangian [13,19]

It is convenient to rescale to dimensionless variables $A_\mu \rightarrow RA_\mu$, $\Psi \rightarrow R^{3/2}\Psi$, and $t \rightarrow R^{-1}t$. Then performing a Wick rotation $t \rightarrow -i\tau$ and $A_0 \rightarrow iA_0$ in L gives us the Euclidean action $S_E = S_E^{\text{YM}} + S_E^F$ [20] with

$$S_E^{\text{YM}} = \frac{1}{g^2 R} \int d\tau [\text{Tr} \mathbb{E}_i \mathbb{E}_i + V_0], \quad S_E^F = \frac{1}{R} \int d\tau \Psi^\dagger i \not{D} \Psi, \quad (4)$$

where $\mathbb{E}_i (\equiv \mathbb{E}_i^a T_a) = (\partial A_i / \partial \tau) - [A_0, A_i]$ is the Euclidean chromoelectric field, and the potential $V_0 = (A_i + (i/2)\epsilon_{ijk}[A_j, A_k])^2$ is obtained by setting $R = 1$ in (3). \not{D} is the Euclidean Dirac operator, $i \not{D} = (\partial / \partial \tau) - A_0 - \gamma^0 \gamma^i A_i - \frac{3}{2}\gamma^5$. As, the R and g dependence of S_E^{YM} and S_E^F is just via multiplicative prefactors, we choose $R = 1$ and $g = 1$ without any loss of generality.

The classical action S_E is indeed invariant under the axial rotation $\Psi \rightarrow e^{i\alpha\gamma^5}\Psi$ and has $U(1)_A$ symmetry. However as we demonstrate below, the quantum effects do not preserve this symmetry.

Classical vacuum configurations of Eq. (3) are given by those A_i which satisfy $V(A) = 0$, i.e., $[A_i, A_j] = i\epsilon_{ijk}A_k$. This has solutions $A_i = L_i$, where L_i are generators of the Lie algebra of $SU(2)$. There are multiple degenerate vacua which correspond to the matrices A_i forming a general N -dimensional representation of L_i .

The tunnelling between degenerate classical minima is captured by instantons, the finite action solutions of equations of motion of S_E^{YM} in Eq. (4). This action is extremized when $\mathbb{E}_i = \pm B_i$, whose solutions with a plus (minus) sign give the self-dual (anti-self-dual) instantons.

Using the gauge freedom, the self-dual equation can be transformed to the temporal gauge $A_0 = 0$ to read

$$\frac{dA_i}{d\tau} = \pm \left(-A_i - \frac{i}{2}\epsilon_{ijk}[A_j, A_k] \right). \quad (5)$$

Substituting the ansatz

$$A_i = \phi(\tau)L_i^{(1)} + [1 - \phi(\tau)]L_i^{(2)}, \quad [L_i^{(1)}, L_j^{(2)}] = 0 \quad (6)$$

into Eq. (5), we obtain

$$\frac{\partial \phi_s}{\partial \tau} = -\phi_s(1 - \phi_s), \quad \frac{\partial \phi_a}{\partial \tau} = \phi_a(1 - \phi_a), \quad (7)$$

which have solutions

$$\phi_s(\tau) = \frac{1}{1 + e^{(\tau - \tau_0)}}, \quad \phi_a(\tau) = \frac{1}{1 + e^{-(\tau - \tau_0)}}. \quad (8)$$

Here, s denotes the self-dual instanton, and a denotes the anti-self-dual instanton. If $[L_i^{(1)}, L_j^{(2)}] \neq 0$, we would have to solve the second order equation of motion instead of the simpler first order Eq. (5). In general, $L_i^{(\alpha)}$ is a direct sum of $r_0^{(\alpha)}$ irreps with each irrep $L_i^{(\alpha),r}$ with $j_r^{(\alpha)} = [(N_r^{(\alpha)} - 1)/2]$.

If we gauge transform the solutions by $h(\tau)$ to obtain a nonzero A_0 , the transformed instanton solutions go between $L_i^{(1,2)}$ and $h_b L_i^{(2,1)} h_b^{-1}$, where $h_b = h(-\infty)$ for the self-dual and $h_b = h(\infty)$ for the anti-self-dual instanton. This does not affect any of the subsequent arguments, since we focus only on gauge-invariant quantities. These instanton solutions have been studied earlier by [21,22] in a different context.

The instanton number $\mathcal{T} = c \int_{-\infty}^{\infty} d\tau \text{Tr} E_i B_i$ is the integral of a total derivative. The normalization factor c is fixed to 4 so that \mathcal{T} is an integer. For the (anti-) self-dual instantons

$$\mathcal{T}_s = -\mathcal{T}_a = \frac{2}{3} \sum_{\alpha=1}^2 \epsilon_{\alpha\beta} \sum_{r=1}^{r_0^{(\alpha)}} j_r^{(\alpha)} [j_r^{(\alpha)} + 1] [2j_r^{(\alpha)} + 1]. \quad (9)$$

However for the (anti-)self-dual instanton, we can construct another charge that is expressible only in terms of $r_0^{(\alpha)}$, the number of irreps at $\tau = \pm\infty$. Defining $E'_i \equiv (d\phi_{s/a}/d\tau)[e_i^{(2)} - e_i^{(1)}]$ with

$$e_i^{(\alpha)} = \bigoplus_{r=1}^{r_0^{(\alpha)}} e_i^{(\alpha),r}, \quad e_i^{(\alpha),r} = \frac{3}{[j_r^{(\alpha)} + 1][2j_r^{(\alpha)} + 1]} L_i^{(\alpha),r}, \quad (10)$$

the new charge is given by

$$\mathcal{T}_{\text{new}} = 4 \int d\tau \text{Tr} E'_i B_i = \pm [r_0^{(2)} - r_0^{(1)}]. \quad (11)$$

\mathcal{T}_{new} is still an integral of a total τ derivative and depends only on $[r_0^{(2)} - r_0^{(1)}]$, rather than the Casimirs or other labels of each individual irrep. As we will demonstrate below,

\mathcal{T}_{new} for a (anti-)self-dual instanton is *equal* to the index of the Dirac operator.

Index of \mathcal{D} .—The Dirac operator obeys $\{\mathcal{D}, \gamma^5\} = 0$ and consequently, for every eigenfunction ψ_n of \mathcal{D} with a nonzero eigenvalue λ_n , there is an eigenfunction $\gamma^5 \psi_n$ with eigenvalue $(-\lambda_n)$. The eigenfunctions with zero eigenvalue (the zero modes) can also be arranged to be eigenfunctions of γ^5 :

$$\mathcal{D}\chi_k^\pm = 0, \quad \gamma^5 \chi_k^\pm = \pm \chi_k^\pm, \quad k = 1, 2, \dots, n_\pm. \quad (12)$$

but n_+ and n_- need not be equal. The index of \mathcal{D} is defined as $\text{ind} \mathcal{D} = n_+ - n_-$.

In the Weyl basis, the Dirac operator is

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{L} \\ \mathcal{L}^\dagger & 0 \end{pmatrix}, \quad \mathcal{L} \equiv -i \left(\frac{d}{d\tau} - A_0 + \sigma_i A_i + \frac{3}{2} \right). \quad (13)$$

We solve for the zero modes $\Psi_{\alpha A}$ of \mathcal{D} in the temporal gauge. If $A_0 \neq 0$, the corresponding zero mode is obtained via a gauge transformation on Ψ . In any case, n_\pm and the index do not change under the gauge transformation, and we can make our entire argument in temporal gauge.

For the (anti-) self-dual instanton, $\sigma_i A_i(\tau) + 3/2$ is a $2N$ -dimensional Hermitian matrix whose eigenvalues $\xi_i(\tau)$ are time dependent, but the eigenvectors are not. So \mathcal{L} can be diagonalized and \mathcal{D} brought to the form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{L}_d \\ \mathcal{L}_d^\dagger & 0 \end{pmatrix}, \quad \mathcal{L}_d = -i \left(\frac{d}{d\tau} + \Sigma \right), \quad (14)$$

where $\Sigma = \text{diag}[\xi_1(\tau), \xi_2(\tau), \dots, \xi_{2N}(\tau)]$. In the new basis, the positive and negative chirality zero modes are of the form $\chi^+ = (0, \psi^+)^T$ and $\chi^- = (\psi^-, 0)^T$, respectively, where the ψ^\pm satisfy $\mathcal{L}_d \psi^+ = 0$ and $\mathcal{L}_d^\dagger \psi^- = 0$. The index is given by $\text{ind} \mathcal{D} = \dim \text{Ker}(\mathcal{L}_d) - \dim \text{Ker}(\mathcal{L}_d^\dagger)$.

For the Dirac operator [Eq. (14)], there is a neat way to determine the index from $\xi_i(\tau \rightarrow \pm\infty)$ using Callias' index theorem [23]:

$$\text{ind} \mathcal{D} = \frac{1}{2} \sum_{i=1}^{2N} [\text{sgn} \xi_i(\tau \rightarrow -\infty) - \text{sgn} \xi_i(\tau \rightarrow \infty)]. \quad (15)$$

Let us apply Eq. (15) to the simple case when $A_i = \phi_{s/a}(\tau)L_i$, i.e., when $L_i^{(2)} = 0$ and $L_i^{(1)} = L_i$. Say L_i consists of r_0 irreducible blocks with each block of dimension $N_r \equiv (2j_r + 1)$. In each block, $\sigma_i A_i + 3/2$ is a $2N_r$ -dimensional matrix and has $(N_r - 1)$ degenerate eigenvalues $\lambda_{1,s/a} \equiv \frac{1}{2}[3 - \phi_{s/a}(\tau)(N_r + 1)]$, and $(N_r + 1)$ degenerate eigenvalues $\lambda_{2,s/a} \equiv \frac{1}{2}[3 + \phi_{s/a}(\tau)(N_r - 1)]$. As $\lambda_{2,s/a} > 0$ has same sign at both $\tau = -\infty$ and $\tau = \infty$, their contribution in Eq. (15) vanishes, and consequently the index in (anti-) self-dual background is

$$\text{ind}\mathcal{D}_s = \frac{1}{2} \sum_{r=1}^{r_0} (N_r - 1) \left[1 - \frac{2 - N_r}{|2 - N_r|} \right] = -\text{ind}\mathcal{D}_a. \quad (16)$$

For each block with $N_r > 2$, the term in the square brackets above evaluates to 2. There exist $N_r - 1$ normalizable zero modes of \mathcal{D} with one chirality (+1 for the self-dual instanton background and -1 for anti-self-dual one) and none with the opposite chirality. For blocks with $N_r = 1$, the index evaluates to 0 and consequently there are no normalizable zero modes.

For the blocks with $N_r = 2$, it seems at first sight that the index [Eq. (16)] is undefined. For this case, one eigenvalue of Σ vanishes at either $\tau = -\infty$ or $+\infty$, depending on whether the background is self-dual or anti-self-dual. Thus for large $|\tau|$, the Dirac operator resembles that of a free particle. The corresponding zero mode is not strictly normalizable but is delta-function normalizable: it is a zero energy resonance, or a threshold state.

We show the normalizable and the non-normalizable zero modes for $N_r = 2$ and $N_r = 3$ in Fig. 1 (explicit form of the solutions $\psi_{s/a,n}^\pm$ is given in the Supplemental Material [18]).

As we shall see below, both zero modes and zero energy resonance states have a nonzero contribution to the axial anomaly. To correctly take into account all contributions, one must extend the definition of $\text{ind}\mathcal{D}$ to include zero modes as well as zero energy resonances with positive and negative chirality [24].

In general for the ansatz [Eq. (6)], we find that $\text{ind}\mathcal{D}_s = -\text{ind}\mathcal{D}_a = r_0^{(2)} - r_0^{(1)}$, where $r_0^{(a)}$ is the number of irreps in $L_i^{(a)}$. Thus the index is equal to \mathcal{T}_{new} defined in Eq. (11). This yields the matrix model version of the Atiyah-Singer index theorem

$$\text{ind}\mathcal{D} = \mathcal{T}_{\text{new}}. \quad (17)$$

We emphasize that \mathcal{T}_{new} is a quantity computed from the pure gauge sector, while $\text{ind}\mathcal{D}$ counts the difference between the number of fermion zero modes of opposite chiralities. It is remarkable that there exists such a simple relation between the two: *a priori*, there is no reason to expect this equality. Moreover, as we will show further below, this relation between the charge and the index can be

suitably adapted for adjoint fermions as well, hinting toward its universal nature in the context of the matrix model. Despite the demonstration of this equality, we are not aware of an explanation of its geometric or topological origin.

Noninvariance of fermion measure.—We adapt Fujikawa’s method to demonstrate the axial anomaly and its relation to $\text{ind}\mathcal{D}$. Specifically, we show that under a $U(1)_A$ transformation, the measure of the fermionic path integral is not invariant. The Jacobian of the transformation gives the integrated anomaly.

In the fermionic path integral $\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{-S_E^F}$, we expand the fermionic variables $\Psi = \sum a_n \Phi_n$ and $\bar{\Psi} = \sum b_n \Phi_n$, where Φ_n ’s are eigenfunctions of \mathcal{D} with nonzero eigenvalues, zero modes as well as zero-energy resonances. In this basis, the fermionic path integral measure is given by $d\mu \equiv \mathcal{D}\bar{\Psi}\mathcal{D}\Psi = \prod_{n,m} da_n db_m$. Under a $U(1)_A$ rotation with an angle α , the transformed variables Ψ' can be expanded as $\Psi' = \sum a'_n \Phi_n$, $\bar{\Psi}' = \sum b'_n \Phi_n$ where the coefficients transform linearly: $a'_n = C_{mn} a_m$ and $b'_n = C_{mn} b_m$.

Consequently $d\mu$ transforms as $d\mu \rightarrow [\det C]^{-2} d\mu$. The Jacobian is $[\det C]^{-1} = e^{-i \int d\tau \mathcal{A}(\tau)}$, where the anomaly function $\mathcal{A}(\tau)$ is defined as $\mathcal{A}(\tau) = \sum_n \Phi_n^\dagger \gamma^5 \Phi_n$. In $\mathcal{A}(\tau)$, the summation is over an infinite number of modes and hence divergent. We introduce a gauge-invariant regulator $e^{-\beta \psi^2}$ and formally take the limit $\beta \rightarrow 0$ at the end: $\mathcal{A}(\tau) = \lim_{\beta \rightarrow 0} \sum_n \Phi_n^\dagger \gamma^5 e^{-\beta \psi^2} \Phi_n$.

We already saw that for nonzero eigenvalues, the eigenfunctions Φ_n and $\gamma^5 \Phi_n$ of \mathcal{D} are orthogonal, and hence do not contribute to the summation. However, the zero modes and the zero-energy scattering states can have nonzero contribution to $\mathcal{A}(\tau)$.

Setting $\alpha = \text{constant}$, the integrated anomaly function is

$$\int d\tau \mathcal{A}(\tau) = \int d\tau \left[\sum_{k=1}^{n_+} \chi_k^{+\dagger} \gamma^5 \chi_k^+ + \sum_{k=1}^{n_-} \chi_k^{-\dagger} \gamma^5 \chi_k^- \right]. \quad (18)$$

Using Eq. (12), we get $\int d\tau \mathcal{A}(\tau) = (n_+ - n_-) = \text{ind}\mathcal{D}$.

Thus in a background gauge configuration where the Dirac operator has a nonzero index, the fermion measure is

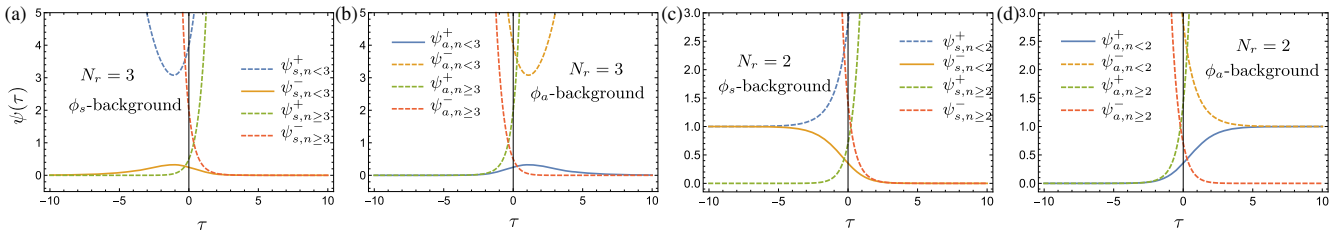


FIG. 1. The solutions of $\mathcal{D}\psi_{s/a,n}^\pm = 0$ with (a) $N_r = 3$ in self-dual background, (b) $N_r = 3$ in anti-self-dual background, (c) $N_r = 2$ in self-dual background, (d) $N_r = 2$ in anti-self-dual background.

not invariant under axial transformations, and axial symmetry is anomalously broken.

Because $\text{ind}\mathcal{D}$ is always an integer, $e^{\mp 2i\alpha \int d\tau A(\tau)} = 1$ when $\alpha = n\pi$ ($n \in \mathbb{Z}$) for any N . For axial rotations with $\alpha = n\pi$, the anomaly vanishes. Thus the $U(1)_A$ is anomalously broken to \mathbb{Z}_2 , the residual axial symmetry.

For N_f flavors, the Dirac operator is diagonal in flavor. Its spectrum is N_f copies of the spectrum of \mathcal{D} for a single flavor. The axial symmetry is now broken to \mathbb{Z}_{2N_f} .

Adjoint Weyl Fermion.—For supersymmetric gauge matrix models, the adjoint Weyl fermions λ are of relevance [25,26]. We show that axial symmetry is anomalously broken in this case too.

The Euclidean fermionic action can be written as

$$S_E^F = \int dt \lambda^\dagger \left(\partial_\tau - \mathcal{F}_0 + \sigma^i \mathcal{F}_i + \frac{3}{2} \right) \lambda \quad (19)$$

where $\mathcal{F}_0 = M_{0a} G_a$, $\mathcal{F}_i = M_{ia} G_a$, and $G_a = -if_{abc}$ are the $SU(N)$ generators in the adjoint representation. As before, we choose the temporal gauge.

For the ansatz [Eq. (6)], \mathcal{F}_i takes the form $\mathcal{F}_i = \phi_{s/a} \mathcal{J}_i^{(1)} + (1 - \phi_{s/a}) \mathcal{J}_i^{(2)}$ where $\mathcal{J}_i^{(\alpha)}$ are the representations of $SU(2)$ obtained by embedding $L_i^{(\alpha)}$ in $(N^2 - 1)$ dimensions. It is straightforward to show that \mathcal{J}_i are given by one singlet representation removed from the following direct sum

$$\bigoplus_{r,q} [(N_r + N_q - 1) \oplus (N_r + N_q - 3) \oplus \dots \oplus (|N_r - N_q| + 1)]. \quad (20)$$

Thus $\mathcal{J}_i^{(\alpha)}$ is a direct sum of $q_0^{(\alpha)}$ irreps $\mathcal{J}_i^{(\alpha),q}$ of dimension $\mathcal{N}_q^{(\alpha)}$, with $\sum_{q=1}^{q_0^{(\alpha)}} \mathcal{N}_q^{(\alpha)} = N^2 - 1$. Just as before, we define $\mathcal{E}'_i = (d\phi_{s/a}/d\tau)[e_i^{(2)} - e_i^{(1)}]$ and $\mathcal{B}_i = -\mathcal{F}_i - (i/2)\epsilon_{ijk}[\mathcal{F}_j, \mathcal{F}_k]$ where $e_i^{(\alpha)} = \bigoplus_{q=1}^{q_0^{(\alpha)}} e_i^{(\alpha),q}$ [the $e_i^{(\alpha),q}$ are as in Eq. (10)]. The instanton charge for the embedding is given by

$$\mathcal{T}_{\text{new}}^{\text{adj}} = \int d\tau \text{Tr} \mathcal{E}'_i \mathcal{B}_i = \pm [q_0^{(2)} - q_0^{(1)}]. \quad (21)$$

The index calculation is exactly as in Eqs. (15) and (16), by replacing $N_r^{(\alpha)}$ with $\mathcal{N}_q^{(\alpha)}$. By correctly taking into account the zero modes as well as the zero-energy resonances, the index is now given by $\text{ind}\mathcal{D} = \mathcal{T}_{\text{new}}^{\text{adj}}$.

In the adjoint case, all eigenvalues of \mathcal{D} are doubly degenerate. For every eigenvector φ_n of \mathcal{D} with eigenvalue λ_n , there exists an independent eigenvector $(\sigma_2 \otimes \mathbf{1})\varphi_n^*$ with the same eigenvalue. Consequently, the index is always an even integer, and so $e^{-2i\alpha \text{ind}\mathcal{D}} = 1$ when

$\alpha = (n\pi/2)$, $n \in \mathbb{Z}$. Therefore a single adjoint Weyl fermion breaks the $U(1)_A$ axial symmetry to a residual \mathbb{Z}_4 subgroup. For N_f flavors, the residual symmetry is \mathbb{Z}_{4N_f} .

Discussion.—Though the Yang-Mills matrix model is very different from the corresponding field theory, it nevertheless retains important nonperturbative features of the field theory. As we have demonstrated here, the axial anomaly is one such feature. In the usual discussion of the axial anomaly in gauge field theories, only the irreducible connections are considered, and it is the instanton number of such connections that is related to the fermion zero modes. *A priori*, there is no reason that the residual axial symmetry in the matrix model should match the corresponding field theory result, and it is surprising that it matches for the case of fundamental fermions. Whether this coincidence has a deeper significance requires further investigation.

Our result on the anomaly provides a strong conceptual support for the numerical investigations of the matrix model [15]. In addition to reproducing the masses of light hadrons with surprising accuracy, the numerics also show that the pseudoscalar mesons are much lighter than their scalar counterparts. Furthermore, it also finds the η' meson to be considerably heavier than the η meson. The result on the axial anomaly presented here serves to strengthen the position of the $SU(3)$ gauge matrix model as an effective low-energy approximation of QCD.

The axial anomaly is present for any $SU(N)$ gauge group, and there is no reason to expect that it is washed out in the large N limit.

There is a plethora of matrix models inspired by [27–29] that have been studied in the literature and which remain a subject of continuing interest [25,30–34]. These models have a non-Abelian gauge symmetry and fermions, and in light of the results presented here, the implications of the axial anomaly for these models need to be understood. Similarly, whether this axial anomaly has a role to play in quantum systems with emergent non-Abelian gauge symmetry like the multimodal Rabi model [35] is an interesting and open question.

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