On the Davis-Wielandt shell of an operator and the Davis-Wielandt index of a normed linear space

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Abstract

We study the Davis-Wielandt shell and the Davis-Wielandt radius of an operator on a normed linear space $\mathcal{X}$. We show that after a suitable modification, the modified Davis-Wielandt radius defines a norm on $\mathcal{L}(\mathcal{X})$ which is equivalent to the usual operator norm on $\mathcal{L}(\mathcal{X})$. We introduce the Davis-Wielandt index of a normed linear space and compute its value explicitly in case of some particular polyhedral Banach spaces. We also present a general method to estimate the Davis-Wielandt index of any polyhedral finite-dimensional Banach space.

Keywords Davis-Wielandt shell · Davis-Wielandt radius · Davis-Wielandt index · Inequalities in normed spaces · Polyhedral Banach spaces · Linear operators

Mathematics Subject Classification Primary 47A12 · Secondary 46B20 · 47L05

1 Introduction

The purpose of this article is to study the Davis-Wielandt shell of an operator between normed linear spaces, and some related geometric and analytic properties. Let us first establish the relevant notations and terminologies to be used throughout the article.

For a complex number $c \in \mathbb{C}$, we write $\mathcal{R}(c)$ and $\mathcal{I}(c)$ for the real part of c and the imaginary part of c , respectively. Let $\mathcal{X}$ be a normed linear space, real or complex and let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$. Let $B_{\mathcal{X}} = \{x \in \mathcal{X} : \|x\| \leq 1\}$ and $S_{\mathcal{X}} = \{x \in \mathcal{X} : \|x\| = 1\}$ be the unit ball and the unit sphere of $\mathcal{X}$, respectively. Given a convex subset K of $\mathcal{X}$, let $\text{ext } K$ denote the set of all extreme points of K . The Krein-Milman

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theorem asserts that $\text{ext } K \neq \emptyset$, if K is non-empty compact. For each $x \in S_{\mathcal{X}}$, let us define $\mathcal{J}(x) = \{f \in \mathcal{X}^* : f(x) = 1 = \|f\|\}$. We denote by Π the set $\{(x, f) : x \in S_{\mathcal{X}}, f \in \mathcal{J}(x)\}$. Let $\mathcal{L}(\mathcal{X})$ be the normed linear space of all bounded linear operators on $\mathcal{X}$, endowed with the usual operator norm. For a complex Hilbert space $\mathcal{H}$, the Davis-Wielandt shell and the Davis-Wielandt radius of an operator $T \in \mathcal{L}(\mathcal{H})$ (see [4, 15]) are defined respectively as:

$$DW(T) = \{(\langle Tx, x \rangle, \|Tx\|^2) : x \in \mathcal{H}, \|x\| = 1\},$$

$$dw(T) = \sup \left\{ \sqrt{|\langle Tx, x \rangle|^2 + \|Tx\|^4} : x \in \mathcal{H}, \|x\| = 1 \right\}.$$

For recent articles about the Davis-Wielandt shell and the Davis-Wielandt radius of an operator, we refer to see [2, 5]. The above concepts are closely related to the numerical range of the operator T , studied initially by Bauer [1] and Lumer [8], in the more general setting of normed linear spaces. Given a $T \in \mathcal{L}(\mathcal{X})$, the numerical range and the numerical radius of T are defined respectively as:

$$W(T) = \{x^*(Tx) : (x, x^*) \in \Pi\},$$

$$w(T) = \sup \{|x^*(Tx)| : (x, x^*) \in \Pi\}.$$

Computation of the numerical index of a normed linear space involves the interaction between the geometry of the space and linear operators acting on it. We refer the readers to [3, 10–14] and references therein for more information on this topic of interest. Given a normed linear space $\mathcal{X}$, the numerical index of $\mathcal{X}$ is defined as:

$$n(\mathcal{X}) = \inf\{w(T) : T \in S_{\mathcal{L}(\mathcal{X})}\}.$$

Motivated by the study on the numerical index of a normed linear space, we introduce another numerical constant associated with a given space, which is related to the study of the Davis-Wielandt shell of an operator and is christened the Davis-Wielandt index. To this end, we first introduce the Davis-Wielandt set of an operator at a point of unit norm. For $T \in \mathcal{L}(\mathcal{X})$, the Davis-Wielandt set of T at the point $x \in S_{\mathcal{X}}$ is defined as:

$$DW(T_x) = \{(x^*(Tx), \|Tx\|^2) : x^* \in \mathcal{J}(x)\}.$$

Next we recall that the Davis-Wielandt shell and corresponding the Davis-Wielandt radius of an operator T on $\mathcal{X}$. The Davis-Wielandt shell and the Davis-Wielandt radius of $T \in \mathcal{L}(\mathcal{X})$ are defined respectively as:

$$DW(T) = \{(x^*(Tx), \|Tx\|^2) : (x, x^*) \in \Pi\},$$

$$dw(T) = \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^4} : (x, x^*) \in \Pi \right\}.$$

It is easy to observe that the Davis-Wielandt radius $dw(\cdot)$ does not define a norm on $\mathcal{L}(\mathcal{X})$. However, the following modification allows us to define a norm on $\mathcal{L}(\mathcal{X})$. For $T \in \mathcal{L}(\mathcal{X})$, the modified Davis-Wielandt shell and the modified Davis-Wielandt radius of T are defined respectively as:

$$DW^*(T) = \{(x^*(Tx), \|Tx\|) : (x, x^*) \in \Pi\},$$

$$dw^*(T) = \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^2} : (x, x^*) \in \Pi \right\}.$$

Now we are in a position to introduce the Davis-Wielandt index and the modified Davis-Wielandt index of a normed linear space. Given a normed linear space $\mathcal{X}$, the Davis-Wielandt index and the modified Davis-Wielandt index of $\mathcal{X}$ are defined respectively as:

$$\eta_{dw}(\mathcal{X}) = \inf \{dw(T) : T \in S_{\mathcal{L}(\mathcal{X})}\},$$

$$\eta_{dw^*}(\mathcal{X}) = \inf \{dw^*(T) : T \in S_{\mathcal{L}(\mathcal{X})}\}.$$

We are especially interested in computing the (modified) Davis-Wielandt index of polyhedral normed linear spaces. Let us first recall that a finite-dimensional real Banach space is said to be polyhedral if $\text{ext } B_{\mathcal{X}}$ is finite. The following definition, introduced in [14], is useful in studying the geometry of polyhedral Banach spaces.

Definition 1.1 Let $\mathcal{X}$ be a polyhedral Banach space. Let F be a facet of the unit ball $B_{\mathcal{X}}$ of $\mathcal{X}$. A functional $f \in S_{\mathcal{X}^*}$ is said to be a *supporting functional corresponding to the facet F of the unit ball $B_{\mathcal{X}}$* if the following two conditions are satisfied:

- (i) f attains norm at some point v of F ,
- (ii) $F = (v + \ker f) \cap S_{\mathcal{X}}$.

For a polyhedral Banach space $\mathcal{X}$, it was shown in [14] that $f \in S_{\mathcal{X}^*}$ is an extreme point of $B_{\mathcal{X}^*}$ if and only if f is a supporting functional corresponding to a facet of $B_{\mathcal{X}}$.

Now we recall the basic definitions related to the concept of smoothness in a normed linear space $\mathcal{X}$. A normed linear space $\mathcal{X}$ is said to be smooth at a non-zero point $x \in \mathcal{X}$ if $\mathcal{J}\left(\frac{x}{\|x\|}\right)$ is singleton. $\mathcal{X}$ is said to be smooth if $\mathcal{X}$ is smooth at each non-zero point $x \in \mathcal{X}$.

This article is demarcated into three sections, including the introductory one. In the second section we explore the convexity properties of the Davis-Wielandt shell of an operator, both from the local and the global perspective. We also study the norming properties of the modified Davis-Wielandt radius in $\mathcal{L}(\mathcal{X})$ and its connection with the usual operator norm. The third section is devoted to the investigation of the Davis-Wielandt index of a normed linear space. We estimate the Davis-Wielandt index of polyhedral Banach spaces. We also obtain the exact values of the Davis-Wielandt index in case of some particular polyhedral Banach spaces.

2 On the Davis-Wielandt shell

We begin this section with the study of the local convexity properties of the Davis-Wielandt set of a bounded linear operator at a point of unit norm. First we prove the following theorem.

Theorem 2.1 *Let $\mathcal{X}$ be a normed linear space. Let $T \in \mathcal{L}(\mathcal{X})$ and let $x \in S_{\mathcal{X}}$. Then $DW(T_x)$ is convex. If $\mathcal{X}$ is finite-dimensional, then $DW(T_x)$ is the convex hull of $\{(x^*(Tx), \|Tx\|^2) : x^* \in \text{ext } B_{\mathcal{X}^*}, x^*(x) = 1\}$.*

Proof Without loss of generality we assume that $DW(T_x)$ is not singleton. Let $(x^*(Tx), \|Tx\|^2), (y^*(Tx), \|Tx\|^2) \in DW(T_x)$, where $x^*, y^* \in \mathcal{J}(x)$. It is easy to see that $\lambda x^* + (1 - \lambda)y^* \in S_{\mathcal{X}^*}$ and $(\lambda x^* + (1 - \lambda)y^*)(x) = 1$, for all $\lambda \in [0, 1]$. Therefore,

$\lambda(x^*(Tx), \|Tx\|^2) + (1 - \lambda)(y^*(Tx), \|Tx\|^2) = ((\lambda x^* + (1 - \lambda)y^*)(Tx), \|Tx\|^2) \in DW(T_x)$, for all $\lambda \in [0, 1]$. So, $DW(T_x)$ is convex.

Let $\dim(\mathcal{X}) = n$ and let $x^* \in \mathcal{J}(x)$. Then x^* can be written as a convex combination of extreme supporting functionals at x . Let x_i^* be the extreme supporting functionals at x , for $i = 1, 2, \dots, m$. Therefore, there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $x^* = \sum_{i=1}^m \lambda_i x_i^*$. So, $x^*(Tx) = \sum_{i=1}^m \lambda_i x_i^*(Tx)$. This proves that $DW(T_x)$ is the convex hull of $\{(x^*(Tx), \|Tx\|^2) : x^* \in \text{ext } B_{\mathcal{X}^*}, x^*(x) = 1\}$. $\square$

Remark 2.2 We would like to remark that if $\mathcal{X}$ is a real normed linear space, then it is easy to observe that $DW(T_x)$ is the line segment joining $(\inf\{x^*(Tx) : x^* \in \text{ext } B_{\mathcal{X}^*}, x^*(x) = 1\}, \|Tx\|^2)$ and $(\sup\{x^*(Tx) : x^* \in \text{ext } B_{\mathcal{X}^*}, x^*(x) = 1\}, \|Tx\|^2)$.

Next we study the convexity properties of the Davis-Wielandt shell of an operator T defined on a normed linear space $\mathcal{X}$. In the case of Hilbert space, this convexity property is well-known (see [7, Th. 2.2 and Th. 2.3], as given in the following theorem.

Theorem 2.3 Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$. Then the following hold true:

- (i) If $\dim(\mathcal{H}) = 2$, then $DW(T)$ is convex if and only if T is normal.
- (ii) If $\dim(\mathcal{H}) \geq 3$, then $DW(T)$ is convex.

However, the above convexity property may not hold true in the more general setting of normed linear spaces. To illustrate this, we first make note of the following proposition.

Proposition 2.4 Let $\mathcal{X}$ be a normed linear space and let $T \in \mathcal{L}(\mathcal{X})$. If $W(T)$ is not convex then $DW(T)$ is not convex.

Proof Since $W(T)$ is not convex, there exist $x^*(Tx), y^*(Ty) \in W(T)$ and $\lambda_0 \in (0, 1)$ such that $\lambda_0 x^*(Tx) + (1 - \lambda_0)y^*(Ty) \notin W(T)$, where $x, y \in S_{\mathcal{X}}$ and $x^* \in \mathcal{J}(x)$, $y^* \in \mathcal{J}(y)$. Therefore, there exists no $z^* \in \mathcal{J}(z)$, $z \in S_{\mathcal{X}}$ such that $\lambda_0 x^*(Tx) + (1 - \lambda_0)y^*(Ty) = z^*(Tz)$. Clearly, $(x^*(Tx), \|Tx\|^2), (y^*(Ty), \|Ty\|^2) \in DW(T)$, but $\lambda_0(x^*(Tx), \|Tx\|^2) + (1 - \lambda_0)(y^*(Ty), \|Ty\|^2) \notin DW(T)$. So, $DW(T)$ is not convex. $\square$

We now show that $DW(T)$ is not necessarily convex for $T \in \mathcal{L}(\mathcal{X})$, even if $\dim(\mathcal{X}) \geq 3$. This illustrates that Theorem 2.3 (ii) cannot be extended to the setting of Banach spaces.

Example 2.5 Let $\mathcal{X} = \ell_p^n$, $p \neq 1, 2, \infty$ and $n \geq 2$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus 0_{n-2 \times n-2}, \quad a, d \in \mathbb{R} \setminus \{0\}, \quad b, c \in \mathbb{C} \setminus \{0\}$$

and (i) $a + d = 0$, (ii) $|b| = |c|$, (iii) $\mathcal{R}(b)\mathcal{I}(c) + \mathcal{R}(c)\mathcal{I}(b) = 0$. Then $DW(T)$ is not convex.

Proof Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Mandal et al. proved in [9, Th. 2.11] that $W(S)$ is not convex. By similar computations we can show that $W(T)$ is not convex. Therefore, it follows from Proposition 2.4 that $DW(T)$ is not convex. $\square$

Our next result relates the smoothness of a normed linear space at a point of unit norm with the local convexity properties of the Davis-Wielandt set of operators on the space.

Proposition 2.6 *Let $\mathcal{X}$ be a normed linear space and let $x \in S_{\mathcal{X}}$. Then x is a smooth point of $\mathcal{X}$ if and only if $DW(T_x)$ is singleton, for every $T \in \mathcal{L}(\mathcal{X})$.*

Proof The necessary part follows trivially. Let us prove only the sufficient part. Let $DW(T_x)$ be singleton, for every $T \in \mathcal{L}(\mathcal{X})$. Suppose on the contrary that x is not a smooth point. Then there exist distinct supporting functionals $x^*, y^* \in \mathcal{J}(x)$. Let $z \in S_{\mathcal{X}}$ be such that $x^*(z) \neq y^*(z)$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that $Tx = z$. Therefore, $x^*(Tx) \neq y^*(Tx)$. Thus, there is an operator $T \in \mathcal{L}(\mathcal{X})$ such that $DW(T_x)$ is not singleton, a contradiction. Hence, x is a smooth point. $\square$

In the next result we show that the modified Davis-Wielandt radius $dw^*(\cdot)$ defines a norm on $\mathcal{L}(\mathcal{X})$, for any normed linear space $\mathcal{X}$.

Theorem 2.7 *Let $\mathcal{X}$ be a normed linear space. Then $dw^*(\cdot)$ defines a norm on $\mathcal{L}(\mathcal{X})$. It is equivalent to the usual operator norm $\|\cdot\|$, satisfying the following inequality:*

$$\|T\| \leq dw^*(T) \leq \sqrt{2}\|T\|, \quad \text{for every } T \in \mathcal{L}(\mathcal{X}).$$

Proof First we prove that $dw^*(\cdot)$ defines a norm on $\mathcal{L}(\mathcal{X})$. Let $T \in \mathcal{L}(\mathcal{X})$. Clearly, $dw^*(T) = 0$ if and only if $T = 0$ and $dw^*(cT) = |c|dw^*(T)$, for all scalars c . Let $T_1, T_2 \in \mathcal{L}(\mathcal{X})$. Then

$$\begin{aligned} & dw^*(T_1 + T_2) \\ &= \sup \left\{ \sqrt{|x^*((T_1 + T_2)x)|^2 + \|(T_1 + T_2)x\|^2} : (x, x^*) \in \Pi \right\} \\ &\leq \sup \left\{ \sqrt{|x^*(T_1x) + x^*(T_2x)|^2 + (\|T_1x\| + \|T_2x\|)^2} : (x, x^*) \in \Pi \right\} \\ &\leq \sup \left\{ \sqrt{|x^*(T_1x)|^2 + \|T_1x\|^2} + \sqrt{|x^*(T_2x)|^2 + \|T_2x\|^2} : (x, x^*) \in \Pi \right\}, \\ &\quad (\text{by Minkowski inequality}) \\ &\leq \sup \left\{ \sqrt{|x^*(T_1x)|^2 + \|T_1x\|^2} : (x, x^*) \in \Pi \right\} \\ &\quad + \sup \left\{ \sqrt{|x^*(T_2x)|^2 + \|T_2x\|^2} : (x, x^*) \in \Pi \right\} \\ &= dw^*(T_1) + dw^*(T_2). \end{aligned}$$

Therefore, $dw^*(T_1 + T_2) \leq dw^*(T_1) + dw^*(T_2)$ for every $T_1, T_2 \in \mathcal{L}(\mathcal{X})$. Hence, $dw^*(\cdot)$ defines a norm on $\mathcal{L}(\mathcal{X})$. Next we prove the other part of the theorem. It follows from the definition of the modified Davis-Wielandt radius that

$$\begin{aligned} dw^*(T) &= \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^2} : (x, x^*) \in \Pi \right\} \\ &\leq \sqrt{w^2(T) + \|T\|^2} \leq \sqrt{\|T\|^2 + \|T\|^2} \leq \sqrt{2}\|T\|. \end{aligned}$$

Also,

$$\begin{aligned} dw^*(T) &= \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^2} : (x, x^*) \in \Pi \right\} \\ &\geq \max \{w(T), \|T\|\} = \|T\|. \end{aligned}$$

This completes the proof. $\square$

Remark 2.8 We would like to remark that the inequality in Theorem 2.7 is sharp. For the identity operator T on $\mathcal{X}$, the second inequality becomes an equality. For a skew-symmetric operator T on a real Hilbert space $\mathcal{H}$, the first inequality becomes an equality.

Now we would like to estimate the Davis-Wielandt radius of an arbitrary operator on a given finite-dimensional Banach space.

Theorem 2.9 Let $\mathcal{X}$ be a finite-dimensional Banach space and let $T \in \mathcal{L}(\mathcal{X})$. Then

$$dw(T) \leq \max \left\{ \sqrt{|x^*(Tx)|^2 + \|T\|^4} : (x, x^*) \in G \right\},$$

where $G = \{(x, x^*) : x \in \text{ext } B_{\mathcal{X}}, x^* \in \text{ext } B_{\mathcal{X}^*}, x^*(x) = 1\}$.

Proof It follows from the definition of the Davis-Wielandt radius of T that

$$\begin{aligned} dw(T) &= \sup \left\{ \sqrt{|y^*(Ty)|^2 + \|Ty\|^4} : (y, y^*) \in \Pi \right\} \\ &\leq \sup \left\{ \sqrt{|y^*(Ty)|^2 + \|T\|^4} : (y, y^*) \in \Pi \right\}. \end{aligned}$$

Since $\mathcal{X}$ is finite dimensional, it follows from Minkowski's theorem that

$$w(T) = \max \{|x^*(Tx)| : (x, x^*) \in G\}.$$

Therefore, for a fixed $(y, y^*) \in \Pi$, we have that

$$|y^*(Ty)|^2 + \|T\|^4 \leq \max \{|x^*(Tx)|^2 + \|T\|^4 : (x, x^*) \in G\}$$

which combined with $dw(T) \leq \sup \left\{ \sqrt{|y^*(Ty)|^2 + \|T\|^4} : (y, y^*) \in \Pi \right\}$ gives the desired inequality. $\square$

3 On the Davis-Wielandt index

We begin this section with the following inequalities for the Davis-Wielandt index and the numerical index of a normed linear space $\mathcal{X}$.

Proposition 3.1 *Let $\mathcal{X}$ be a normed linear space. Then*

$$0 \leq n(\mathcal{X}) \leq 1 \leq \eta_{dw}(\mathcal{X}) \leq \sqrt{n^2(\mathcal{X}) + 1} \leq \sqrt{2}.$$

Proof The inequalities $0 \leq n(\mathcal{X}) \leq 1 \leq \eta_{dw}(\mathcal{X})$ and $\sqrt{n^2(\mathcal{X}) + 1} \leq \sqrt{2}$ follow trivially. We only prove that $\eta_{dw}(\mathcal{X}) \leq \sqrt{n^2(\mathcal{X}) + 1}$. Let $T \in S_{\mathcal{L}(\mathcal{X})}$. Then it follows from the definition of the Davis-Wielandt radius of T that

$$dw(T) = \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^4} : (x, x^*) \in \Pi \right\} \leq \sqrt{w^2(T) + 1}.$$

Taking infimum over all $T \in S_{\mathcal{L}(\mathcal{X})}$, we get $\eta_{dw}(\mathcal{X}) \leq \sqrt{n^2(\mathcal{X}) + 1}$. $\square$

Remark 3.2 (i) For the inequalities $0 \leq n(\mathcal{X}) \leq 1$ to be equalities (in separate cases), we refer the readers to the study of normed spaces with numerical index 0 [11, 12] and normed spaces with numerical index 1 [13], respectively.

(ii) It follows from Proposition 3.1 that if $n(\mathcal{X}) = 0$ then $\eta_{dw}(\mathcal{X}) = 1$. However, $\eta_{dw}(\mathcal{X}) = 1$ does not always imply $n(\mathcal{X}) = 0$. As for example, we consider $\mathcal{H}$ to be the n -dimensional complex Hilbert space, where $n \geq 2$. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0$, be the usual matrix representation of $T \in \mathcal{L}(\mathcal{H})$. It is easy to see that $\|T\| = dw(T) = 1$. Since $\eta_{dw}(\mathcal{H}) \geq 1$, it is now immediate that $\eta_{dw}(\mathcal{H}) = 1$. On the other hand, it is well known that $n(\mathcal{H}) = \frac{1}{2}$.

In our next result we prove that the following equivalent conditions for the Davis-Wielandt index of a finite-dimensional Banach space to be $\sqrt{2}$.

Proposition 3.3 *Let $\mathcal{X}$ be a Banach space. Then the following conditions are equivalent:*

- (i) $\eta_{dw}(\mathcal{X}) = \sqrt{2}$.
- (ii) $n(\mathcal{X}) = 1$.
- (iii) $w(T) = \|T\|$ for all $T \in \mathcal{L}(\mathcal{X})$.

Moreover, if $\mathcal{X}$ is finite-dimensional, then each of the first three conditions are also equivalent to

- (iv) $|x^*(x)| = 1$ for every $x \in \text{ext } B_{\mathcal{X}}$ and for every $x^* \in \text{ext } B_{\mathcal{X}^*}$.

Proof The equivalence of (ii) and (iii) follows trivially. We next prove the equivalence of (i) and (iii).

(i) $\implies$ (iii): Let $\eta_{dw}(\mathcal{X}) = \sqrt{2}$. Then $dw(T) \geq \sqrt{2}$ for all $T \in S_{\mathcal{L}(\mathcal{X})}$. Therefore, $\sqrt{2} \leq dw(T) \leq \sqrt{w^2(T) + \|T\|^4} \leq \sqrt{2}$ for all $T \in S_{\mathcal{L}(\mathcal{X})}$. This implies that $w(T) = 1$ when $\|T\| = 1$. Hence, $w(T) = \|T\|$ for all $T \in \mathcal{L}(\mathcal{X})$.

(iii) $\implies$ (i): Let $T \in \mathcal{L}(\mathcal{X})$. From the definition of $w(T)$ there exists a sequence $(x_n^*, x_n) \in \Pi$ such that $w(T) = \lim_{n \rightarrow \infty} |x_n^*(Tx_n)|$. Since $\{\|Tx_n\|\}$ is a bounded sequence of real numbers so it has a convergent subsequence, say, $\{\|Tx_{n_k}\|\}$. Therefore, $\|T\| = w(T) = \lim_{k \rightarrow \infty} |x_{n_k}^*(Tx_{n_k})| \leq \lim_{k \rightarrow \infty} \|Tx_{n_k}\| \leq \|T\|$. This implies that $\lim_{k \rightarrow \infty} \|Tx_{n_k}\| = \|T\|$. Thus for $T \in S_{\mathcal{L}(\mathcal{X})}$, we get

$$dw(T) \geq \lim_{k \rightarrow \infty} \sqrt{|x_{n_k}^*(Tx_{n_k})|^2 + \|Tx_{n_k}\|^4} = \sqrt{w^2(T) + \|T\|^4} = \sqrt{2}.$$

So, $\eta_{dw}(\mathcal{X}) \geq \sqrt{2}$. Also, it follows from Proposition 3.1 that $\eta_{dw}(\mathcal{X}) \leq \sqrt{2}$. Therefore, $\eta_{dw}(\mathcal{X}) = \sqrt{2}$.

If the space $\mathcal{X}$ is finite-dimensional then the equivalence of (ii) and (iv) follows from [13, Th. 3.1]. $\square$

Remark 3.4 The implication (iv) $\implies$ (ii) may not hold true if $\mathcal{X}$ is infinite-dimensional. As for example, we note from [6] that $\text{ext } B_{c_0(\ell_2)} = \emptyset$ but $n(c_0(\ell_2)) < 1$.

In our next result we obtain a lower bound of the Davis-Wielandt index of polyhedral Banach spaces.

Theorem 3.5 *Let $\mathcal{X}$ be an n -dimensional polyhedral Banach space such that $\text{ext } B_{\mathcal{X}} = \{\pm v_i : i \leq i \leq m\}$. Let $F_{i1}, F_{i2}, \dots, F_{in}$ be any n facets of $B_{\mathcal{X}}$ meeting at v_i . Let $f_{i1}, f_{i2}, \dots, f_{in}$ be the n supporting functionals corresponding to the facets $F_{i1}, F_{i2}, \dots, F_{in}$, respectively. For each $i = 1, 2, \dots, m$, let*

$$\xi_i = \min_{x \in S_{\mathcal{X}}} \left\{ \max_{1 \leq r \leq n} \sqrt{|f_{ir}(x)|^2 + 1} \right\}.$$

Then

$$\eta_{dw}(\mathcal{X}) \geq \min\{\xi_1, \xi_2, \dots, \xi_m\}.$$

Proof It is easy to observe that $x \longrightarrow \max_{1 \leq r \leq n} \left\{ \sqrt{|f_{ir}(x)|^2 + 1} \right\}$, for each $i = 1, 2, \dots, m$, is a continuous function from $S_{\mathcal{X}}$ to $\mathbb{R}$. Since $S_{\mathcal{X}}$ is compact, $\xi_i = \min_{x \in S_{\mathcal{X}}} \left\{ \max_{1 \leq r \leq n} \sqrt{|f_{ir}(x)|^2 + 1} \right\}$ exists. Let $T \in S_{\mathcal{L}(\mathcal{X})}$ be arbitrary. T must attain its norm at an extreme point of $B_{\mathcal{X}}$. Thus, there exists v_j such that $\|Tv_j\| = \|T\| = 1$ for some $j \in \{1, 2, \dots, m\}$. Therefore, we have

$$\begin{aligned} dw(T) &= \sup \left\{ \sqrt{|x^*(Tx)|^2 + \|Tx\|^4} : (x, x^*) \in \Pi \right\} \\ &\geq \max_{1 \leq r \leq n} \left\{ \sqrt{|f_{jr}(Tv_j)|^2 + 1} \right\} \\ &\geq \xi_j. \end{aligned}$$

This gives the desired inequality. $\square$

For the inequality $\eta_{dw}(\mathcal{X}) \leq \sqrt{n^2(\mathcal{X}) + 1}$ to be an equality, we note that a sufficient condition is $n(\mathcal{X}) \in \{0, 1\}$. However, there exist polyhedral Banach spaces $\mathcal{X}$, for which $n(\mathcal{X}) \neq 0, 1$, but the above inequality turns out to be an equality. Before discussing such examples, we refer the readers to [14], where explicit computation of the exact numerical index of certain 3-dimensional Banach spaces was done. Since the computation of the Davis-Wielandt index resembles the computation of the numerical index to a considerable extent, we will not repeat the calculations. However, to show that $\eta_{dw}(\mathcal{X}) = \sqrt{n^2(\mathcal{X}) + 1}$ for the following spaces, denoted in each case by $\mathcal{X}$, we follow the same strategy:

(1) The value of $n(\mathcal{X})$ is known from [14]. We show that $dw(T) \geq \sqrt{n^2(\mathcal{X}) + 1}$, for each $T \in S_{\mathcal{L}(\mathcal{X})}$. In particular, this implies that $\eta_{dw}(\mathcal{X}) \geq \sqrt{n^2(\mathcal{X}) + 1}$.

(2) We explicitly demonstrate an operator $T_0 \in S_{\mathcal{L}(\mathcal{X})}$ for which $dw(T_0) = \sqrt{n^2(\mathcal{X}) + 1}$. This combined with (1) gives the desired equality.

In each case, the operator T_0 is chosen such that $w(T_0) = n(\mathcal{X})$, as given in [14].

We state the following theorems that determine the exact value of the Davis-Wielandt index of some special 3-dimensional polyhedral Banach spaces, the proof of which is clear from the above observations coupled with the results obtained in [14].

Theorem 3.6 *Let $\mathcal{X}$ be a 3-dimensional polyhedral Banach space such that $B_{\mathcal{X}}$ is a polyhedron obtained by gluing two pyramids at the opposite base faces of a right prism having square base, with vertices $\pm(1, 1, 1), \pm(-1, 1, 1), \pm(-1, -1, 1), \pm(1, -1, 1), \pm(0, 0, 2)$. Then*

$$\eta_{dw}(\mathcal{X}) = \frac{\sqrt{5}}{2}.$$

Theorem 3.7 *Let $\mathcal{X}$ be a 3-dimensional polyhedral Banach space such that $B_{\mathcal{X}}$ is a polyhedron with vertices $(\cos(j-1)\frac{\pi}{n}, \sin(j-1)\frac{\pi}{n}, \pm 1), (0, 0, \pm 2), j \in \{1, 2, \dots, 2n\}, n \geq 3$. Then*

$$\eta_{dw}(\mathcal{X}) = \begin{cases} \sqrt{\sin^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is odd} \\ \sqrt{\tan^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is even.} \end{cases}$$

Theorem 3.8 *Let $\mathcal{X}$ be a 3-dimensional polyhedral Banach space such that $B_{\mathcal{X}}$ is a right prism whose base is a regular polygon having $2n$ sides. Then*

$$\eta_{dw}(\mathcal{X}) = \begin{cases} \sqrt{\sin^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is odd} \\ \sqrt{\tan^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is even.} \end{cases}$$

We observe that Theorem 3.8 does not depend on the height h (say) of the prism. In particular, taking $h = 0$, we obtain the Davis-Wielandt index of the corresponding two-dimensional polyhedral Banach spaces. This fact is recorded in the following theorem.

Theorem 3.9 *Let $\mathcal{X}$ be a 2-dimensional polyhedral Banach space such that $B_{\mathcal{X}}$ is a regular $2n$ -gon with vertices $(\cos(j-1)\frac{\pi}{n}, \sin(j-1)\frac{\pi}{n}), j \in \{1, 2, \dots, 2n\}$. Then*

$$\eta_{dw}(\mathcal{X}) = \begin{cases} \sqrt{\sin^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is odd} \\ \sqrt{\tan^2 \frac{\pi}{2n} + 1}, & \text{when } n \text{ is even.} \end{cases}$$

Remark 3.10 Comparing the polyhedral Banach spaces considered in Theorem 3.6 and Theorem 3.7 with $n = 3$, we conclude that non-isometric Banach spaces may have the same Davis-Wielandt index.

Now as an illustrative application of Theorem 3.5 in estimating the Davis-Wielandt index of polyhedral Banach spaces, we consider a family of Banach spaces considered in [10]. For each $\gamma \in (0, 1)$, define a norm $\|\cdot\|_\gamma$ on $\mathcal{X}_\gamma = \{(x, y, 0) : x, y \in \mathbb{R}\}$ as

$$\|(x, y, 0)\|_\gamma = \max \{|y|, |x| + (1 - \gamma)|y|\} \quad \forall (x, y, 0) \in \mathbb{R}^3.$$

The unit ball $B_{\mathcal{X}_\gamma}$ is a hexagon with vertices $\pm(\gamma, 1, 0), \pm(1, 0, 0), \pm(\gamma, -1, 0)$. We can estimate the Davis-Wielandt index of the 3-dimensional polyhedral Banach space $\mathcal{Y}_\gamma$ such that $B_{\mathcal{Y}_\gamma}$ is a right prism whose base is a hexagon $B_{\mathcal{X}_\gamma}$. We identify the extreme functionals of the unit ball of the dual space as the functionals corresponding to the facets of the unit ball of the space and then study the action of these functionals on the unit sphere to obtain a lower bound for the Davis-Wielandt index.

Theorem 3.11 *Let $\mathcal{Y}_\gamma$ be a 3-dimensional polyhedral Banach space such that $B_{\mathcal{Y}_\gamma}$ is a right prism whose base is a hexagon $B_{\mathcal{X}_\gamma}$ defined as above. Then*

$$\eta_{dw}(\mathcal{Y}_\gamma) \geq \begin{cases} \sqrt{\frac{1}{(3-2\gamma)^2} + 1}, & \text{when } 0 < \gamma \leq \frac{1}{2} \\ \sqrt{(1-\gamma)^2 + 1}, & \text{when } \frac{1}{2} \leq \gamma < 1. \end{cases}$$

Proof First we consider the case $0 < \gamma \leq \frac{1}{2}$. Let the vertices of $B_{\mathcal{Y}_\gamma}$ be $v_{\pm j}, j = 1, 2, \dots, 6$, where $v_{\pm 1} = (1, 0, \pm h), v_{\pm 2} = (\gamma, 1, \pm h), v_{\pm 3} = (-\gamma, 1, \pm h), v_{\pm 4} = (-1, 0, \pm h), v_{\pm 5} = (-\gamma, -1, \pm h)$ and $v_{\pm 6} = (\gamma, -1, \pm h), (h > 0)$. The unit sphere $S_{\mathcal{Y}_\gamma}$ is shown in Fig. 1. Let G_j denote the facet of $B_{\mathcal{Y}_\gamma}$ containing $v_j, v_{j+1}, v_{-j}, v_{-j-1}$ where $v_7 = v_1$ and $v_{-7} = v_{-1}$. Let $F_{\pm 1}$ denote the facet of $B_{\mathcal{Y}_\gamma}$ containing $v_{\pm 1}, v_{\pm 2}, v_{\pm 3}, v_{\pm 4}, v_{\pm 5}, v_{\pm 6}$. For each $j \in \{1, 2, \dots, 6\}$, let $g_j, f_{\pm 1}$ be the supporting functionals corresponding to the facets $G_j, F_{\pm 1}$, respectively, i.e., $(v_j + \ker g_j) \cap S_{\mathcal{Y}_\gamma} = G_j$ and $(v_{\pm j} + \ker f_{\pm 1}) \cap S_{\mathcal{Y}_\gamma} = F_{\pm 1}$. For each $j = 1, 2, \dots, 6$, let

Fig. 1 Unit sphere $S_{\mathcal{Y}_\gamma}$

$\xi_j = \min_{x \in S_{y_j}} \max \left\{ \sqrt{|g_j(x)|^2 + 1}, \sqrt{|g_{j-1}(x)|^2 + 1}, \sqrt{|f_1(x)|^2 + 1} \right\}$ where $g_0 = g_6$. We show that $\min\{\xi_j : j = 1, 2, \dots, 6\} \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$. Then, it will follow from Theorem 3.5 that $\eta_{dw}(\mathcal{Y}_\gamma) \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$.

Claim 1. $\xi_1 \geq \sqrt{(1-\gamma)^2 + 1}$.

We have that $\xi_1 = \min_{x \in S_{y_1}} \max \left\{ \sqrt{|g_1(x)|^2 + 1}, \sqrt{|g_6(x)|^2 + 1}, \sqrt{|f_1(x)|^2 + 1} \right\}$. We consider the following cases depending on which part of the unit sphere x belongs to.

Case 1. Let $x \in F_1 \cup F_{-1}$. Since $F_1 = -F_{-1}$, so it is easy to see that $|f_1(x)| = 1$. Therefore, $\sqrt{|f_1(x)|^2 + 1} = \sqrt{2} > \sqrt{(1-\gamma)^2 + 1}$.

Case 2. Let $x \in G_1 \cup G_3 \cup G_4 \cup G_6$. Since $G_1 = -G_4$ and $G_6 = -G_3$, so it is easy to observe that if $x \in G_1 \cup G_4$, then $|g_1(x)| = 1$, i.e., $\sqrt{|g_1(x)|^2 + 1} = \sqrt{2} > \sqrt{(1-\gamma)^2 + 1}$ and if $x \in G_3 \cup G_6$, then $|g_6(x)| = 1$, i.e., $\sqrt{|g_6(x)|^2 + 1} = \sqrt{2} > \sqrt{(1-\gamma)^2 + 1}$.

Case 3. Let $x \in G_2 \cup G_5$. Consider $x = (a, b, c) \in G_2$. Then, $-\gamma \leq a \leq \gamma$ and $b = 1$. Clearly,

$$g_1(1, 0, h) = 1, g_1\left(-\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) = g_1\left(\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right) = 0.$$

$$\text{Now, let } (a, b, c) = \alpha_1(1, 0, h) + \beta_1\left(-\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) + \gamma_1\left(\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right).$$

Then,

$$g_1(x) = g_1(a, b, c) = \alpha_1 = a + (1-\gamma). \text{ Again, clearly, } g_6(1, 0, h) = 1, g_6\left(\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) = g_6\left(-\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right) = 0.$$

Now, let

$$(a, b, c) = \alpha_2(1, 0, h) + \beta_2\left(\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) + \gamma_2\left(-\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right).$$

Then, $g_6(x) = g_6(a, b, c) = \alpha_2 = a - (1-\gamma)$. Thus, for $x = (a, b, c) \in G_2$, $|g_1(x)| \geq 1-\gamma$ if $0 \leq a \leq \gamma$ and $|g_6(x)| \geq 1-\gamma$ if $-\gamma \leq a \leq 0$. So,

$$\sqrt{|g_1(x)|^2 + 1} \geq \sqrt{(1-\gamma)^2 + 1} \text{ if } 0 \leq a \leq \gamma \text{ and}$$

$$\sqrt{|g_6(x)|^2 + 1} \geq \sqrt{(1-\gamma)^2 + 1} \text{ if } -\gamma \leq a \leq 0. \text{ Hence, for all } x \in G_2, \max \left\{ \sqrt{|g_1(x)|^2 + 1}, \sqrt{|g_6(x)|^2 + 1} \right\} \geq \sqrt{(1-\gamma)^2 + 1}.$$

$$\text{Also, } \max \left\{ \sqrt{|g_1(x)|^2 + 1}, \sqrt{|g_6(x)|^2 + 1} \right\} \geq \sqrt{(1-\gamma)^2 + 1} \text{ for all } x \in G_5, \text{ as } G_2 = -G_5.$$

Therefore, combining the above cases we conclude that $\xi_1 \geq \sqrt{(1-\gamma)^2 + 1}$.

Claim 2. $\xi_4 \geq \sqrt{(1-\gamma)^2 + 1}$.

Proceeding as in Claim 1 we can prove that $\xi_4 \geq \sqrt{(1-\gamma)^2 + 1}$.

Claim 3. $\xi_2 \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$.

We have that $\xi_2 = \min_{x \in S_{y_2}} \max \left\{ \sqrt{|g_2(x)|^2 + 1}, \sqrt{|g_1(x)|^2 + 1}, \sqrt{|f_1(x)|^2 + 1} \right\}$. We consider the following cases depending on which part of the unit sphere x belongs to.

Case 1. Let $x \in F_1 \cup F_{-1}$. Since $F_1 = -F_{-1}$, so it is easy to see that $|f_1(x)| = 1$. Therefore, $\sqrt{|f_1(x)|^2 + 1} = \sqrt{2} > \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$.

Case 2. Let $x \in G_1 \cup G_2 \cup G_4 \cup G_5$. Since $G_1 = -G_4$ and $G_2 = -G_5$, so it is easy to observe that if $x \in G_1 \cup G_4$, then $|g_1(x)| = 1$, i.e., $\sqrt{|g_1(x)|^2 + 1} = \sqrt{2} > \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$ and if $x \in G_2 \cup G_5$, then $|g_2(x)| = 1$, i.e., $\sqrt{|g_2(x)|^2 + 1} = \sqrt{2} > \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$.

Case 3. Let $x \in G_3 \cup G_6$. Consider $x = (a, b, c) \in G_3$. Then, $-1 \leq a \leq -\gamma$. Clearly, $g_2(\gamma, 1, h) = 1, g_2(1, 0, h) = g_1(-1, 0, h) = 0$.

Now, let

$$(a, b, c) = \alpha_1(\gamma, 1, h) + \beta_1(1, 0, h) + \gamma_1(-1, 0, h).$$

Then, we have $g_2(x) = g_2(a, b, c) = \alpha_1 = \frac{1}{1-\gamma}(1+a)$. Again,

$$g_1(1, 0, h) = 1,$$

$$g_1\left(-\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) = g_1\left(\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right) = 0.$$

Now, let

$$(a, b, c) = \alpha_2(1, 0, h) + \beta_2\left(-\frac{1}{2}, \frac{1}{2(1-\gamma)}, h\right) + \gamma_2\left(\frac{1}{2}, -\frac{1}{2(1-\gamma)}, h\right).$$

Then, $g_1(x) = g_1(a, b, c) = \alpha_2 = 1 + 2a$. Now, it is easy to see that $\max\{|\alpha_1|, |\alpha_2|\} \geq \frac{1}{3-2\gamma}$.

Therefore, for all $x \in G_3$, $\max\left\{\sqrt{|g_2(x)|^2 + 1}, \sqrt{|g_1(x)|^2 + 1}\right\} \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$. Also, for all $x \in G_6$, $\max\left\{\sqrt{|g_2(x)|^2 + 1}, \sqrt{|g_1(x)|^2 + 1}\right\} \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$, as $G_3 = -G_6$.

Therefore, combining the above three cases we conclude that $\xi_2 \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$.

Claim 4. $\xi_j \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$ for $j = 3, 5, 6$.

Proceeding as in Claim 3 we can prove that $\xi_j \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1}$ for $j = 3, 5, 6$.

Combining the above four claims we have that

$$\min_{1 \leq j \leq 6} \xi_j \geq \min\left\{\sqrt{(1-\gamma)^2 + 1}, \sqrt{\frac{1}{(3-2\gamma)^2} + 1}\right\} = \sqrt{\frac{1}{(3-2\gamma)^2} + 1}.$$

Thus, it follows from Theorem 3.5 that

$$\eta_{dw}(\mathcal{V}_\gamma) \geq \sqrt{\frac{1}{(3-2\gamma)^2} + 1},$$

for $0 < \gamma \leq \frac{1}{2}$. The proof of the theorem can be completed by using analogous arguments when $\frac{1}{2} \leq \gamma < 1$. $\square$

Similarly, for $\xi \in (0, 1)$ we consider $\mathcal{X}_\xi = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and define a norm $\|\cdot\|_\xi$ on $\mathcal{X}_\xi$ as

$$\|(x, y, 0)\|_\xi = \max\left\{|x|, |y|, \frac{|x| + |y|}{1 + \xi}\right\} \quad \forall (x, y, 0) \in \mathbb{R}^3.$$

The unit ball $B_{\mathcal{X}_\xi}$ is an octagon with vertices $\pm(1, \xi, 0), \pm(1, -\xi, 0), \pm(\xi, 1, 0), \pm(\xi, -1, 0)$. Now we state the following theorem, proof of which follows from the similar arguments as in Theorem 3.11.

Theorem 3.12 *Let $\mathcal{Y}_\xi$ be a 3-dimensional polyhedral Banach space such that $B_{\mathcal{Y}_\xi}$ is a right prism whose base is an octagon $B_{\mathcal{X}_\xi}$ defined as above. Then*

$$\eta_{dw}(\mathcal{V}_\xi) \geq \begin{cases} \sqrt{\frac{1}{(2+\xi)^2} + 1}, & \text{when } \xi > \frac{1-\xi}{1+\xi} \\ \sqrt{\left(\frac{1+\xi}{3+\xi}\right)^2 + 1}, & \text{when } \xi \leq \frac{1-\xi}{1+\xi}. \end{cases}$$

We note that all the relevant results in Sect. 3 remain unchanged in case of computing the modified Davis-Wielandt index of the corresponding spaces.

Computation of the exact Davis-Wielandt index of a given normed linear space seems to be a non-trivial problem in the general most case. We end the article with the following open question in this regard, that we could not answer:

Open question: Let $\mathcal{X}$ be a normed linear space. Find separate complete characterizations of the following properties:

- (i) $\eta_{dw}(\mathcal{X}) = 1$,
- (ii) $\eta_{dw}(\mathcal{X}) = \sqrt{n^2(\mathcal{X}) + 1}$.

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