

CARLEMAN ESTIMATES FOR A CLASS OF VARIABLE COEFFICIENT DEGENERATE ELLIPTIC OPERATORS WITH APPLICATIONS TO UNIQUE CONTINUATION

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ABSTRACT. In this paper, we obtain new Carleman estimates for a class of variable coefficient degenerate elliptic operators whose constant coefficient model at one point is the so called Baouendi-Grushin operator. This generalizes the results obtained by the two of us with Garofalo in [9] where similar estimates were established for the "constant coefficient" Baouendi-Grushin operator. Consequently, we obtain: (i) a Bourgain-Kenig type quantitative uniqueness result in the variable coefficient setting; (ii) and a strong unique continuation property for a class of degenerate sublinear equations. We also derive a subelliptic version of a scaling critical Carleman estimate proven by Regbaoui in the Euclidean setting using which we deduce a new unique continuation result in the case of scaling critical Hardy type potentials.

1. INTRODUCTION

In this paper, we study some ad-hoc $L^2 - L^2$ Carleman estimates for operators of the type

$$(1.1) \quad \mathcal{L} = \sum_{i=1}^N X_i(a_{ij}(z, t)X_j),$$

where $(z, t) \in \mathbb{R}^m \times \mathbb{R}^k$, $N = m + k$ and the vector fields X_1, \dots, X_N are given by

$$(1.2) \quad X_i = \partial_{z_i}, i = 1, \dots, m \quad X_{m+j} = |z|^\gamma \partial_{t_j}, j = 1, \dots, k, \gamma > 0.$$

Besides ellipticity, the $N \times N$ matrix valued function $A(z, t) = [a_{ij}(z, t)]$ is required to satisfy some structural assumptions that will be specified in (H) in Section 2 below. Such conditions reduce to the standard Lipschitz continuity when the dimension $k = 0$ or when $\gamma = 0$. One should note that when $A = \mathbb{I}$, the operator in (1.1) reduces to the well known Baouendi-Grushin operator given by

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$$(1.3) \quad \mathcal{B}_\gamma = \Delta_z + |z|^{2\gamma} \Delta_t.$$

The operator \mathcal{B}_γ is degenerate elliptic along $\{z = 0\}$ and it is not translation invariant in \mathbb{R}^N . We recall that a more general class of operators modelled on \mathcal{B}_γ was first introduced by Baouendi who studied the Dirichlet problem in some appropriate weighted Sobolev space in [6]. Subsequently in [28], [29], Grushin analyzed the hypoellipticity of this operator when $\gamma \in \mathbb{N}$. We also refer to [25, 26, 27, 48] for other interesting works related to \mathcal{B}_γ . Remarkably, the operator \mathcal{B}_γ also plays an important role in the work [36] on the higher regularity of the free boundary in the classical Signorini problem.

To provide the reader with some perspective we mention that when $\gamma = 1$ the operator \mathcal{B}_γ is intimately connected to the sub-Laplacians in groups of Heisenberg type. In such Lie groups, in fact, in the exponential coordinates with respect to a fixed orthonormal basis of the Lie algebra, the sub-Laplacian is given by

$$(1.4) \quad \Delta_H = \Delta_z + \frac{|z|^2}{4} \Delta_t + \sum_{\ell=1}^k \partial_{t_\ell} \sum_{i < j} b_{ij}^\ell (z_i \partial_{z_j} - z_j \partial_{z_i}),$$

where b_{ij}^ℓ indicate the group constants. If u is a solution of Δ_H that further annihilates the symplectic vector field $\sum_{\ell=1}^k \partial_{t_\ell} \sum_{i < j} b_{ij}^\ell (z_i \partial_{z_j} - z_j \partial_{z_i})$, then, up to a normalisation factor of 4, u solves the operator \mathcal{B}_γ obtained by letting $\gamma = 1$ in (1.3) above.

Concerning the question of interest for this paper, the unique continuation property, we mention that for general uniformly elliptic equations there are essentially two known methods for proving it. The former is based on Carleman inequalities, which are appropriate weighted versions of Sobolev-Poincaré inequalities. This method was first introduced by T. Carleman in his fundamental work [15] in which he showed that strong unique continuation holds for equations of the type

$$-\Delta u + V u = 0, \quad V \in L_{loc}^\infty(\mathbb{R}^2).$$

In his pioneering work [2], Aronszajn extended such estimates to higher dimensions and uniformly elliptic operators with C^2 principal part. Subsequently, in [4] the authors generalised this result to uniformly elliptic equations with Lipschitz coefficients in the principal part, see also [33]. We stress that unique continuation fails in general when the coefficients of the principal part are only Hölder continuous, see Plis' counterexample in [40], and also [39]. The second approach came up in the works of Garofalo and Lin, see [23], [24]. Their method is based on the almost monotonicity of a generalisation of the frequency function, first introduced by Almgren in [1] for harmonic functions. Using this approach, they were able to obtain new quantitative information on the zero set of solutions to divergence form elliptic equations with Lipschitz coefficients.

The unique continuation property for the degenerate operators \mathcal{B}_γ is much subtler than the one for the Laplacian. It was first established by Garofalo in [21]. In that work he introduced a Almgren type frequency function associated with \mathcal{B}_γ , and proved that such function is monotone non-decreasing on solutions of $\mathcal{B}_\gamma = 0$. These results were extended to more general variable coefficient equations by Garofalo and Vassilev in [32]. One should also see the related works [22] and [30] on the Heisenberg and more general Carnot groups. We also note that a version of the monotonicity formula for \mathcal{B}_γ played an extensive role in the recent work [17] on the obstacle problem for the fractional Laplacian.

Using some ad hoc Carleman estimates in [31] the authors were able to establish for the first time some strong unique continuation results for $\mathcal{B}_\gamma u + V u = 0$ in the difficult situation when V satisfies appropriate L^p integrability hypothesis. Their analysis, which is closer in spirit to the

works [35], [34], [16], [37] to name a few, only covers the special case when $\gamma = k = 1$ in (1.3), and ultimately rests on delicate boundedness properties of certain projector operators generalising some of the results in [46]. We also refer to the recent work of one of us with Mallick in [10] where, using such projector operator estimates, a new $L^2 - L^2$ Carleman estimate is derived. Using the latter, the authors deduce strong unique continuation when the potential V satisfies Hardy type growth assumptions. It is worth mentioning at this point that the general situation of the results in [31] presently remains a challenging open question.

$L^2 - L^2$ Carleman estimates with singular weights for the general Baouendi-Grushin operators \mathcal{B}_γ which are analogous to the ones in [2] have been established very recently by two of us with Garofalo in [9] by using elementary arguments based on integration by parts and by an appropriate application of Rellich type identity. In the same paper, quantitative uniqueness result of Bourgain-Kenig type (see [14]) and a strong unique continuation for a class of sublinear equations of the type (2.26) (when $A \equiv \mathbb{I}$) are also proven.

In the present work, we generalize the results in [9] to variable coefficient principal part where the matrix valued function A is assumed to be Lipschitz continuous with respect to a suitable pseudo-distance associated to the system of vector fields $\{X_i\}$. We refer to (H) below for the precise assumptions. This framework was first introduced by Garofalo and Vassilev in the above cited paper [32]. It is easily seen that in the situation when $k = 0$ the hypothesis (H) below coincides with the usual Lipschitz continuity at the origin of the coefficients a_{ij} . Our Carleman estimates thus encompass those in the cited paper [4]. Our main results Theorem 2.12 and Theorem 2.15 can be seen as the variable coefficient analogues of the corresponding results in [9]. The key to the proof of such results are the Carleman estimates in (3.1) and (3.36) below that we derive. As the reader will see, the proof of these estimates are more involved than that for \mathcal{B}_γ because of the additional error terms that are incurred due to the Lipschitz perturbation of the principal part. Such error terms are eventually handled by a delicate interpolation type argument in the proof of the respective estimates. As an application of our techniques, we also show how to obtain a further refined estimate for zero-order C^1 perturbations of the operator as in (2.24) below which in particular implies a quantitative uniqueness result of Donnelly-Fefferman type (see Theorem 2.14). We mention that the result in Theorem 2.14 has however been previously obtained by one of us with Garofalo in [8] by an adaptation of the Almgren's frequency function approach. Therefore this part of our work can be thought of as an alternate approach to the Donnelly-Fefferman type quantitative uniqueness in this degenerate setting. As a further extension of our techniques, we also establish a subelliptic version of a critical Carleman estimate proven by Regbaoui in [41] for uniformly elliptic operators which in turn implies a certain unique continuation result for equations of the type (2.30) where the potential V satisfies the Hardy type growth assumption as in (2.31) (see Theorem 2.29). We mention that proof of the corresponding estimate in [41] uses in a crucial way the polar decomposition of the frozen constant coefficient operator. Our proof of (2.29) is quite different from that in [41] and is instead based on a suitable adaptation of a Rellich type identity as stated in (3.16) below. Therefore in that sense, the proof of all the Carleman estimates in this paper have a universal character. Over here, we would like to mention that Theorem 2.29 is however slightly weaker than the strong unique continuation property because the hypothesis of the theorem involves a somewhat different notion of vanishing (see (2.32) below). Nevertheless it provides an improvement of Theorem 4.4 in [21]. We refer to Section 2 for further discussions on this topic. Finally we would like to point out that a somewhat technical level, our work also differs additionally from [32] and [8] (which uses the frequency function approach in this variable coefficient setting) in the sense that for the proof of the Carleman estimates, a third derivative estimate of the gauge function ρ as in Lemma 2.8 below is crucial for our analysis. We provide a proof of such an estimate in the Appendix because it involves a long and delicate computation.

The paper is organized as follows. In Section 2, we introduce relevant notions, gather some known results and then state our main results. In Section 3, we prove our main results. In the Appendix, we give a proof of Lemma 2.8.

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2. NOTATIONS AND PRELIMINARY RESULTS

Henceforth in this paper we follow the notations adopted in [21] and [32], with one notable proviso: the parameter $\gamma > 0$ in (1.3), etc. in this paper plays the role of $\alpha > 0$ in [21] and [32]. The reason for this is that we have reserved the greek letter α for the powers of the singular weights in our Carleman estimates. Throughout the paper, whenever convenient, we will use the summation convention over repeated indices. Given a function f , we respectively denote

$$Xf = (X_1f, \dots, X_Nf), \quad |Xf|^2 = \langle Xf, Xf \rangle = \sum_{i=1}^N (X_i f)^2,$$

the intrinsic (degenerate) gradient of a function f , and the square of its length where the vector fields $\{X_i\}$ are defined as in (1.2). We note that the vector fields X_i are homogeneous of degree one with respect to the following family of anisotropic dilations

$$(2.1) \quad \delta_\lambda(z, t) = (\lambda z, \lambda^{\gamma+1}t), \quad \lambda > 0.$$

Consequently, the Baouendi-Grushin operator \mathcal{B}_γ as defined in (1.3) is homogeneous of degree two with respect to (2.1). Let $dzdt$ denote the Lebesgue measure in \mathbb{R}^N . Since $d(\delta_\lambda(z, t)) = \lambda^Q dzdt$, where

$$(2.2) \quad Q = m + (\gamma + 1)k,$$

such number plays the role of a dimension in the analysis of the operator \mathcal{B}_γ as well as \mathcal{L} as in (1.1). For instance, one has the following remarkable fact (see [21]) that the fundamental solution Γ of \mathcal{B}_γ with pole at the origin is given by the formula

$$\Gamma(z, t) = \frac{C}{\rho(z, t)^{Q-2}}, \quad (z, t) \neq (0, 0),$$

where $C > 0$ is suitably chosen and ρ is the pseudo-gauge

$$(2.3) \quad \rho(z, t) = (|z|^{2(\gamma+1)} + (\gamma + 1)^2 |t|^2)^{\frac{1}{2(\gamma+1)}}.$$

A function v is δ_λ -homogeneous of degree κ if and only if $Zv = \kappa v$. Since ρ in (2.3) is homogeneous of degree one, we have

$$(2.4) \quad Z\rho = \rho.$$

We respectively denote by

$$B_r = \{(z, t) \in \mathbb{R}^N \mid \rho(z, t) < r\}, \quad S_r = \{(z, t) \in \mathbb{R}^N \mid \rho(z, t) = r\},$$

the gauge pseudo-ball and sphere centered at 0 with radius r . The infinitesimal generator of the family of dilations (2.1) is given by the vector field

$$(2.5) \quad Z = \sum_{i=1}^m z_i \partial_{z_i} + (\gamma + 1) \sum_{j=1}^k t_j \partial_{t_j}.$$

We note the important facts that

$$(2.6) \quad \operatorname{div} Z = Q, \quad [X_i, Z] = X_i, \quad i = 1, \dots, N.$$

We also need the angle function ψ introduced in [21]

$$(2.7) \quad \psi = |X\rho|^2 = \frac{|z|^{2\gamma}}{\rho^{2\gamma}}.$$

The function ψ vanishes on the characteristic manifold $M = \mathbb{R}^n \times \{0\}$ and clearly satisfies $0 \leq \psi \leq 1$. Since ψ is homogeneous of degree zero with respect to (2.1), one has

$$(2.8) \quad Z\psi = 0.$$

If $f \in C^2(\mathbb{R})$ and $v \in C^2(\mathbb{R}^N)$, then we have the important identities (see [21])

$$(2.9) \quad \mathcal{B}_\gamma f(\rho) = \psi \left(f''(\rho) + \frac{Q-1}{\rho} f'(\rho) \right),$$

and

$$(2.10) \quad \langle Xv, X\rho \rangle = \sum_{i=1}^N X_i v X_i \rho = \frac{\psi}{\rho} Zv.$$

Henceforth, for any two vector fields U and W , $[U, W] = UW - WU$ denotes their commutator.

A first basic assumption on the matrix-valued function $A = [a_{ij}]$ is that it be symmetric and uniformly elliptic. i.e., $a_{ij} = a_{ji}$, $i, j = 1, \dots, N$, and there exists $\lambda > 0$ such that for every $(z, t) \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$ one has

$$(2.11) \quad \lambda|\eta|^2 \leq \langle A(z, t)\eta, \eta \rangle \leq \lambda^{-1}|\eta|^2.$$

Throughout the paper we assume that

$$(2.12) \quad A(0, 0) = I_N,$$

where I_N indicates the identity matrix in \mathbb{R}^N . In order to state our main assumptions (H) on the matrix A it will be useful to represent the latter in the following block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

Here, the entries are respectively $m \times m$, $m \times k$, $k \times m$ and $k \times k$ matrices, and we assume that $A_{12}^t = A_{21}$. We shall denote by B the matrix

$$B = A - I_N,$$

and thus

$$(2.13) \quad B(0, 0) = O_N,$$

thanks to (2.12). We now state the structural assumptions on the matrix A .

HYPOTHESIS. *There exists a positive constant Λ such that one has in B_1 the following estimates*

$$|b_{ij}| = |a_{ij} - \delta_{ij}| \leq \begin{cases} \Lambda\rho, & \text{for } 1 \leq i, j \leq m, \\ \Lambda\psi^{\frac{1}{2} + \frac{1}{2\gamma}}\rho = \Lambda\frac{|z|^{\gamma+1}}{\rho^\gamma}, & \text{otherwise,} \end{cases}$$

(H)

$$|X_k b_{ij}| = |X_k a_{ij}| \leq \begin{cases} \Lambda, & \text{for } 1 \leq k \leq m, \text{ and } 1 \leq i, j \leq m \\ \Lambda\psi^{1 + \frac{1}{2\gamma}} & \text{when } k > m \text{ and } \max\{i, j\} > m \\ \Lambda\psi^{1/2} & \text{otherwise.} \end{cases}$$

An interesting typical example of a matrix A satisfying the conditions (H) is

$$A = \begin{pmatrix} 1 + \rho f(z, t) & |z|^{\gamma+1} g(z, t) \\ |z|^{\gamma+1} g(z, t) & 1 + |z|^{\gamma+1} h(z, t) \end{pmatrix},$$

where f, g and h are Lipschitz continuous near the origin in \mathbb{R}^2 . In this example $m = k = 1$.

We next collect several preliminary results that will be important in our proof. We first consider the quantity

$$(2.14) \quad \mu = \langle AX\rho, X\rho \rangle.$$

In view of the uniform ellipticity of A , we have

$$(2.15) \quad \lambda\psi \leq \mu \leq \lambda^{-1}\psi.$$

The following vector field F will play an important role in the paper:

$$(2.16) \quad F = \frac{\rho}{\mu} \sum_{i,j=1}^N a_{ij} X_i \rho X_j.$$

We note that

$$(2.17) \quad F\rho = \rho.$$

Definition 2.1. *We have*

$$B = A - Id, \quad \sigma = \langle BX\rho, X\rho \rangle.$$

One more notation we will use is: $(b_{ij}) = B$.

In the next theorem we collect several important estimates that have been established in [21] and [32] which will be useful throughout the work.

Theorem 2.2. *There exists a constant $C(\beta, \lambda, \Lambda, N) > 0$ such that for any function u one has:*

- (i) $|Q - \operatorname{div} F| \leq C\rho$;
- (ii) $|F\mu|, |F\psi| \leq C\rho\psi$;
- (iii) $\operatorname{div}\left(\frac{\sigma Z}{\mu}\right) \leq C\rho$;
- (iv) $|X_i \rho| \leq \psi^{1+\frac{1}{2\gamma}}, \quad i = 1, \dots, m, \quad |X_{m+j} \rho| \leq (\gamma+1)\psi^{1/2}, \quad j = 1, \dots, k$;
- (v) $|F - Z| \leq C\rho^2$;
- (vi) $|\langle FAXu, Xu \rangle| \leq C\rho|Xu|^2$;
- (vii) $|[X_i, F]u - X_i u| \leq C\rho|Xu|, \quad i = 1, \dots, N$;
- (viii) $|\sigma| \leq C\rho\psi^{3/2+\frac{1}{2\gamma}}, \quad |X\sigma| \leq C\psi^{3/2}$;
- (ix) $\left|\frac{b_{ij} X_j \rho X_i}{\mu}\right| \leq C|z|$;
- (x) $|X_i \psi| \leq \frac{C\gamma\psi}{|z|}, \quad i = 1, \dots, m, \quad |X_{n+j} \psi| \leq \frac{C\gamma\psi}{\rho}, \quad j = 1, \dots, k$;
- (xi) $\left|\frac{\sigma}{\mu}\right| \leq C\rho\psi, \quad |Z\sigma| \leq C\rho\psi, \quad |X_k \sigma| \leq C\psi^{3/2}$;
- (xii) $|[X_i, -\frac{\sigma Z}{\mu}]u| \leq C\rho|Xu|, \quad (\text{Lemma 2.7 in [32]});$
- (xiii) $|[X_\ell, \frac{\rho}{\mu} \sum_{i,j=1}^N \frac{b_{ij} X_j \rho}{X} X_i]u| \leq C\rho|Xu|, \quad \ell = 1, \dots, N.$

We also need the following lemmas.

Proposition 2.3 (Proposition 3.1, [32]). *We have*

$$X_l \rho = \begin{cases} \psi \frac{z_l}{\rho}, & \text{for } 1 \leq l \leq m \\ (\gamma+1) \psi^{1/2} \frac{t_{l-m}}{\rho^{\gamma+1}}, & \text{for } m+1 \leq l \leq N. \end{cases}$$

Consequently it follows that,

- (1) $|X_i \rho| \leq \psi^{1+\frac{1}{2\gamma}}$ for $1 \leq i \leq m$.
(2) $|X_{n+i} \rho| \leq (\gamma + 1) \psi^{\frac{1}{2}}$ for $1 \leq i \leq k$.

Lemma 2.4. *We have the expressions for the second derivatives of ρ (See, Proposition 3.3 in [32]):*

- (1) For $1 \leq i, j \leq m$, we have:

$$X_i X_j \rho = -(2\gamma + 1) z_i z_j \frac{\psi^2}{\rho^3} + \left[2\gamma \frac{z_i z_j}{|z|^2} + \delta_{ij} \right] \frac{\psi}{\rho}.$$

- (2) For $1 \leq i \leq m$ and $1 \leq j \leq k$, we have:

$$X_i X_{m+j} \rho = -(2\gamma + 1)(\gamma + 1) \frac{z_i t_j \psi^2}{|z|^\gamma \rho^3} + \frac{\psi}{\rho} \left[\gamma(\gamma + 1) \frac{z_i t_j}{|z|^{\gamma+2}} \right].$$

- (3) For $1 \leq i \leq m$ and $1 \leq j \leq k$ we have:

$$X_{m+j} X_i \rho = -(2\gamma + 1)(\gamma + 1) \frac{z_i t_j \psi^2}{|z|^\gamma \rho^3}.$$

- (4) For $1 \leq i, j \leq k$ we have:

$$X_{m+i} X_{m+j} \rho = -(2\gamma + 1)(\gamma + 1)^2 \frac{t_j t_i \psi^2}{|z|^{2\gamma} \rho^3} + (\gamma + 1) \delta_{ij} \frac{\psi}{\rho}.$$

Lemma 2.4 in particular implies the following bounds.

Proposition 2.5 (Proposition 3.3, [32]).

$$\begin{aligned} |X_i X_j \rho| &\leq C \frac{\mu}{\rho} \text{ for } 1 \leq i, j \leq m \text{ or } m+1 \leq i, j \leq N, \\ |X_i X_{m+j} \rho| &\leq C \frac{\mu^{\frac{1}{2}}}{|z|} = C \frac{\mu^{\frac{1}{2}-\frac{1}{2\gamma}}}{\rho} \text{ for } 1 \leq i \leq m, 1 \leq j \leq k, \\ |X_{m+j} X_i \rho| &\leq C \frac{\mu^{\frac{3}{2}|z|}}{\rho^2} = C \frac{\mu^{\frac{3}{2}+\frac{1}{2\gamma}}}{\rho} \text{ for } 1 \leq i \leq m, 1 \leq j \leq k. \end{aligned}$$

Remark 2.6. *It is easily seen that*

$$|X_i b_{ij} X_i \rho| \leq C \psi.$$

We also have (See [32], page 653)

$$\sum_{i,j=1}^N |X_i b_{ij} X_j \rho| \leq C \mu, \quad \text{and} \quad \sum_{i,j=1}^N |b_{ij} X_i X_j \rho| \leq C \mu.$$

Lemma 2.7 (Lemma 3.9, [32]). *If (H) holds then:*

$$|b_{kj} X_j \rho| \leq C \rho \mu^{1+\frac{1}{2\gamma}}.$$

We also need the following third derivative estimate in our analysis.

Lemma 2.8. *Let $F = \frac{\rho}{\mu} \sum a_{qr} X_q \rho X_r$. Then, we have*

$$(2.18) \quad |F(b_{ij} X_i X_j \rho)| \leq C \psi.$$

The proof of the Lemma 2.8 which is based on a long computation is postponed to the appendix.

We then define the relevant function space that is repeatedly used in our work.

Definition 2.9. For a given domain Ω , we denote by $S^{2,2}(\Omega)$, the completion of $C^\infty(\overline{\Omega})$ under the norm

$$\|f\|_{S^{2,2}(\Omega)} = \int_{\Omega} f^2 + |Xf|^2 + \sum_{i,j=1}^N |X_i X_j f|^2$$

We instead indicate with $S_0^{2,2}(\Omega)$ the completion of $C_0^\infty(\Omega)$ under the same norm.

We now introduce the relevant notion of vanishing to infinite order.

Definition 2.10. We say that u vanishes to infinite order at the origin $(0,0)$, if for every $\ell > 0$ one has

$$(2.19) \quad \int_{B_r} |u|^2 \psi = O(r^\ell), \quad \text{as } r \rightarrow 0.$$

Remark 2.11. Throughout this paper, when we say that a constant is universal, we mean that it depends exclusively on m, k, β , on the ellipticity bound λ on $A(z, t)$, see (2.11) above, and on the Lipschitz bound Λ in (H). Likewise, we will say that $O(1)$, $O(r)$, etc. are universal if $|O(1)| \leq C$, $|O(r)| \leq Cr$, etc., with $C \geq 0$ universal.

2.1. Statement of the main results. We now state the main results of the paper. Our first result is the subelliptic analogue of the corresponding quantitative uniqueness result of Bourgain and Kenig for the Euclidean Laplacian, see [14]. We also refer to the work of Bakri [13] for a generalisation of their result to Laplace Beltrami operators on compact manifolds. From now onwards, by \mathcal{L} , we refer to the operator defined in (1.1) where A satisfies the assumptions in (H).

Theorem 2.12. Let $u \in S^{2,2}(B_1)$ with $|u| \leq 1$ be a solution to

$$(2.20) \quad \mathcal{L}u = Vu,$$

where the potential V satisfies the following bound

$$(2.21) \quad |V(z, t)| \leq K\psi.$$

Then, there exists universal $R_0 \in (0, 1/2]$ and constants C_1, C_2 depending also on $\int_{B_{\frac{R_0}{4}}} u^2 \psi$, such that for all $0 < r < \frac{R_0}{9}$ one has

$$(2.22) \quad \|u\|_{L^\infty(B_r)} \geq C_1 \left(\frac{r}{R_0} \right)^{C_2(K^{2/3+1})}.$$

It is worth emphasizing that, when $k = 0$, we have $N = m$ and then from (2.7) we have $\psi \equiv 1$. In such a case the constant K in (2.21) can be taken to be $\|V\|_{L^\infty}$, and Theorem 2.12 reduces to the cited Euclidean result in [14]. We note that the sharpness of the estimate corresponding to (2.22) follows from counterexamples due to Meshkov, see [38].

For zero order C^1 perturbation of the operator \mathcal{L} , we also obtain the following analogue of a Carleman estimate proven by Bakri in [12] for Laplace Beltrami operators on manifolds.

Theorem 2.13. Let $0 < \varepsilon < 1$ be fixed. There exists a universal $R_0 > 0$, depending also on ε , such that for all $R \leq R_0$, $u \in S_0^{2,2}(B_R)$, and V satisfying

$$(2.23) \quad |V| + |FV| \leq K\psi,$$

one has

$$(2.24) \quad \begin{aligned} & \alpha^3 \int \rho^{-2\alpha-4+\varepsilon} u^2 e^{2\alpha\rho^\varepsilon} \mu + \alpha \int_{B_R} \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle dz dt \\ & \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u + Vu)^2 \mu^{-1}, \end{aligned}$$

for universal constants $C, C_1 > 0$ depending also on ε such that

$$\alpha \geq C_1(K^{1/2} + 1).$$

As a consequence of the estimate (2.24), we deduce the following quantitative uniqueness result for "C¹" type potentials V by repeating the arguments as in the proof of Theorem 1.3 in [9].

Theorem 2.14. *Let u solve*

$$\mathcal{L}u + Vu = 0$$

in B_1 where V satisfies (2.23). Then there exists R_0 universal such that for all $r \leq R_0$, we have

$$(2.25) \quad \|u\|_{L^\infty(B_r)} \geq C_1 \left(\frac{r}{R_0} \right)^{C_2(\sqrt{K}+1)},$$

where C_1, C_2 have the same dependence as in Theorem 2.12.

It is to be noted that in the Euclidean case, the constant C can be taken to be the C^1 norm of V . As previously mentioned, Theorem 2.14 has already been proven in [8] differently using a variant of the frequency function approach which in turn is inspired by the work of Zhu in [49] for the standard Laplacian (see also [7] for the extension of the frequency function approach of Zhu to variable coefficients at the boundary). For a historical account, we note that such an estimate in the Euclidean case was first established by Bakri in the above cited paper [12] using the Euclidean version of the Carleman estimate (2.24). Such result generalised the sharp vanishing order estimate of Donnelly and Fefferman in [18] for eigenfunctions of the Laplacian on compact manifolds. Finally, we refer to the paper [43] for an interesting generalisation to nonlocal equations of the quantitative uniqueness result in [12], and also to [5] for a generalisation to Carnot groups of arbitrary step.

We then study strong unique continuation for sublinear equations of the type

$$(2.26) \quad -\mathcal{L}u = f((z, t), u)\psi + Vu,$$

where V satisfies (2.21), and f and its primitive, $G((z, t), s) = \int_0^s f((z, t)s)ds$, satisfies the following assumptions analogous to those in [42] and [45] in the uniformly elliptic case:

$$(2.27) \quad \begin{cases} f((z, t), 0) = 0, \\ 0 < sf((z, t), s) \leq qG((z, t), s), \text{ for some } q \in (1, 2) \text{ and } s \in (-1, 1) \setminus \{0\}, \\ |\nabla_{(z,t)} f| \leq K|f|, \quad |\nabla_{(z,t)} G| \leq KG, \\ f((z, t), s) \leq \kappa s^{p-1} \text{ for some } p \in (1, 2), \\ G((z, t), 1) \geq \varepsilon_0 \text{ for some } \varepsilon_0 > 0. \end{cases}$$

We note that the conditions in (2.27) imply that for some constant c_0, c_1 , we have

$$(2.28) \quad c_1 s^p \geq G(., s) \geq c_0 s^q, \text{ for } s \in (-1, 1).$$

A prototypical f satisfying (2.27) is

$$f((z, t), u) = \sum_{i=1}^{\ell} c_i(z, t)|u|^{q_i-2}u, \text{ where for each } i, q_i \in (1, 2), 0 < k_0 < c_i < k_1, \text{ and } |\nabla c_i| < K,$$

for some k_0, k_1 and K . In this case, we can take $q = \max\{q_i\}$ and $p = \min\{q_i\}$.

Unique continuation properties for uniformly elliptic nonlinear equations of the type

$$-\operatorname{div}(A(x)\nabla u) = f(x, u),$$

with f satisfying the assumptions in (2.27), have been recently studied in [45] and [42]. More precisely, weak unique continuation properties for such sublinear equations were first obtained

in [45] using the frequency function approach. Subsequently, in [42] the author established the strong unique continuation property (see also [44] and [47]). In this work we generalise Rüländ's result to degenerate elliptic equations of the type (2.26). For related results in the parabolic setting, we refer to [3] and [11]. We have the following generalization of the result of Rüländ for the variable coefficient Baouendi-Grushin operators.

Theorem 2.15. *Let $u \in S^{2,2}(B_1)$ be a solution to (2.26) in B_1 where f satisfies the bounds in (2.27) and V satisfies (2.21). Furthermore, assume that $\|u\|_{L^\infty(B_1)} \leq 1$. If u vanishes to infinite order at $(0,0)$ in the sense of Definition 2.10, then $u \equiv 0$.*

The reader should note that the assumption on the sign of f, G in (2.27) is not restrictive because, even in the Euclidean case, the strong unique continuation property fails when $f = -|u|^{q-2}u$ and $A = \mathbb{I}$. This follows from a one dimensional counterexample in [45].

Our final result concerns the following non-Euclidean analogue of a Carleman estimate due to Regbaoui in [41].

Theorem 2.16. *There exists universal $C > 0$ such that for every $\beta > 0$ sufficiently large, R_0 sufficiently small and $u \in S_0^{2,2}(B_R \setminus \{0\})$ with $\text{supp } u \subset (B_R \setminus \{0\})$ for $R \leq R_0$, one has*

$$(2.29) \quad \begin{aligned} & \beta^3 \int_{B_R} \rho^{-Q} e^{\beta(\log \rho)^2} u^2 \mu \, dzdt + \beta \int_{B_R} \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle A Xu, Xu \rangle dzdt \\ & \leq C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} \, dzdt. \end{aligned}$$

As a corollary of the estimate (2.29), we obtain the following unique continuation result for a class of scaling critical zero order perturbations of \mathcal{L} (as in (2.30) below) by an obvious modification of the arguments in [10].

Theorem 2.17. *Let $u \in S^{2,2}(B_1)$ with $|u| \leq 1$ be a solution to*

$$(2.30) \quad \mathcal{L}u = Vu,$$

where the potential V satisfies the following Hardy type growth assumption,

$$(2.31) \quad |V(z, t)| \leq C \frac{\psi}{\rho^2}.$$

Assume that u vanishes at $(0,0)$ in the following sense,

$$(2.32) \quad \int_{B_r} u^2 \psi = O(e^{-k(\log r)^2}), \text{ for all } k > 0.$$

Then $u \equiv 0$.

We note that Theorem 2.17 can be regarded as a slight improvement of Theorem 4.4 in [21] where it is instead assumed that $V = V^+ - V^-$ with

$$0 \leq V^+(z, t) \leq C \frac{\psi}{\rho^2} \text{ and } 0 \leq V^-(z, t) \leq \delta \frac{\psi}{\rho^2}$$

where δ is sufficiently small. Theorem 2.17 thus gets rid of such a smallness assumption. It is to be noted that Theorem 2.17 gives a unique continuation result for (2.30) which is somewhat weaker than strong unique continuation because the notion of vanishing in (2.32) is stronger than the notion of vanishing to infinite order as in Definition 2.10. As previously mentioned in the introduction, the strong unique continuation result for the case of the Hardy type potentials has only been proven when $\gamma = k = 1$ in [10]. The proof of the main Carleman estimate in [10] however relies on some delicate projection operator estimates established previously in [31]. Such estimates are not available for the general Baouendi-Grushin operators and as of now, the

strong unique continuation property for Hardy type potentials in this degenerate setting remains an open question.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 2.12. The proof of Theorem 2.12 is a consequence of the following Carleman estimate following which one can repeat the arguments as in the proof of Theorem 1.3 in [9]. In the Euclidean case, such a Carleman estimate was first established by Escauriaza and Vessella in [20].

Theorem 3.1. *For every $\varepsilon \in (0, 1)$, there exists $C > 0$ sufficiently large and $R_0 > 0$ sufficiently small universal depending also on ε such that for every $\alpha > 0$ sufficiently large (depending also on ε) and $u \in S_0^{2,2}(B_R \setminus \{0\})$ with $\text{supp } u \subset (B_R \setminus \{0\})$ for $R \leq R_0$, one has*

$$(3.1) \quad \begin{aligned} & \alpha^3 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \mu + \alpha \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ & \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u)^2 \mu^{-1}. \end{aligned}$$

Proof. First by a limiting type argument, it suffices to establish the estimate when u is smooth. Let $v = \rho^{-\beta} e^{\alpha\rho^\varepsilon} u$. With such choice we have

$$u = \rho^\beta e^{-\alpha\rho^\varepsilon} v,$$

where β is to be determined later (β would finally depend on α and Q !). Then we have that

$$\begin{aligned} \mathcal{L}u &= X_i(a_{ij}X_ju) = X_i(a_{ij}X_j(\rho^\beta e^{-\alpha\rho^\varepsilon} v)) = X_i \left(a_{ij} \left[(\rho^\beta e^{-\alpha\rho^\varepsilon}) X_j v + X_j (\rho^\beta e^{-\alpha\rho^\varepsilon}) v \right] \right) \\ &= X_i \left(a_{ij} (\rho^\beta e^{-\alpha\rho^\varepsilon}) X_j v \right) + X_i \left(a_{ij} X_j (\rho^\beta e^{-\alpha\rho^\varepsilon}) v \right) \\ &= \mathcal{L}v (\rho^\beta e^{-\alpha\rho^\varepsilon}) + 2a_{ij} X_j v X_i (\rho^\beta e^{-\alpha\rho^\varepsilon}) + X_i (a_{ij} X_j (\rho^\beta e^{-\alpha\rho^\varepsilon})) v \\ &= \mathcal{L}v (\rho^\beta e^{-\alpha\rho^\varepsilon}) + 2a_{ij} X_j v X_i (\rho^\beta e^{-\alpha\rho^\varepsilon}) + \mathcal{L}(\rho^\beta e^{-\alpha\rho^\varepsilon}) v \end{aligned}$$

Now using

$$\mathcal{L} = \mathcal{B}_\gamma + X_i(b_{ij}X_j),$$

we note that the above expression can be further rewritten as

$$\mathcal{L}u = \mathcal{L}v (\rho^\beta e^{-\alpha\rho^\varepsilon}) + 2a_{ij} X_j v X_i (\rho^\beta e^{-\alpha\rho^\varepsilon}) + \mathcal{B}_\gamma (\rho^\beta e^{-\alpha\rho^\varepsilon}) v + X_i(b_{ij}X_j \rho^\beta e^{-\alpha\rho^\varepsilon}) v.$$

Now we calculate last two terms of the right hand side of the above equation. As in [9], we have

$$\mathcal{B}_\gamma (\rho^\beta e^{-\alpha\rho^\varepsilon}) = \left(\alpha^2 \varepsilon^2 \rho^{\beta+2\varepsilon-2} + \beta(\beta + Q - 2) \rho^{\beta-2} - \alpha\varepsilon ((2\beta + \varepsilon + Q - 2)) \rho^{\beta+\varepsilon-2} \right) e^{-\alpha\rho^\varepsilon} \psi,$$

and similarly we have

$$X_i (\rho^\beta e^{-\alpha\rho^\varepsilon}) = \rho^{\beta-1} e^{-\alpha\rho^\varepsilon} (\beta - \alpha\varepsilon\rho^\varepsilon) X_i \rho,$$

and also

$$\begin{aligned} X_i X_j (\rho^\beta e^{-\alpha\rho^\varepsilon}) &= X_i \left[\rho^{\beta-1} e^{-\alpha\rho^\varepsilon} (\beta - \alpha\varepsilon\rho^\varepsilon) X_j \rho \right] \\ &= \rho^{\beta-2} e^{-\alpha\rho^\varepsilon} (\beta(\beta-1) - \alpha\varepsilon(2\beta + \varepsilon - 1) + (\alpha\varepsilon)^2 \rho^{2\varepsilon}) X_i \rho X_j \rho + \rho^{\beta-1} e^{-\alpha\rho^\varepsilon} (\beta - \alpha\varepsilon\rho^\varepsilon) X_i X_j \rho. \end{aligned}$$

Thus it follows,

$$\begin{aligned}
\mathcal{L}u &= \mathcal{L}v(\rho^\beta e^{-\alpha\rho^\epsilon}) + 2a_{ij} X_j v X_i \rho \left[\rho^{\beta-1} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right] \\
&+ \left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta + Q - 2) \rho^{\beta-2} - \alpha\epsilon((2\beta + \epsilon + Q - 2)) \rho^{\beta+\epsilon-2} \right) e^{-\alpha\rho^\epsilon} \psi \right] v \\
&+ \left[X_i b_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) \rho^{\beta-1} e^{-\alpha\rho^\epsilon} X_j \rho \right] v + \left[b_{ij} \left(\rho^{\beta-1} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right) X_i X_j \rho \right] v \\
&+ \left[b_{ij} [\rho^{\beta-2} e^{-\alpha\rho^\epsilon} (\beta(\beta - 1) - \alpha\epsilon\rho^\epsilon(2\beta + \epsilon - 1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho \right] v.
\end{aligned}$$

Now in terms of F and μ , we note that $\mathcal{L}u$ can be written in the following way,

$$\begin{aligned}
(3.2) \quad \mathcal{L}u &= \mathcal{L}v(\rho^\beta e^{-\alpha\rho^\epsilon}) + 2\mu F v \left[\rho^{\beta-2} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right] \\
&+ \left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta + Q - 2) \rho^{\beta-2} - \alpha\epsilon((2\beta + \epsilon + Q - 2)) \rho^{\beta+\epsilon-2} \right) e^{-\alpha\rho^\epsilon} \psi \right] v \\
&+ \left[X_i b_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) \rho^{\beta-1} e^{-\alpha\rho^\epsilon} X_j \rho \right] v + \left[a_{ij} \left(\rho^{\beta-1} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right) X_i X_j \rho \right] v \\
&+ \left[b_{ij} [\rho^{\beta-2} e^{-\alpha\rho^\epsilon} (\beta(\beta - 1) - \alpha\epsilon\rho^\epsilon(2\beta + \epsilon - 1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho \right] v.
\end{aligned}$$

Before proceeding further, we make the following discursive remark.

Remark 3.2. *From now on unless otherwise specified, we will be following the Einstein notation for summation over repeated indices.*

Now using $(a + b)^2 \geq a^2 + 2ab$ with $a = 2\beta \mu F v \rho^{\beta-2}$ and with b being the rest of the terms in (3.2) above, we obtain

$$\begin{aligned}
(3.3) \quad e^{2\alpha\rho^\epsilon} (\mathcal{L}u)^2 &\geq 4\beta^2 \rho^{2\beta-4} \mu^2 (Fv)^2 + 4\beta \mu \rho^{2\beta-2} Fv \mathcal{L}v - 8\alpha\beta \epsilon \mu^2 (Fv)^2 \rho^{2\beta-4+\epsilon} \\
&+ 4\beta \mu \rho^{\beta-2} \left[\left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta + Q - 2) \rho^{\beta-2} - \alpha\epsilon((2\beta + \epsilon + Q - 2)) \rho^{\beta+\epsilon-2} \right) \psi \right] \right. \\
&+ \left[X_i b_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) \rho^{\beta-1} X_j \rho \right] + \left[b_{ij} \left(\rho^{\beta-1} (\beta - \alpha\epsilon\rho^\epsilon) \right) X_i X_j \rho \right] \\
&\left. + \left[b_{ij} [\rho^{\beta-2} (\beta(\beta - 1) - \alpha\epsilon\rho^\epsilon(2\beta + \epsilon - 1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho \right] \right] Fv \cdot v.
\end{aligned}$$

Hence from (3.3) we have,

(3.4)

$$\begin{aligned}
& \int \rho^{-2\alpha} e^{2\alpha\rho^\epsilon} (\mathcal{L}u)^2 \mu^{-1} \\
& \geq \int [4\beta^2 \rho^{-2\alpha+2\beta-4} \mu(Fv)^2 - 8\alpha\beta \epsilon \rho^{-2\alpha+2\beta-4+\epsilon} \mu(Fv)^2] + \int 4\beta \rho^{-2\alpha+2\beta-2} Fv \mathcal{L}v \\
& + \int 4\beta \rho^{-2\alpha+\beta-2} \left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta+Q-2) \rho^{\beta-2} - \alpha\epsilon((2\beta+\epsilon+Q-2)) \rho^{\beta+\epsilon-2} \right) \psi \right] \\
& + [X_i b_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) \rho^{\beta-1} X_j \rho] + [a_{ij} (\rho^{\beta-1} (\beta - \alpha\epsilon\rho^\epsilon)) X_i X_j \rho] \\
& + [b_{ij} [\rho^{\beta-2} (\beta(\beta-1) - \alpha\epsilon\rho^\epsilon (2\beta+\epsilon-1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho] Fv \cdot v \\
& = \int [4\beta^2 \rho^{-2\alpha+2\beta-4} \mu(Fv)^2 - 8\alpha\beta \epsilon \mu(Fv)^2 \rho^{-2\alpha+2\beta-4+\epsilon}] + \int 4\beta \rho^{-2\alpha+2\beta-2} Fv \mathcal{L}v \\
& + \int 4\beta \rho^{-2\alpha+2\beta-3} [X_i b_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) X_j \rho] Fv \cdot v, \\
& + \int 4\beta \rho^{-2\alpha+\beta-2} \left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta+Q-2) \rho^{\beta-2} - \alpha\epsilon(2\beta+\epsilon+Q-2t) \rho^{\beta+\epsilon-2} \right) \psi \right] \\
& + [b_{ij} [\rho^{\beta-2} (\beta(\beta-1) - \alpha\epsilon\rho^\epsilon (2\beta+\epsilon-1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho] + [b_{ij} (\rho^{\beta-1} (\beta - \alpha\epsilon\rho^\epsilon)) X_i X_j \rho] F \left(\frac{v^2}{2} \right)
\end{aligned}$$

The following integral in (3.4) above, i.e.

(3.5)

$$\begin{aligned}
& \int 4\beta \rho^{-2\alpha+\beta-2} \left[\left(\alpha^2 \epsilon^2 \rho^{\beta+2\epsilon-2} + \beta(\beta+Q-2) \rho^{\beta-2} - \alpha\epsilon((2\beta+\epsilon+Q-2)) \rho^{\beta+\epsilon-2} \right) \psi \right] \\
& + [b_{ij} [\rho^{\beta-2} (\beta(\beta-1) - \alpha\epsilon\rho^\epsilon (2\beta+\epsilon-1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho] + [b_{ij} (\rho^{\beta-1} (\beta - \alpha\epsilon\rho^\epsilon)) X_i X_j \rho] F \left(\frac{v^2}{2} \right).
\end{aligned}$$

is handled using integration by parts as follows.

We first observe from (i) in Theorem 2.2 and also by using $F\rho = \rho$ that for every $\gamma \geq 0$,

$$(3.6) \quad \operatorname{div}(\rho^{-Q+\gamma} F) = \gamma \rho^{-Q+\gamma} + O(1) \rho^{-Q+\gamma+1}.$$

Now we look at each individual term in (3.5). We let

$$(3.7) \quad \beta = \frac{2\alpha + 4 - Q}{2},$$

which gives $2\beta - 2\alpha - 4 = -Q$. With such a choice, it follows using Theorem 2.2 i) and ii) that the following holds,

(3.8)

$$2\beta^2 (\beta+Q-2) \int \rho^{2\beta-2\alpha-4} F(v^2) \psi = 2\beta^2 (\beta+Q-2) \int \operatorname{div}(\rho^{-Q} F \psi) v^2 = O(1) \beta^2 (\beta+Q-2) \int \rho^{-Q+1} v^2 \psi.$$

Similarly we have,

(3.9)

$$-2\beta\alpha\epsilon(2\beta+\epsilon+Q-2) \int \rho^{2\beta-2\alpha-4+\epsilon} F(v^2) \psi = 2\beta\alpha\epsilon^2(2\beta+\epsilon+Q-2) \left(\int \rho^{-Q+\epsilon} v^2 \psi + O(1) \int \rho^{-Q+\epsilon+1} v^2 \psi \right),$$

and

$$(3.10) \quad 2\beta(\alpha\epsilon)^2 \int \rho^{2\beta-2\alpha-4+2\epsilon} F(v^2) \psi = -2\beta\alpha^2\epsilon^3 \left(\int \rho^{-Q+2\epsilon} v^2 \psi + O(1) \int \rho^{-Q+2\epsilon+1} v^2 \psi \right).$$

Since $\varepsilon < 1$, thus from (3.8)-(3.10) we deduce the following estimate for all R small enough depending also on ε ,

$$(3.11) \quad \begin{aligned} & \int 4\beta\rho^{-2\alpha+\beta-2} \left[\left[\left(\alpha^2\varepsilon^2\rho^{\beta+2\varepsilon-2} + \beta(\beta+Q-2)\rho^{\beta-2} - \alpha\varepsilon(2\beta+\varepsilon+Q-2)\rho^{\beta+\varepsilon-2} \right) \psi \right] Fv \cdot v \right. \\ & \geq c\beta^3\varepsilon^2 \int \rho^{-Q+\varepsilon}v^2\psi, \end{aligned}$$

for some universal $c > 0$.

Again by integrating by parts and by using $|F(b_{ij}X_i\rho X_j\rho)| \leq C\rho\psi$ and the Hypothesis (H), we note that the following holds,

$$\begin{aligned} & \int 4\beta\rho^{-2\alpha+\beta-2} \left[b_{ij}[\rho^{\beta-2}(\beta(\beta-1) - \alpha\varepsilon\rho^\varepsilon(2\beta+\varepsilon-1) + (\alpha\varepsilon)^2\rho^{2\varepsilon})]X_i\rho X_j\rho \right] F \left(\frac{v^2}{2} \right) \\ & = - \int \operatorname{div} \left(4\beta\rho^{-2\alpha+\beta-2} \left[b_{ij}[\rho^{\beta-2}(\beta(\beta-1) - \alpha\varepsilon\rho^\varepsilon(2\beta+\varepsilon-1) + (\alpha\varepsilon)^2\rho^{2\varepsilon})]X_i\rho X_j\rho \right] F \right) \frac{v^2}{2} \\ & \geq -C\beta^3 \int \rho^{-Q+1}v^2\psi, \end{aligned}$$

for some universal C . Likewise, we have that

$$(3.12) \quad \begin{aligned} & \int 4\beta\rho^{-2\alpha+\beta-2} \left[b_{ij} \left(\rho^{\beta-1} (\beta - \alpha\varepsilon\rho^\varepsilon) \right) X_i X_j \rho \right] F \left(\frac{v^2}{2} \right) \\ & = - \int 4\beta \operatorname{div} \left(\rho^{-2\alpha+\beta-2} \left[b_{ij} \left(\rho^{\beta-1} (\beta - \alpha\varepsilon\rho^\varepsilon) \right) X_i X_j \rho \right] F \right) \frac{v^2}{2} \\ & \geq -C\beta^2 \int \rho^{-Q+1}v^2\psi. \end{aligned}$$

We note that in (3.12) above, we also used the estimate from Lemma 2.8 and also that $|b_{ij}X_iX_j\rho| \leq C\psi$. Thus from (3.11)-(3.12) it follows that

$$(3.13) \quad \begin{aligned} & \int 4\beta\rho^{-2\alpha+\beta-2} \left[\left[\left(\alpha^2\varepsilon^2\rho^{\beta+2\varepsilon-2} + \beta(\beta+Q-2)\rho^{\beta-2} - \alpha\varepsilon((2\beta+\varepsilon+Q-2))\rho^{\beta+\varepsilon-2} \right) \psi \right] \right. \\ & + \left. \left[b_{ij}[\rho^{\beta-2}(\beta(\beta-1) - \alpha\varepsilon\rho^\varepsilon(2\beta+\varepsilon-1) + (\alpha\varepsilon)^2\rho^{2\varepsilon})]X_i\rho X_j\rho \right] + \left[b_{ij} \left(\rho^{\beta-1} (\beta - \alpha\varepsilon\rho^\varepsilon) \right) X_i X_j \rho \right] \right] F \left(\frac{v^2}{2} \right) \\ & \geq c\beta^3\varepsilon^2 \int \rho^{-Q+\varepsilon}v^2\psi, \end{aligned}$$

provided R is chosen small enough depending also on ε .

Now using $\sum_{i,j=1}^N |X_i b_{ij} X_j \rho| \leq C\mu$, we obtain by an application of Cauchy-Schwartz inequality that the following holds

$$(3.14) \quad \begin{aligned} & \int 4\beta\rho^{-2\alpha+2\beta-3} [X_i b_{ij} \cdot (\beta - \alpha\varepsilon\rho^\varepsilon) X_j \rho] Fv \cdot v \\ & \leq \frac{1}{2} \left[\beta^2 \int \rho^{-2\alpha+2\beta-4} \mu(Fv)^2 dz dt + 8\beta^2 \int \rho^{-2\alpha+2\beta-2} v^2 \mu \right] \\ & = \frac{1}{2} \left[\beta^2 \int \rho^{-Q} \mu(Fv)^2 + 8\beta^2 \int \rho^{-Q+2} v^2 \mu \right]. \end{aligned}$$

Again by using the Cauchy-Schwartz inequality, we note that if R is chosen sufficiently small depending also on ε , then for all large enough β we can ensure that

$$(3.15) \quad \begin{aligned} & \int [4\beta^2 \rho^{-2\alpha+2\beta-4} \mu(Fv)^2 - 8\alpha\beta \epsilon \rho^{-2\alpha+2\beta-4+\epsilon} \mu(Fv)^2] \\ & \geq 3\beta^2 \int \rho^{-Q} (Fv)^2 \mu. \end{aligned}$$

We finally estimate the remaining integral in (3.4), i.e.

$$\int 4\beta \rho^{-2\alpha+2\beta-2} Fv \mathcal{L}v.$$

This is accomplished using an appropriate Rellich type identity similar to that for the constant coefficient case considered in [9]. However in the present scenario of variable coefficients, as the reader will see that an error term is incurred after the application of such an identity which is then eventually handled by an interpolation type argument. We first note that with our choice of α as in (3.7), such an integral equals

$$\int 4\beta \rho^{-Q+2} Fv \mathcal{L}v.$$

We now state the relevant Rellich type identity(see for instance Lemma 2.11 in [32]) which will be used:

$$(3.16) \quad \begin{aligned} & \int_{\partial B_R} \langle AXu, Xu \rangle \langle G, \nu \rangle = 2 \int_{\partial B_R} a_{ij} X_i u \langle X_j, \nu \rangle G u \\ & - 2 \int_{B_R} a_{ij} (\operatorname{div} X_i) X_j u G u - 2 \int_{B_R} a_{ij} X_i u [X_j, G] u \\ & + \int_{B_R} \operatorname{div} G \langle AXu, Xu \rangle + \int_{B_R} \langle (GA)Xu, Xu \rangle - 2 \int_{B_R} G u X_i (a_{ij} X_j u), \end{aligned}$$

where G is a vector field, GA is the matrix with coefficients Ga_{ij} , ν denotes the outer unit normal to B_r , and the summation convention over repeated indices has been adopted. Therefore with $G = \rho^{-Q+2}F$, we obtain

$$(3.17) \quad \begin{aligned} 4\beta \int \rho^{-Q+2} Fv \mathcal{L}v & = 2\beta \int \operatorname{div}[\rho^{-Q+2}F] \langle AXv, Xv \rangle - 4\beta \int a_{ij} X_i v [X_i, \rho^{-Q+2}F] v \\ & + 2\beta \int \langle (\rho^{-Q+2}FA)Xv, Xv \rangle \\ & \geq 4\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle - C\beta \int \rho^{-Q+3} \langle AXv, Xv \rangle - 4\beta \int a_{ij} X_i v [X_i, \rho^{-Q+2}F] v. \end{aligned}$$

In the last step we used the fact that $\operatorname{div}[\rho^{-Q+2}F] = 2\rho^{-Q+2} + O(1)\rho^{-Q+3}$ and $|F a_{rs}| \leq C\rho$. We next note that

$$\begin{aligned} [X_i, G]v & = \rho^{-Q+2} [X_i, F]v + X_i [\rho^{-Q+2}] Fv \\ & = \rho^{-Q+2} [X_i, F]v + (-Q+2)\rho^{-Q+1} X_i \rho Fv. \end{aligned}$$

This gives

$$\begin{aligned} a_{ij} X_i v [X_j, G]v & = (-Q+2)\rho^{-Q+1} \langle AX\rho, Xv \rangle Fv + \rho^{-Q+2} a_{ij} X_i v [X_j, F]v \\ & = (-Q+2)\rho^{-Q} (Fv)^2 \mu + \rho^{-Q+2} a_{ij} X_i v ([X_j, F]v - X_j v) \\ & \quad + \rho^{-Q+2} \langle AXv, Xv \rangle, \end{aligned}$$

where we have used the fact that

$$\rho \langle AX\rho, Xu \rangle = \mu Fu.$$

Therefore using the estimate in Theorem 2.2 vii) we obtain that for some universal C the following holds,

$$(3.18) \quad 4\beta \int \rho^{-Q+2} Fv \mathcal{L}v \geq 4\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle \\ - 4\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle - C\beta \int \langle AXv, Xv \rangle \rho^{-Q+3} + 2\beta(Q-2) \int_{B_r} (Fv)^2 \rho^{-Q} \mu.$$

Since $Q \geq 2$, therefore from (3.18) we deduce that the following holds,

$$(3.19) \quad 4\beta \int \rho^{-Q+2} Fv \mathcal{L}v + C\beta \int \rho^{-Q+3} \langle AXv, Xv \rangle \geq 0.$$

Therefore, by combining (3.13), (3.14), (3.15) and (3.19), we finally obtain that

$$(3.20) \quad \int \rho^{-2\alpha} e^{2\alpha\rho^\epsilon} (\mathcal{L}u)^2 \mu^{-1} \\ \geq 3\beta^2 \int \rho^{-Q} (Fv)^2 \mu - C\beta \int \rho^{-Q+3} \langle AXv, Xv \rangle + c\beta^3 \epsilon^2 \int \rho^{-Q+\epsilon} v^2 \mu$$

where C and c are universal constants. We now estimate the term

$$C\beta \int \rho^{-Q+3} \langle AXv, Xv \rangle$$

in (3.20) using an interpolation type argument. We first rewrite such an integral in terms of $\int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle AXu, AXu \rangle$ as follows,

$$(3.21) \quad \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle = \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle AX(\rho^\beta e^{-\alpha\rho^\epsilon} v), X(\rho^\beta e^{-\alpha\rho^\epsilon} v) \rangle \\ = \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle A(\beta - \alpha\epsilon\rho^\epsilon)\rho^{\beta-1} e^{-\alpha\rho^\epsilon} X\rho v + \rho^\beta e^{-\alpha\rho^\epsilon} AXv, (\beta - \alpha\epsilon\rho^\epsilon)\rho^{\beta-1} e^{-\alpha\rho^\epsilon} X\rho v + \rho^\beta e^{-\alpha\rho^\epsilon} Xv \rangle \\ = \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \left[\mu \rho^{2\beta-2} e^{-2\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon)^2 v^2 + 2\mu Fv v (\rho^{2\beta-2} (\beta - \alpha\epsilon\rho^\epsilon) e^{-2\alpha\rho^\epsilon}) + (e^{-2\alpha\rho^\epsilon} \rho^{2\beta}) \langle AXv, Xv \rangle \right] \\ = \int \rho^{-Q+3} \left[\frac{\mu (\beta - \alpha\epsilon\rho^\epsilon)^2}{\rho^2} v^2 + 2\mu \frac{(\beta - \alpha\epsilon\rho^\epsilon)}{\rho^2} Fv v + \langle AXv, Xv \rangle \right] \\ = \int \rho^{-Q+3} \left[\frac{\mu (\beta - \alpha\epsilon\rho^\epsilon)^2}{\rho^2} v^2 - \mu [\rho^{-2} (\beta - \alpha\epsilon\rho^\epsilon) \operatorname{div}(F) + \rho^{-2} (\beta(-Q+1) + \alpha\epsilon(-Q+1+\epsilon))] v^2 + \langle AXv, Xv \rangle \right] \\ = \int \rho^{-Q+3} \left[\frac{\mu (\beta - \alpha\epsilon\rho^\epsilon)^2}{\rho^2} v^2 - \mu [\rho^{-2} (\beta - \alpha\epsilon\rho^\epsilon) (Q + O(\rho)) \right. \\ \left. + \rho^{-2} (\beta(-Q+1) + \alpha\epsilon(-Q+1+\epsilon))] v^2 + \langle AXv, Xv \rangle \right].$$

Thus we get for some universal C_1 that

$$(3.22) \quad \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle \geq \int \rho^{-Q+3} \langle AXv, Xv \rangle - C_1 \beta \int \rho^{-Q+1} v^2.$$

Using (3.22) in (3.20) we obtain that for all large enough β depending also on ε we have,

$$(3.23) \quad \begin{aligned} & \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u)^2 \mu^{-1} + C_1 \beta \int \rho^{-2\alpha-1} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ & \geq 3\beta^2 \int \rho^{-Q} (Fv)^2 \mu + c\beta^3 \varepsilon^2 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \mu. \end{aligned}$$

We now show how to incorporate the integral $C\beta \int_{B_R} \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle$ in the left hand side of (3.1) by interpolation.

We have by integration by parts,

$$(3.24) \quad \begin{aligned} & \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle = \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AX(\rho^\beta e^{-\alpha\rho^\varepsilon} v), X(\rho^\beta e^{-\alpha\rho^\varepsilon} v) \rangle \\ & = - \int \langle X(\rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon}), AX(\rho^\beta e^{-\alpha\rho^\varepsilon} v) \rangle [\rho^\beta e^{-\alpha\rho^\varepsilon} v] - \int \mathcal{L}(\rho^\beta e^{-\alpha\rho^\varepsilon} v) v [\rho^{-2\alpha+\beta-2+\varepsilon} e^{\alpha\rho^\varepsilon}]. \end{aligned}$$

Now we look at each individual term on the right hand side of (3.24). We have

$$(3.25) \quad \begin{aligned} & - \int \langle X(\rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon}), AX(\rho^\beta e^{-\alpha\rho^\varepsilon} v) \rangle \rho^\beta e^{-\alpha\rho^\varepsilon} v = - \int e^{2\alpha\rho^\varepsilon} \\ & [\rho^{-2\alpha-3+\varepsilon} (-2\alpha-2+\varepsilon+2\alpha\varepsilon\rho^\varepsilon)] \langle X\rho, A[\rho^\beta e^{-\alpha\rho^\varepsilon} Xv + (\beta-\alpha\varepsilon\rho^\varepsilon) \rho^{\beta-1} e^{-\alpha\rho^\varepsilon} X\rho v] (\rho^\beta e^{-\alpha\rho^\varepsilon} v) \rangle \\ & = - \int \rho^{-2\alpha+2\beta-4+\varepsilon} (-2\alpha-2+\varepsilon+2\alpha\varepsilon\rho^\varepsilon) [(\beta-\alpha\varepsilon\rho^\varepsilon) \mu v^2 + \mu Fv \cdot v] dzdt \\ & = \int \rho^{-Q+\varepsilon} (2\alpha+2-\varepsilon-2\alpha\varepsilon\rho^\varepsilon) (\beta-\alpha\varepsilon\rho^\varepsilon) \mu v^2 \\ & + \int \rho^{-Q+\varepsilon} (2\alpha+2-\varepsilon-2\alpha\varepsilon\rho^\varepsilon) \mu Fv \cdot v. \end{aligned}$$

Let $c_0 = \frac{c}{100}$ where c is as in (3.23). From (3.25) it follows that

$$(3.26) \quad \begin{aligned} & c_0 \beta \varepsilon^2 \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \leq c_0 \beta \varepsilon^2 \int \rho^{-Q+\varepsilon} (2\alpha+2-\varepsilon-2\alpha\varepsilon\rho^\varepsilon) (\beta-\alpha\varepsilon\rho^\varepsilon) \mu v^2 dzdt \\ & + c_0 \beta \varepsilon^2 \left| \int \rho^{-Q+\varepsilon} (2\alpha+2-\varepsilon-2\alpha\varepsilon\rho^\varepsilon) \mu Fv \cdot v \right| + c_0 \beta \varepsilon^2 \left| \int \mathcal{L}(\rho^\beta e^{-\alpha\rho^\varepsilon} v) v [\rho^{-2\alpha+\beta-2+\varepsilon} e^{\alpha\rho^\varepsilon}] \right|. \end{aligned}$$

Now by applying Cauchy-Schwartz inequality to the integrals

$$c_0 \beta \varepsilon^2 \left| \int \rho^{-Q+\varepsilon} (2\alpha+2-\varepsilon-2\alpha\varepsilon\rho^\varepsilon) \mu Fv \cdot v \right| \text{ and } c_0 \beta \varepsilon^2 \left| \int \mathcal{L}(\rho^\beta e^{-\alpha\rho^\varepsilon} v) v [\rho^{-2\alpha+\beta-2+\varepsilon} e^{\alpha\rho^\varepsilon}] \right|,$$

we obtain from (3.26) that the following inequality holds for all large enough β ,

$$(3.27) \quad \begin{aligned} & c_0 \beta \varepsilon^2 \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \leq 4c_0 \beta^3 \varepsilon^2 \int \rho^{-Q+\varepsilon} v^2 \mu \\ & + 4c_0 \beta^2 \varepsilon^2 \int \rho^{-Q} (Fv)^2 \mu + c_0 \varepsilon^2 \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u)^2 \mu^{-1} \end{aligned}$$

Using (3.23) into (3.27) above, we obtain

$$(3.28) \quad \begin{aligned} & c_0 \beta \varepsilon^2 \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ & \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u)^2 \mu^{-1} + C\beta \int \rho^{-2\alpha-1} \langle AXu, Xu \rangle. \end{aligned}$$

Now since $\varepsilon < 1$, therefore if R is chosen small enough depending also on ε , then the following integral in (3.28), i.e.

$$C\beta \int \rho^{-2\alpha-1} \langle AXu, Xu \rangle,$$

can be absorbed in the left hand side of (3.28) since $\varepsilon < 1$ and we thus deduce that the following holds for some new c_0, C ,

$$(3.29) \quad \begin{aligned} c_0\beta\varepsilon^2 \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u)^2 \mu^{-1}. \end{aligned}$$

The estimate (3.1) now follows by using (3.29) in (3.23). \square

Proof of Theorem 2.13.

Proof. As before, we let $u = \rho^\beta e^{\alpha\rho^\varepsilon} v$ where α and β are related as in (3.7). In terms of v , we have that

$$(3.30) \quad \begin{aligned} \mathcal{L}u + Vu &= \mathcal{L}v(\rho^\beta e^{-\alpha\rho^\varepsilon}) + 2\mu Fv \left[\rho^{\beta-2} e^{-\alpha\rho^\varepsilon} (\beta - \alpha\varepsilon\rho^\varepsilon) \right] + \left[X_i a_{ij} \cdot (\beta - \alpha\varepsilon\rho^\varepsilon) \rho^{\beta-1} e^{-\alpha\rho^\varepsilon} X_j \rho \right] v \\ &\quad + \left[a_{ij} [\rho^{\beta-2} e^{-\alpha\rho^\varepsilon} (\beta(\beta-1) - \alpha\varepsilon\rho^\varepsilon (2\beta + \varepsilon - 1) + (\alpha\varepsilon)^2 \rho^{2\varepsilon})] X_i \rho X_j \rho \right] v \\ &\quad + \left[a_{ij} \left(\rho^{\beta-1} e^{-\alpha\rho^\varepsilon} (\beta - \alpha\varepsilon\rho^\varepsilon) \right) X_i X_j \rho \right] v + V\rho^\beta e^{-\alpha\rho^\varepsilon} v. \end{aligned}$$

Now again the integral

$$\int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u + Vu)^2 \mu^{-1}$$

is estimated from below by using $(a+b)^2 \geq a^2 + 2ab$, with $a = 2\beta\rho^{\beta-2}\mu Fv$ and b being the rest of the terms in (3.30). Arguing as in the proof of Theorem 3.1 we obtain,

$$(3.31) \quad \begin{aligned} \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u + Vu)^2 \mu^{-1} + C_1\beta \int \rho^{-2\alpha-1} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ \geq 3\beta^2 \int \rho^{-Q} (Fv)^2 \mu + c\beta^3\varepsilon^2 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \mu + 4\beta \int \rho^{-2\alpha+2\beta-2} FvVv. \end{aligned}$$

We note that the additional integral in (3.31) is incurred due to the presence of the additional term $V\rho^\beta e^{-\alpha\rho^\varepsilon} v$ in (3.30) (that is not present in (3.2) !). Such an integral is estimated as follows. We have

$$(3.32) \quad \begin{aligned} 4\beta \int \rho^{-2\alpha+2\beta-2} FvVv &= 2\beta \int V\rho^{-Q+2} F(v^2) \\ &= -2\beta \int FV\rho^{-Q+2} v^2 + 4\beta \int V\rho^{-Q+2} v^2. \end{aligned}$$

Recall V satisfying the bound

$$|V(z, t)| \leq K\psi \text{ and } |ZV(z, t)| \leq K\psi.$$

Thus both the integrals in (3.32) can be controlled by the following term in (3.31), i.e.

$$c\beta^3\varepsilon^2 \int \rho^{-2\alpha-4+\varepsilon} e^{2\alpha\rho^\varepsilon} u^2 \mu,$$

provided

$$\frac{c\beta^3\varepsilon^2}{2} > CK\beta$$

which in turn can be ensured by choosing

$$(3.33) \quad \alpha \geq C_1(K^{1/2} + 1)$$

(in view of (3.7)) where C_1 is some universal constant.

Finally, we show how to incorporate the integral $\beta \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle dxdt$ in the left hand side of (2.24) by an interpolation type argument as before.

We have by integration by parts,

$$(3.34) \quad \begin{aligned} \beta \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle &= \beta \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AX(\rho^\beta e^{-\alpha\rho^\varepsilon} v), X(\rho^\beta e^{-\alpha\rho^\varepsilon} v) \rangle \\ &= -\beta \int X(\rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon}) \cdot AX(\rho^\beta e^{-\alpha\rho^\varepsilon} v) [\rho^\beta e^{-\alpha\rho^\varepsilon} v] - \beta \int \mathcal{L}(\rho^\beta e^{-\alpha\rho^\varepsilon} v) [\rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon}] [\rho^\beta e^{-\alpha\rho^\varepsilon} v] \end{aligned}$$

Now by writing $\mathcal{L}u = (\mathcal{L}u + Vu) - Vu$, we obtain from (3.34) that the following inequality holds,

$$(3.35) \quad \begin{aligned} &\beta \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \\ &\leq \beta \int \rho^{-Q+\varepsilon} (2\alpha + 2 - \varepsilon - 2\alpha\varepsilon\rho^\varepsilon) (\beta - \alpha\varepsilon\rho^\varepsilon) \mu v^2 + \beta \left| \int \rho^{-Q+\varepsilon} (2\alpha + 2 - \varepsilon - 2\alpha\varepsilon\rho^\varepsilon) \mu Fv \cdot v \right| \\ &+ \beta \left| \int (\mathcal{L}u + Vu) v [\rho^{-2\alpha+\beta-2+\varepsilon} e^{\alpha\rho^\varepsilon}] \right| + C\beta^3 \int \rho^{-Q+2+\varepsilon} v^2 \mu, \end{aligned}$$

where in the last inequality above, we used that $|V| \leq \beta^2\psi$ which follows from (3.33). Therefore, at this point by suitably applying Cauchy Schwartz inequality to the integrals,

$$\left| \int \rho^{-Q+\varepsilon} (2\alpha + 2 - \varepsilon - 2\alpha\varepsilon\rho^\varepsilon) \mu Fv \cdot v \right|$$

and

$$\beta \left| \int (\mathcal{L}u + Vu) v [\rho^{-2\alpha+\beta-2+\varepsilon} e^{\alpha\rho^\varepsilon}] \right|,$$

we can argue as in the proof of Theorem 3.1 using the estimate (3.31) instead of (3.23) to get to the desired conclusion. \square

Proof of Theorem 2.15. The proof of Theorem 2.15 is a consequence of the following Carleman estimate after which one can repeat the arguments in [9].

Theorem 3.3. *Let $\varepsilon \in (0, 1)$, $1 < q < 2$ and let f satisfy the assumptions in (2.27). Then for every $\varepsilon > 0$, there exists C universal depending also on ε and the constants in (2.27) such that for every $\alpha > 0$ sufficiently large and $u \in S_0^{2,2}(B_R \setminus \{0\})$ with $\text{supp } u \subset (B_R \setminus \{0\})$, one has*

$$(3.36) \quad \begin{aligned} &\alpha^3 \int e^{2\alpha\rho^\varepsilon} [\rho^{-2\alpha-4+\varepsilon} u^2 \mu \rho^{-2\alpha-2} |u|^q \mu] \\ &+ \alpha \int \rho^{-2\alpha-2+\varepsilon} e^{2\alpha\rho^\varepsilon} \langle AXu, Xu \rangle \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\varepsilon} (\mathcal{L}u + f((z, t), u)\psi)^2 \mu^{-1}, \end{aligned}$$

provided $R \leq R_0$ where R_0 is sufficiently small.

Proof. The proof is similar to that of Theorem 3.1 except that we additionally exploit the intrinsic nature of the sublinearity $f((z, t), u)$ and the structural assumptions in (2.27). As before, we let $u = \rho^\beta e^{\alpha\rho^\epsilon} v$ where α and β are related as in (3.7). In terms of v , we have that

$$(3.37) \quad \begin{aligned} \mathcal{L}u + f((z, t), u)\psi &= \mathcal{L}v(\rho^\beta e^{-\alpha\rho^\epsilon}) + 2\mu Fv \left[\rho^{\beta-2} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right] + \left[X_i a_{ij} \cdot (\beta - \alpha\epsilon\rho^\epsilon) \rho^{\beta-1} e^{-\alpha\rho^\epsilon} X_j \rho \right] v \\ &+ \left[a_{ij} [\rho^{\beta-2} e^{-\alpha\rho^\epsilon} (\beta(\beta-1) - \alpha\epsilon\rho^\epsilon(2\beta + \epsilon - 1) + (\alpha\epsilon)^2 \rho^{2\epsilon})] X_i \rho X_j \rho \right] v \\ &+ \left[a_{ij} \left(\rho^{\beta-1} e^{-\alpha\rho^\epsilon} (\beta - \alpha\epsilon\rho^\epsilon) \right) X_i X_j \rho \right] v + f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi. \end{aligned}$$

Now again the integral

$$\int \rho^{-2\alpha} e^{2\alpha\rho^\epsilon} (\mathcal{L}u + f((z, t), u)\psi)^2 \mu^{-1}$$

is estimated from below by using $(a+b)^2 \geq a^2 + 2ab$, with $a = 2\beta\rho^{\beta-2}\mu Fv$ and b being the rest of the terms in (3.37). In this case, we note that all the other terms (with the exception of (3.38) below) are handled in the same way as before and therefore we only need to focus our attention on the following additional term which is incurred due to the presence of the additional term $f((z, t)\rho^\beta e^{-\alpha\rho^\epsilon} v)\psi$ in (3.37) (that is not present in (3.2) !), i.e.

$$(3.38) \quad 4\beta \int \rho^{-2\alpha+\beta-2} e^{\alpha\rho^\epsilon} Fv f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi.$$

Now from the fact that G is the "s-antiderivative" of f we have

$$(3.39) \quad \begin{aligned} F \left(G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \right) &= Fv f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \rho^\beta e^{-\alpha\rho^\epsilon} + (\beta - \alpha\epsilon\rho^\epsilon) \rho^\beta e^{-\alpha\rho^\epsilon} v f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \\ &+ \langle \nabla_{(z,t)} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v), F \rangle. \end{aligned}$$

Note that in (3.39) above, we used the fact that $F\rho^\beta = \beta\rho^\beta$. Then by using (3.39) we obtain

$$(3.40) \quad \begin{aligned} 4\beta \int \rho^{-2\alpha+\beta-2} e^{\alpha\rho^\epsilon} Fv f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi &= 4\beta \int F \left(G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \right) \rho^{-2\alpha-2} e^{\alpha\rho^\epsilon} \psi \\ &- 4\beta \int (\beta - \alpha\epsilon\rho^\epsilon) \rho^{-2\alpha-2} f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \rho^\beta e^{\alpha\rho^\epsilon} v \psi - 4\beta \int \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} \langle \nabla_{(z,t)} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v), F \rangle \psi. \end{aligned}$$

Now from the third condition in (2.27) we have

$$(3.41) \quad f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \rho^\beta e^{-\alpha\rho^\epsilon} v \leq qG((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)$$

and the fourth condition in (2.27) implies

$$(3.42) \quad \left\langle \nabla_{z,t} G, F \right\rangle \leq C_2 G.$$

Thus by using (3.41) and (3.42) in (3.40) we get the following inequality,

$$(3.43) \quad \begin{aligned} 4\beta \int \rho^{-2\alpha+\beta-2} e^{\alpha\rho^\epsilon} Fv f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi &\geq 4\beta \int F \left(G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \right) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} \psi - 4\beta q \int (\beta - \alpha\epsilon\rho^\epsilon) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi \\ &- 4C_2\beta \int \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\psi. \end{aligned}$$

where in order to estimate the last integral, we used that $\mu \sim \psi$. Now the first term in the right hand side of (3.43), i.e. the integral

$$4\beta \int F\left(G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\right) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} \psi$$

is handled using integration by parts in the following way using the estimates in Theorem 2.2 i) and ii).

$$\begin{aligned} (3.44) \quad & 4\beta \int F\left(G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v)\right) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} \psi \\ &= -4\beta \int G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \operatorname{div}(\rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} F\psi) \\ &= \int (8\beta(\beta-1) - 8\alpha\beta\epsilon\rho^\epsilon - 4\beta O(\rho)) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), \rho^\beta v) \psi. \end{aligned}$$

We note that over here we used (3.7) which implies that

$$\begin{aligned} \operatorname{div}(\rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} F) &= [-2\alpha-2 + 2\alpha\epsilon\rho^\epsilon] \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} + \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} \operatorname{div} F \\ &= (-2(\beta-1) + 2\alpha\epsilon\rho^\epsilon + O(\rho)) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon}. \end{aligned}$$

Now since $q < 2$, by using (3.43) and (3.44) we obtain that

$$\begin{aligned} (3.45) \quad & \int \rho^{-2\alpha+\beta-2} e^{\alpha\rho^\epsilon} 4\beta F v f((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \psi \\ &\geq \int (4\beta^2(2-q) + 4\alpha\beta\epsilon\rho^\epsilon(q-2) - C\rho - 4C_2\beta - 8\beta) \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \psi \\ &\geq c\beta^2 \int \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), \rho^\beta e^{-\alpha\rho^\epsilon} v) \psi, \text{ for large enough } \beta \text{ provided } R \text{ is small enough.} \end{aligned}$$

Thus it follows from the computations as in the proof of Theorem 3.1 and by using (3.45) that the following inequality holds,

$$\begin{aligned} (3.46) \quad & \int \rho^{-2\alpha} e^{2\alpha\rho^\epsilon} (\mathcal{L}u + f((z, t), u)\psi)^2 \mu^{-1} + C_1\beta \int \rho^{-2\alpha-1} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle \\ &\geq 3\beta^2 \int \rho^{-Q} (Fv)^2 \mu + c\beta^3 \epsilon^2 \int \rho^{-2\alpha-4+\epsilon} e^{2\alpha\rho^\epsilon} u^2 \mu + c\beta^2 \int \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), u) \psi. \end{aligned}$$

Finally as before, we show how to incorporate the integral $\beta \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle$ in the left hand side of (3.36) by interpolation.

We have,

$$\begin{aligned} (3.47) \quad & \beta \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle = \beta \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} \langle AX(\rho^\beta e^{-\alpha\rho^\epsilon} v), X(\rho^\beta e^{-\alpha\rho^\epsilon} v) \rangle \\ &= -\beta \int X(\rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon}) \cdot AX(\rho^\beta e^{-\alpha\rho^\epsilon} v) [\rho^\beta e^{-\alpha\rho^\epsilon} v] - \beta \int \mathcal{L}u v [\rho^{-2\alpha+\beta-2+\epsilon} e^{\alpha\rho^\epsilon}]. \end{aligned}$$

Now by rewriting $\mathcal{L}u = (\mathcal{L}u + f((z, t), u)\psi) - f((z, t), u)\psi$ we obtain from (3.47) that

$$\begin{aligned}
(3.48) \quad & \beta \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle \\
& \leq \beta \int \rho^{-Q+\epsilon} (2\alpha + 2 - \epsilon - 2\alpha\epsilon\rho^\epsilon) (\beta - \alpha\epsilon\rho^\epsilon) \mu v^2 + \beta \left| \int \rho^{-Q+\epsilon} (2\alpha + 2 - \epsilon - 2\alpha\epsilon\rho^\epsilon) \mu Fv \cdot v \right| \\
& + \beta \left| \int (\mathcal{L}u + f((z, t), u)) v [\rho^{-2\alpha+\beta-2+\epsilon} e^{\alpha\rho^\epsilon}] \right| + \beta \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} f((z, t), u) u \mu \\
& \leq \beta \int \rho^{-Q+\epsilon} (2\alpha + 2 - \epsilon - 2\alpha\epsilon\rho^\epsilon) (\beta - \alpha\epsilon\rho^\epsilon) \mu v^2 + \beta \left| \int \rho^{-Q+\epsilon} (2\alpha + 2 - \epsilon - 2\alpha\epsilon\rho^\epsilon) \mu Fv \cdot v \right| \\
& + \beta \left| \int (\mathcal{L}u + f((z, t), u)) v [\rho^{-2\alpha+\beta-2+\epsilon} e^{\alpha\rho^\epsilon}] \right| + \beta q \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} G((z, t), u) \psi,
\end{aligned}$$

where in the last inequality in (3.48) above, we used that $uf((z, t), u) \leq qG((z, t), u)$. We now note that the last integral in (3.48) above, i.e.

$$\beta q \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} G((z, t), u) \mu$$

can be estimated from above by the following integral in (3.46), i.e.

$$c\beta^2 \int \rho^{-2\alpha-2} e^{2\alpha\rho^\epsilon} G((z, t), u) \psi,$$

provided β is sufficiently large. At this point, the rest of the argument is similar to that for Theorem 3.1 where we use the inequality (3.46) instead of (3.23) and we finally arrive at the following estimate

$$\begin{aligned}
(3.49) \quad & \alpha^3 \int e^{2\alpha\rho^\epsilon} [\rho^{-2\alpha-4+\epsilon} u^2 \mu + \rho^{-2\alpha-2} G((z, t), u) \mu] \\
& + \alpha \int \rho^{-2\alpha-2+\epsilon} e^{2\alpha\rho^\epsilon} \langle AXu, Xu \rangle \leq C \int \rho^{-2\alpha} e^{2\alpha\rho^\epsilon} (\mathcal{L}u + f((z, t), u)\psi)^2 \mu^{-1}.
\end{aligned}$$

The desired inequality (3.36) now follows by using (2.28) in (3.49). □

Proof of Theorem 2.16.

Proof. Let $v = e^{\frac{\beta}{2}(\log \rho)^2} u$. Then it follows,

$$\mathcal{L}u = \mathcal{L}v(e^{-\frac{\beta}{2}(\log \rho)^2}) + 2a_{ij} X_j v X_i(e^{-\frac{\beta}{2}(\log \rho)^2}) + \mathcal{L}(e^{-\frac{\beta}{2}(\log \rho)^2})v$$

Since $a_{ij} = \delta_{ij} + b_{ij}$, therefore we have,

$$\mathcal{L}u = \mathcal{L}v(e^{-\frac{\beta}{2}(\log \rho)^2}) + 2a_{ij} X_j v X_i(e^{-\frac{\beta}{2}(\log \rho)^2}) + \mathcal{B}_\gamma(e^{-\frac{\beta}{2}(\log \rho)^2})v + X_i(b_{ij} X_j(e^{-\frac{\beta}{2}(\log \rho)^2}))v.$$

Now, we compute the last two terms in the right hand side of the above expression. By a standard calculation we obtain,

$$\mathcal{B}_\gamma(e^{-\frac{\beta}{2}(\log \rho)^2}) = \psi e^{-\frac{\beta}{2}(\log \rho)^2} \rho^{-2} ((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho),$$

and

$$X_i(e^{-\frac{\beta}{2}(\log \rho)^2}) = -\beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} X_i \rho.$$

Similarly we have,

$$\begin{aligned} X_i X_j (e^{-\frac{\beta}{2}(\log \rho)^2}) &= X_i \left[-\beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} X_j \rho \right] \\ &= \beta \rho^{-2} e^{-\frac{\beta}{2}(\log \rho)^2} \left(-1 + (\log \rho) + \beta(\log \rho)^2 \right) X_i \rho X_j \rho - \beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} X_i X_j \rho. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}u &= \mathcal{L}v(e^{-\frac{\beta}{2}(\log \rho)^2}) + 2a_{ij} X_j v X_i \rho \left[-\beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} \right] \\ &+ \left[\psi e^{-\frac{\beta}{2}(\log \rho)^2} \rho^{-2} \left((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho \right) \right] v \\ &- \left[X_i b_{ij} \beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} X_j \rho \right] v + \left[b_{ij} \left(-\beta(\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} \right) X_i X_j \rho \right] v \\ &+ \left[b_{ij} \left(\beta \rho^{-2} e^{-\frac{\beta}{2}(\log \rho)^2} \left(-1 + (\log \rho) + \beta(\log \rho)^2 \right) \right) X_i \rho X_j \rho \right] v. \end{aligned}$$

Consequently in terms of the vector field F , we observe that $\mathcal{L}u$ can be written as,

$$\begin{aligned} (3.50) \quad \mathcal{L}u &= \mathcal{L}v(e^{-\frac{\beta}{2}(\log \rho)^2}) + 2\mu Fv \left[-\beta(\log \rho) \rho^{-2} e^{-\frac{\beta}{2}(\log \rho)^2} \right] \\ &+ \left[\psi e^{-\frac{\beta}{2}(\log \rho)^2} \rho^{-2} \left((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho \right) \right] v \\ &+ \left[X_i b_{ij} \cdot \beta(-\log \rho) \rho^{-1} e^{-\frac{\beta}{2}(\log \rho)^2} X_j \rho \right] v + \left[b_{ij} \left(-\beta(\log \rho) e^{-\frac{\beta}{2}(\log \rho)^2} \rho^{-1} \right) X_i X_j \rho \right] v \\ &+ \left[b_{ij} \left(\beta \rho^{-2} e^{-\frac{\beta}{2}(\log \rho)^2} \left(-1 + (\log \rho) + \beta(\log \rho)^2 \right) \right) X_i \rho X_j \rho \right] v. \end{aligned}$$

Now using $(a+b)^2 \geq a^2 + 2ab$ with $a = 2\beta(-\log \rho) \mu Fv \rho^{-2}$ and with b being the rest of the terms on (3.50) above, we obtain,

$$\begin{aligned} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 &\geq 4\beta^2 \rho^{-4} (\log \rho)^2 \mu^2 (Fv)^2 + 4\beta \mu \rho^{-2} (-\log \rho) Fv \mathcal{L}v \\ &+ 4\beta (-\log \rho) \mu \rho^{-2} \left[\psi \rho^{-2} \left((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho \right) \right. \\ &+ \left. [X_i b_{ij} \cdot \beta(-\log \rho) \rho^{-1} X_j \rho] + [b_{ij} (\beta(-\log \rho) \rho^{-1}) X_i X_j \rho] \right. \\ &+ \left. [b_{ij} (\beta \rho^{-2} (-1 + (\log \rho) + \beta(\log \rho)^2)) X_i \rho X_j \rho] \right] Fv \cdot v. \end{aligned}$$

Hence,

$$\begin{aligned}
(3.51) \quad & \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} \\
& \geq \int [4\beta^2 \rho^{-Q} (\log \rho)^2 \mu(Fv)^2] dzdt + \int 4\beta \rho^{-Q+2} (-\log \rho) Fv \mathcal{L}v \\
& + \int 4\beta \rho^{-Q+2} (-\log \rho) \left[[\psi \rho^{-2} ((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho)] \right. \\
& + [X_i b_{ij} \cdot \beta (-\log \rho) \rho^{-1} X_j \rho] + [b_{ij} (\beta (-\log \rho) \rho^{-1}) X_i X_j \rho] \\
& \left. + [b_{ij} (\beta \rho^{-2} (-1 + (\log \rho) + \beta (\log \rho)^2)) X_i \rho X_j \rho] \right] Fv \cdot v \\
& = \int [4\beta^2 \rho^{-Q} (\log \rho)^2 \mu(Fv)^2] dzdt + \int 4\beta \rho^{-Q+2} (-\log \rho) Fv \mathcal{L}v \\
& + \int 4\beta \rho^{-Q+2} (-\log \rho) \left[[\psi \rho^{-2} ((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho)] \right. \\
& + [X_i b_{ij} \cdot \beta (-\log \rho) \rho^{-1} X_j \rho] + [b_{ij} (\beta (-\log \rho) \rho^{-1}) X_i X_j \rho] \\
& \left. + [b_{ij} (\beta \rho^{-2} (-1 + (\log \rho) + \beta (\log \rho)^2)) X_i \rho X_j \rho] \right] F \left(\frac{v^2}{2} \right)
\end{aligned}$$

We first handle the following term in the right hand side of (3.51) above,

$$\begin{aligned}
(3.52) \quad & \int 4\beta \rho^{-Q+2} (-\log \rho) \left[[\psi \rho^{-2} ((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho)] + [b_{ij} (\beta (-\log \rho) \rho^{-1}) X_i X_j \rho] \right. \\
& \left. + [b_{ij} (\beta \rho^{-2} (-1 + (\log \rho) + \beta (\log \rho)^2)) X_i \rho X_j \rho] \right] F \left(\frac{v^2}{2} \right).
\end{aligned}$$

We now look at each individual term in (3.52).

First we observe that by applying integration by parts to the integral

$$\int 4\beta [(\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho] (-\log \rho) \rho^{-Q} \psi F \left(\frac{v^2}{2} \right)$$

we get that the following holds,

$$\begin{aligned}
(3.53) \quad & \int 4\beta [(\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho] (-\log \rho) \rho^{-Q} \psi F \left(\frac{v^2}{2} \right) = 4 \int \beta^3 \operatorname{div}[(\log \rho)^3 \rho^{-Q} F \psi] \frac{v^2}{2} \\
& - 4 \int \beta^2 \operatorname{div}((\log \rho) \rho^{-Q} F \psi) \frac{v^2}{2} - 4 \int \beta^2 (Q-2) \operatorname{div}((\log \rho)^2 \rho^{-Q} F \psi) \frac{v^2}{2}
\end{aligned}$$

Now we estimate each individual term in the right hand side of (3.53). We have using the estimates in Theorem 2.2,

$$(3.54) \quad \int \beta^3 \operatorname{div}[(\log \rho)^3 \rho^{-Q} F \psi] \frac{v^2}{2} \geq 6\beta^3 \int \rho^{-Q} (\log \rho)^2 v^2 \psi - C\beta^3 \int \rho^{-Q+1} (\log \rho)^3 v^2 \psi.$$

Likewise it follows that,

$$\begin{aligned} & -4 \int \beta^2 \operatorname{div}((\log \rho) \rho^{-Q} F \psi) \frac{v^2}{2} - 4 \int \beta^2 (Q-2) \operatorname{div}((\log \rho)^2 \rho^{-Q} F \psi) \frac{v^2}{2} \\ & \geq 2\beta^2 \int \rho^{-Q} [2(Q-2)(-\log \rho) - 1] v^2 \psi - C \int \rho^{-Q+1} [\beta^2 (-\log \rho) + 4\beta^2 (Q-2)(\log \rho)^2]. \end{aligned}$$

Next, we see that

$$\begin{aligned} (3.55) \quad & 4\beta^2 \int \rho^{-Q} (-\log \rho) (-1 + (\log \rho) + \beta(\log \rho)^2) b_{ij} X_i \rho X_j \rho F \left(\frac{v^2}{2} \right) \\ & = -2\beta^2 \int \operatorname{div}(\rho^{-Q} (-\log \rho) (-1 + (\log \rho) + \beta(\log \rho)^2) b_{ij} X_i \rho X_j \rho F) v^2 \\ & \geq -C\beta^2 \int \rho^{-Q+2} (-\log \rho) (-1 + (\log \rho) + \beta(\log \rho)^2) v^2 \mu - C\beta^2 \int \rho^{-Q+1} v^2 \mu - C\beta^3 \int \rho^{-Q+1} (\log \rho)^2 v^2 \mu \end{aligned}$$

Finally, the second integral in (3.52) can be estimated using the estimates in Theorem 2.2 as well as the third derivative estimate in Lemma 2.8 in the following way,

$$\begin{aligned} (3.56) \quad & 4\beta \int \rho^{-Q+1} [b_{ij} (-\log \rho) X_i X_j \rho] F \left(\frac{v^2}{2} \right) \\ & \int \operatorname{div} \left(4\beta \rho^{-Q+1} [b_{ij} (\beta(\log \rho)) X_i X_j \rho F] \right) \frac{v^2}{2} \\ & \geq -C\beta^2 \int \rho^{-Q+1} (-\log \rho) v^2 \mu. \end{aligned}$$

Thus from (3.53)-(3.56) it follows that for all R small enough, we have

$$\begin{aligned} (3.57) \quad & \int 4\beta \rho^{-Q+2} (-\log \rho) \left[[\psi \rho^{-2} ((\beta \log \rho)^2 - \beta - (Q-2)\beta \log \rho)] + [b_{ij} (\beta(-\log \rho) \rho^{-1}) X_i X_j \rho] \right. \\ & \left. + [b_{ij} (\beta \rho^{-2} (-1 + (\log \rho) + \beta(\log \rho)^2)) X_i \rho X_j \rho] \right] F \left(\frac{v^2}{2} \right) \\ & \geq 5\beta^3 \int \rho^{-Q} (\log \rho)^2 v^2 \mu. \end{aligned}$$

Now using $\sum_{i,j=1}^N |X_i b_{ij} X_j \rho| \leq C\mu$, we obtain by applying Cauchy-Schwartz inequality that the following holds,

$$\begin{aligned} (3.58) \quad & \int 4\beta \rho^{-Q+1} [X_i b_{ij} \cdot \beta(-\log \rho) X_j \rho] F v \cdot v \\ & \geq -\beta^2 \int \rho^{-Q+1} (Fv)^2 (\log(\rho))^2 \mu - C \int \rho^{-Q+1} v^2 \mu \end{aligned}$$

We now estimate the second integral in (3.51), i.e.

$$\int 4\beta \rho^{-Q+2} (-\log \rho) F v \mathcal{L} v.$$

Now in order to estimate this integral, we use the Rellich type identity as in (3.16) with $G = \rho^{-Q+2} (-\log \rho) F$. It follows using (3.16), the estimates in Theorem 2.2 and by computations

which are analogous to that in (3.17)-(3.19) that the following holds,

$$(3.59) \quad \begin{aligned} 4\beta \int \rho^{-Q+2} (-\log \rho) Fv \mathcal{L}v &\geq 4\beta \int \rho^{-Q} ((Q-2)(-\log \rho) + 1) (Fv)^2 \mu - 2\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle \\ &- C\beta \int \rho^{-Q+3} (-\log \rho) \langle AXv, Xv \rangle \geq 4\beta \int \rho^{-Q} ((Q-2)(-\log \rho) + 1) (Fv)^2 \mu - \frac{5}{2}\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle, \end{aligned}$$

where in the last inequality above, we used that for all small enough ρ ,

$$C\rho^{-Q+3} \log(-\rho) \leq \frac{1}{2}\rho^{-Q+2}.$$

Therefore by combining (3.51), (3.57), (3.58) and (3.59), we finally deduce the following inequality for all β large and R small,

$$(3.60) \quad \begin{aligned} &\int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} + \frac{5}{2}\beta \int \rho^{-Q+2} \langle AXv, Xv \rangle \\ &\geq 3\beta^2 \int \rho^{-Q} (\log \rho)^2 (Fv)^2 \mu + 4\beta^3 \int \rho^{-Q} (\log \rho)^2 v^2 \mu. \end{aligned}$$

We now rewrite the integral

$$\int \rho^{-Q+2} \langle AXv, Xv \rangle$$

as follows. We have,

$$\begin{aligned} &\int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle = \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AX e^{-\beta/2(\log \rho)^2} v, X e^{-\beta/2(\log \rho)^2} v \rangle \\ &= \int \rho^{-Q+2} \left[\frac{(\beta \log \rho)^2}{\rho^2} \mu v^2 + 2 \frac{(-\beta \log \rho)}{\rho^2} v Fv \mu + \langle AXv, Xv \rangle \right] \\ &\geq \int \rho^{-Q+2} \left[\frac{(\beta \log \rho)^2}{\rho^2} \mu v^2 + \frac{\beta}{\rho^2} v^2 - C\beta \rho^{-1} (-\log \rho) v^2 \mu + \langle AXv, Xv \rangle \right]. \\ &\geq \int \rho^{-Q+2} \left[\frac{(\beta \log \rho)^2}{\rho^2} \mu v^2 + \langle AXv, Xv \rangle \right] \quad (\text{provided } R \text{ is small enough}). \end{aligned}$$

Thus, we get

$$(3.61) \quad \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle \geq \int \rho^{-Q+2} \langle AXv, Xv \rangle + \beta^2 \int \rho^{-Q} (\log \rho)^2 v^2 \mu.$$

Using (3.61) in (3.60) we obtain,

$$(3.62) \quad \begin{aligned} &\int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}(e^{-\frac{\beta}{2}(\log \rho)^2} v))^2 \mu^{-1} + \frac{5}{2}\beta \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle \\ &\geq 3\beta^2 \int_{B_R} \rho^{-Q} (\log \rho)^2 (Fv)^2 \mu + \frac{13}{2}\beta^3 \int \rho^{-Q} (\log \rho)^2 v^2 \mu. \end{aligned}$$

Finally, we show how to incorporate the integral $\beta \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle$ in the left hand side of (2.29) by an interpolation type argument as before. We have,

(3.63)

$$\begin{aligned}
& \beta \int \rho^{-Q+2} e^{\beta(\log \rho)^2} (-\log \rho)^\varepsilon \langle AXu, Xu \rangle = -\beta \int \left\langle X(\rho^{-Q+2} e^{\beta(\log \rho)^2}), AX(e^{-\beta/2(\log \rho)^2} v) \right\rangle e^{-\beta/2(\log \rho)^2} v \\
& - \beta \int \mathcal{L}(e^{-\beta/2(\log \rho)^2} v) \rho^{-Q+2} e^{\beta/2(\log \rho)^2} v. \\
& \leq -\beta \int \left\langle X(\rho^{-Q+2} e^{\beta(\log \rho)^2}), AX(e^{-\beta/2(\log \rho)^2} v) \right\rangle e^{-\beta/2(\log \rho)^2} v \\
& + C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} + C\beta^2 \int \rho^{-Q} v^2 \\
& \leq -\beta \int \left\langle X(\rho^{-Q+2} e^{\beta(\log \rho)^2}), AX(e^{-\beta/2(\log \rho)^2} v) \right\rangle e^{-\beta/2(\log \rho)^2} v + C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} \\
& + C \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle,
\end{aligned}$$

where in the last inequality in (3.63) above, we used the estimate (3.62).

Now the following term in (3.63) above, i.e.

$$-\beta \int \left\langle X(\rho^{-Q+2} e^{\beta(\log \rho)^2}), AX(e^{-\beta/2(\log \rho)^2} v) \right\rangle e^{-\beta/2(\log \rho)^2} v$$

is estimated as follows.

We have,

(3.64)

$$\begin{aligned}
& -\beta \int \left\langle X(\rho^{-Q+2} e^{\beta(\log \rho)^2}), AX(e^{-\beta/2(\log \rho)^2} v) \right\rangle e^{-\beta/2(\log \rho)^2} v \\
& = -\beta(-Q+2) \int \rho^{-Q} [\mu\beta(-\log \rho)v^2 + \mu Fv \cdot v] + \beta \int \rho^{-Q} 2\beta(-\log \rho) [(-\beta \log \rho)v^2 \mu + Fv \cdot v\mu] \\
& \leq \frac{5}{2}\beta^3 \int \rho^{-Q} (\log \rho)^2 v^2 \mu + C\beta \int \rho^{-Q} (\log \rho)^2 (Fv)^2 \mu \text{ (for all large } \beta \text{ and } R \text{ small)} \\
& \leq C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} + \left(\frac{25}{26}\beta + C\right) \int \rho^{-Q+2} \langle AXu, Xu \rangle,
\end{aligned}$$

where in the last inequality above, we again used the estimate (3.62). Thus from (3.63) and (3.64) we obtain

(3.65)

$$\beta \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle \leq C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1} + \left(C + \frac{25}{26}\beta\right) \int \rho^{-Q+2} \langle AXu, Xu \rangle.$$

Now for all β large enough, we observe that the following term in (3.65) above, i.e.

$$\left(C + \frac{25}{26}\beta\right) \int \rho^{-Q+2} \langle AXu, Xu \rangle.$$

can be absorbed in the left hand side of (3.65) and we thus infer that the following estimate holds,

$$(3.66) \quad \beta \int \rho^{-Q+2} e^{\beta(\log \rho)^2} \langle AXu, Xu \rangle \leq C \int \rho^{-Q+4} e^{\beta(\log \rho)^2} (\mathcal{L}u)^2 \mu^{-1}.$$

The desired estimate (2.29) now follows from (3.62) and (3.66). \square

4. APPENDIX

Proof of Lemma 2.8. First note that

$$F(b_{ij}X_iX_j\rho) = F(b_{ij})X_iX_j\rho + \left(\frac{\rho}{\mu} \sum a_{qr}X_q\rho\right) b_{ij}X_r(X_iX_j\rho).$$

Now a standard tedious computation which uses the estimates in Lemma 2.3, Proposition 2.5 and the hypothesis (H) shows that

$$(4.1) \quad \sum |(F(b_{ij})X_iX_j\rho)| \leq C\psi.$$

Consequently, we turn our attention to estimating the term $(\frac{\rho}{\mu} \sum a_{qr}X_q\rho) b_{ij}X_r(X_iX_j\rho)$. To do this, we need to compute the third derivatives of ρ . For that, we use the expressions for the second derivatives of ρ as listed in Lemma 2.4. We first recall the expression for the derivatives of ψ as in the proof of Proposition 3.2 in [32].

$$X_l\psi = \begin{cases} 2\gamma\psi \frac{z_l}{|z|^2} - 2\gamma\psi^2 \frac{z_l}{\rho^2}, & \text{for } 1 \leq l \leq m \\ -2\gamma(\gamma+1)\psi \frac{t_{l-m}|z|^\gamma}{\rho^{2\gamma+2}}, & \text{for } m+1 \leq l \leq N. \end{cases}$$

We split our consideration into the following cases.

(1) For $1 \leq r \leq m$ and $1 \leq i, j \leq m$, we have:

$$\begin{aligned} X_r(X_iX_j\rho) &= -(2\gamma+1)X_r(z_iz_j) \frac{\psi^2}{\rho^3} - (2\gamma+1)z_iz_jz_r \frac{\psi^2}{\rho^3} \left[\frac{4\gamma}{|z|^2} - \frac{\psi}{\rho^2}(4\gamma+3) \right] \\ &+ X_r \left(2\gamma \frac{z_iz_j}{|z|^2} + \delta_{ij} \right) \frac{\psi}{\rho} + \left(2\gamma \frac{z_iz_j}{|z|^2} + \delta_{ij} \right) \frac{\psi}{\rho} z_r \left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2}(2\gamma+1) \right] \\ &= -(2\gamma+1)(z_j\delta_{ri} + z_i\delta_{rj}) \frac{\psi^2}{\rho^3} - (2\gamma+1)z_iz_jz_r \frac{\psi^2}{\rho^3} \left[\frac{4\gamma}{|z|^2} - \frac{\psi}{\rho^2}(4\gamma+3) \right] \\ &+ 2\gamma \left(\frac{z_i\delta_{rj} + z_j\delta_{ri}}{|z|^2} - 2 \frac{z_iz_jz_r}{|z|^4} \right) \frac{\psi}{\rho} + \left(2\gamma \frac{z_iz_j}{|z|^2} + \delta_{ij} \right) \frac{\psi}{\rho} z_r \left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2}(2\gamma+1) \right]. \end{aligned}$$

Since $|z| \leq \rho$ and $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$, we have $|X_r(X_iX_j\rho)| \leq C \left[\frac{\psi^2}{\rho^2} + \frac{\psi}{\rho|z|} \right] \leq C \frac{\psi^{1-\frac{1}{2\gamma}}}{\rho^2}$. Thus, we have

$$(4.2) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr}X_q\rho \right) b_{ij}X_r(X_iX_j\rho) \right| \leq C \rho^2 \mu^{\frac{1}{2\gamma}} \frac{\psi^{1-\frac{1}{2\gamma}}}{\rho^2} \leq C\psi.$$

(2) For $m+1 \leq r \leq N$ and $1 \leq i, j \leq m$, we have:

$$\begin{aligned} X_r(X_iX_j\rho) &= (2\gamma+1)(\gamma+1)z_iz_j \frac{\psi^2}{\rho^{2\gamma+5}} t_{r-m} [4\gamma|z|^\gamma + 3\rho^\gamma\psi^{1/2}] \\ &- \left((2\gamma \frac{z_iz_j}{|z|^2} + \delta_{ij})(\gamma+1) \right) \left[\frac{\psi}{\rho^{2\gamma+3}} t_{r-m} [2\gamma|z|^\gamma + \rho^\gamma\psi^{1/2}] \right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_iX_j\rho)| \leq C \frac{\psi^{1+\frac{1}{2}}}{\rho^2}$. Thus, we have

$$(4.3) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr}X_q\rho \right) b_{ij}X_r(X_iX_j\rho) \right| \leq C\psi.$$

(3) For $1 \leq r \leq m$, $1 \leq i \leq m$ and $1 \leq j \leq k$, we have:

$$\begin{aligned} X_r(X_i X_{m+j} \rho) &= -(2\gamma + 1)(\gamma + 1) X_r \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} - (2\gamma + 1)(\gamma + 1) \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} z_r \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma + 3)\psi}{\rho^2} \right] \\ &+ \frac{\psi}{\rho} \left[\gamma(\gamma + 1) X_r \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] + \left[\gamma(\gamma + 1) \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] \frac{\psi}{\rho} z_r \left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2} (2\gamma + 1) \right] \\ &= -(2\gamma + 1)(\gamma + 1) \left(\frac{\delta_{ri} t_j}{|z|^\gamma} - \gamma \frac{z_i z_r t_j}{|z|^{\gamma+2}} \right) \frac{\psi^2}{\rho^3} - (2\gamma + 1)(\gamma + 1) \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} z_r \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma + 3)\psi}{\rho^2} \right] \\ &+ \frac{\psi}{\rho} \left[\gamma(\gamma + 1) \left(\frac{\delta_{ir} t_j}{|z|^{\gamma+2}} - (\gamma + 2) \frac{z_i z_r t_j}{|z|^{\gamma+4}} \right) \right] + \left[\gamma(\gamma + 1) \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] \frac{\psi}{\rho} z_r \left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2} (2\gamma + 1) \right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_i X_{m+j} \rho)| \leq C \left[\frac{\psi^{3/2}}{\rho^2} + \frac{\psi^{1/2}}{|z|^2} \right] \leq C \frac{\psi^{1/2 - \frac{1}{\gamma}}}{\rho^2}$. Thus, we have

$$(4.4) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_q \rho \right) b_{i(m+j)} X_r(X_i X_{m+j} \rho) \right| \leq C \rho^2 \mu^{\frac{1}{2\gamma}} (\psi^{1/2 + \frac{1}{2\gamma}}) \frac{\psi^{1/2 - \frac{1}{\gamma}}}{\rho^2} \leq C\psi.$$

(4) For $m + 1 \leq r \leq N$, $1 \leq i \leq m$ and $1 \leq j \leq k$, we have:

$$\begin{aligned} X_r(X_i X_{m+j} \rho) &= -(2\gamma + 1)(\gamma + 1) X_r \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} + (2\gamma + 1)(\gamma + 1)^2 \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^{2\gamma+5}} t_{r-m} \left[4\gamma |z|^\gamma + 3\rho^\gamma \psi^{1/2} \right] \\ &+ \frac{\psi}{\rho} \left[\gamma(\gamma + 1) X_r \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] - \left[\gamma(\gamma + 1)^2 \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] \frac{\psi}{\rho^{2\gamma+3}} t_{r-m} \left[2\gamma |z|^\gamma + \rho^\gamma \psi^{1/2} \right] \\ &= -(2\gamma + 1)(\gamma + 1) \left(\frac{z_i |z|^\gamma \delta_{rj}}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} + (2\gamma + 1)(\gamma + 1)^2 \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^{2\gamma+5}} t_{r-m} \left[4\gamma |z|^\gamma + 3\rho^\gamma \psi^{1/2} \right] \\ &+ \frac{\psi}{\rho} \left[\gamma(\gamma + 1) \left(\frac{z_i |z|^\gamma \delta_{rj}}{|z|^{\gamma+2}} \right) \right] - \left[\gamma(\gamma + 1)^2 \left(\frac{z_i t_j}{|z|^{\gamma+2}} \right) \right] \frac{\psi}{\rho^{2\gamma+3}} t_{r-m} \left[2\gamma |z|^\gamma + \rho^\gamma \psi^{1/2} \right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_i X_{m+j} \rho)| \leq C \left[\frac{\psi^2}{\rho^2} + \frac{\psi}{\rho |z|} \right] \leq C \frac{\psi^{1 - \frac{1}{2\gamma}}}{\rho^2}$. Thus, we have

$$(4.5) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_q \rho \right) b_{i(m+j)} X_r(X_i X_{m+j} \rho) \right| \leq C \rho^2 \mu^{-\frac{1}{2}} (\psi^{1/2 + \frac{1}{2\gamma}}) \frac{\psi^{1 - \frac{1}{2\gamma}}}{\rho^2} \leq C\psi.$$

(5) For $1 \leq r, i \leq m$ and $1 \leq j \leq k$ we have:

$$\begin{aligned} X_r(X_{m+j} X_i \rho) &= -(2\gamma + 1)(\gamma + 1) X_r \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} - (2\gamma + 1)(\gamma + 1) \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} z_r \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma + 3)\psi}{\rho^2} \right] \\ &= -(2\gamma + 1)(\gamma + 1) \left(\frac{\delta_{ri} t_j}{|z|^\gamma} - \gamma \frac{z_i z_r t_j}{|z|^{\gamma+2}} \right) \frac{\psi^2}{\rho^3} - (2\gamma + 1)(\gamma + 1) \left(\frac{z_i t_j}{|z|^\gamma} \right) \frac{\psi^2}{\rho^3} z_r \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma + 3)\psi}{\rho^2} \right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_{m+j} X_i \rho)| \leq C \frac{\psi^{3/2}}{\rho^2}$. Thus, we have

$$(4.6) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_q \rho \right) b_{(m+j)i} X_r(X_{m+j} X_i \rho) \right| \leq C \rho^2 \frac{\psi^{3/2}}{\rho^2} \leq C\psi.$$

(6) For $m+1 \leq r \leq N$, $1 \leq i \leq m$ and $1 \leq j \leq k$ we have:

$$\begin{aligned} X_r(X_{m+j}X_i\rho) &= -(2\gamma+1)(\gamma+1)X_r\left(\frac{z_it_j}{|z|^\gamma}\right)\frac{\psi^2}{\rho^3} + (2\gamma+1)(\gamma+1)^2\left(\frac{z_it_j}{|z|^\gamma}\right)\frac{\psi^2}{\rho^{2\gamma+5}}t_{r-m}\left[4\gamma|z|^\gamma + 3\rho^\gamma\psi^{1/2}\right] \\ &= -(2\gamma+1)(\gamma+1)\left(\frac{z_i|z|^\gamma\delta_{rj}}{|z|^\gamma}\right)\frac{\psi^2}{\rho^3} + (2\gamma+1)(\gamma+1)^2\left(\frac{z_it_j}{|z|^\gamma}\right)\frac{\psi^2}{\rho^{2\gamma+5}}t_{r-m}\left[4\gamma|z|^\gamma + 3\rho^\gamma\psi^{1/2}\right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_{m+j}X_i\rho)| \leq C\frac{\psi^{3/2}}{\rho^2}$. Thus, we have

$$(4.7) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_{q\rho} \right) b_{(m+j)i} X_r(X_{m+j}X_i\rho) \right| \leq C \rho^2 \psi^{-1/2} \frac{\psi^{3/2}}{\rho^2} \leq C\psi.$$

(7) For $1 \leq r \leq m$ and $1 \leq i, j \leq k$ we have:

$$\begin{aligned} X_r(X_{m+i}X_{m+j}\rho) &= -(2\gamma+1)(\gamma+1)^2X_r\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^3} - (2\gamma+1)(\gamma+1)^2\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^3}z_r \\ &\quad \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma+3)\psi}{\rho^2} \right] + (\gamma+1)\delta_{ij}\frac{\psi}{\rho}z_r\left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2}(2\gamma+1) \right] \\ &= -(2\gamma+1)(\gamma+1)^2\left(-2\gamma\frac{t_iz_rt_i}{|z|^{2\gamma+1}}\right)\frac{\psi^2}{\rho^3} - (2\gamma+1)(\gamma+1)^2\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^3}z_r \\ &\quad \left[\frac{4\gamma}{|z|^2} - \frac{(4\gamma+3)\psi}{\rho^2} \right] + (\gamma+1)\delta_{ij}\frac{\psi}{\rho}z_r\left[\frac{2\gamma}{|z|^2} - \frac{\psi}{\rho^2}(2\gamma+1) \right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_{m+i}X_{m+j}\rho)| \leq C\left[\frac{\psi}{\rho} + \frac{\psi}{\rho|z|}\right] \leq C\frac{\psi^{1-\frac{1}{2\gamma}}}{\rho^2}$. Thus, we have

$$(4.8) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_{q\rho} \right) b_{(m+i)(m+j)} X_r(X_{m+i}X_{m+j}\rho) \right| \leq C \rho^2 \mu^{\frac{1}{2\gamma}} \frac{\psi^{1-\frac{1}{2\gamma}}}{\rho^2} \leq C\psi.$$

(8) For $m+1 \leq r \leq N$ and $1 \leq i, j \leq k$ we have:

$$\begin{aligned} X_r(X_{m+i}X_{m+j}\rho) &= -(2\gamma+1)(\gamma+1)^2X_r\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^3} + (2\gamma+1)(\gamma+1)^3\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^{2\gamma+5}} \\ &\quad t_{r-m}\left[4\gamma|z|^\gamma + 3\rho^\gamma\psi^{1/2}\right] - (\gamma+1)^2\delta_{ij}\frac{\psi}{\rho^{2\gamma+3}}t_{r-m}\left[2\gamma|z|^\gamma + \rho^\gamma\psi^{1/2}\right] \\ &= -(2\gamma+1)(\gamma+1)^2\left(\frac{t_j|z|^\gamma\delta_{ri}}{|z|^{2\gamma}} + \frac{t_i|z|^\gamma\delta_{rj}}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^3} + (2\gamma+1)(\gamma+1)^3\left(\frac{t_jt_i}{|z|^{2\gamma}}\right)\frac{\psi^2}{\rho^{2\gamma+5}} \\ &\quad t_{r-m}\left[4\gamma|z|^\gamma + 3\rho^\gamma\psi^{1/2}\right] - (\gamma+1)^2\delta_{ij}\frac{\psi}{\rho^{2\gamma+3}}t_{r-m}\left[2\gamma|z|^\gamma + \rho^\gamma\psi^{1/2}\right]. \end{aligned}$$

Since $|z| \leq \rho$, $\frac{|z|}{\rho} = \psi^{\frac{1}{2\gamma}}$ and $|t| \leq \rho^{\gamma+1}$, we have $|X_r(X_{m+i}X_{m+j}\rho)| \leq C\frac{\psi^{3/2}}{\rho^2}$. Thus, we have

$$(4.9) \quad \left| \left(\frac{\rho}{\mu} \sum a_{qr} X_{q\rho} \right) b_{(m+i)(m+j)} X_r(X_iX_j\rho) \right| \leq C \rho^2 \mu^{-\frac{1}{2}} \frac{\psi^{3/2}}{\rho^2} \leq C\psi.$$

The estimate (2.18) now follows from (4.1)-(4.9). \square

REFERENCES

- [1] F. J. Almgren, Jr., *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. Minimal submanifolds and geodesics*, (Proc. Japan-United States Sem., Tokyo, 1977), North-Holland, Amsterdam-New York, 1979.
- [2] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235-249.
- [3] V. Arya & A. Banerjee, *Strong backward uniqueness for sublinear parabolic equations*, NoDEA Nonlinear Differential Equations Appl. 27 (2020), no. 6, 50. 35K59 (35A02 35B60 35R25)
- [4] N. Aronszajn, A. Krzywicki & J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Ark. Mat. **4** 1962 417-453 (1962).
- [5] A. Banerjee, *Sharp vanishing order of solutions to stationary Schrödinger equations on Carnot groups of arbitrary step*, J. Math. Anal. Appl. **465** (2018), no. 1, 571-587.
- [6] S. M. Baouendi, *Sur une classe d'opérateurs elliptiques dégénérés* (French) Bull. Soc. Math. France, **95** 1967, 45-87.
- [7] A. Banerjee & N. Garofalo, *Quantitative uniqueness for elliptic equations at the boundary of $C^{1,Dini}$ domains*, J. Differential Equations **261** (2016), no. 12, 6718-6757.
- [8] A. Banerjee & N. Garofalo *Quantitative uniqueness for zero-order perturbations of generalized Baouendi-Grushin operators*, Rend. Istit. Mat. Univ. Trieste **48** (2016), 189-207.
- [9] A. Banerjee, N. Garofalo & R. Manna, *Carleman estimates for Baouendi-Grushin operators with applications to quantitative uniqueness and strong unique continuation*, [arXiv:1903.08382](https://arxiv.org/abs/1903.08382), to appear in Applicable Analysis.
- [10] A. Banerjee & A. Mallick, *On the strong unique continuation of a degenerate elliptic operator with Hardy type potential*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 1, 1-21.
- [11] A. Banerjee & R. Manna, *Space like strong unique continuation for sublinear parabolic equations*, J. Lond. Math. Soc. (2) **102** (2020), no. 1, 205-228.
- [12] L. Bakri, *Quantitative uniqueness for Schrödinger operator*, Indiana Univ. Math. J., **61** (2012), no. 4, 1565-1580.
- [13] L. Bakri, *Carleman estimates for the Schrödinger operator. Applications to quantitative uniqueness*. Comm. Partial Differential Equations **38** (2013), no. 1, 69-91.
- [14] J. Bourgain & C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), no. 2, 389-426.
- [15] T. Carleman, *Sur un problème d'unicité pur les systemes d'équations aux dérivées partielles à deux variables indépendantes*. (French) Ark. Mat., Astr. Fys. **26** (1939). no. 17, 9 pp.
- [16] S. Chanillo & E. Sawyer, *Unique continuation for $\Delta + \nu$ and the C. Fefferman-Phong class*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 275-300.
- [17] L. A. Caffarelli, S. Salsa & L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425-461.
- [18] H. Donnelly & C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math, **93** (1988), 161-183.
- [19] H. Donnelly & C. Fefferman, *Nodal sets of eigenfunctions: Riemannian manifolds with boundary*, Analysis, Et Cetera, Academic Press, Boston, MA, 1(1990,) 251-262.
- [20] L. Escuriaza & S. Vessella, *Optimal three-cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients*. Contemp. Math. **333**, 79-87.
- [21] N. Garofalo, *Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension.*, J. Diff. Equations **104** (1993), no. 1, 117-146.
- [22] N. Garofalo & E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*. Ann. Inst. Fourier (Grenoble) **40** (1990), no. 2, 313-356.
- [23] N. Garofalo & F. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), 245-268.
- [24] N. Garofalo & F. Lin, *Unique continuation for elliptic operators: a geometric-variational approach*, Comm. Pure Appl. Math. **40** (1987), 347-366.
- [25] B. Franchi & E. Lanconelli, *Une métrique associée à une classe d'opérateurs elliptiques dégén'eres*. (French) [A metric associated with a class of degenerate elliptic operators] Conference on linear partial and pseudodifferential operators (Torino, 1982). Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 105-114 (1984).
- [26] B. Franchi & E. Lanconelli, *Hölder regularity theorem for a class of linear non uniformly elliptic operators with measurable coefficients*, Ann. Sc. Norm. Sup. Pisa **4** (1983), 523-541.

- [27] B. Franchi & E. Lanconelli, *An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality*. Comm. Partial Differential Equations **9** (1984), no. 13, 1237-1264.
- [28] V. V. Grushin, *A certain class of hypoelliptic operators*, (Russian) Mat. Sb. (N.S.) **83** (125) (1970), 456-473.
- [29] V. V. Grushin, *A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold*, (Russian) Mat. Sb. (N.S.) **84** (126) (1971), 163-195.
- [30] N. Garofalo & K. Rotz, *Properties of a frequency of Almgren type for harmonic functions in Carnot groups*. Calc. Var. Partial Differential Equations **54** (2015), no. 2, 2197-2238.
- [31] N. Garofalo & Z. Shen, *Carleman estimates for a subelliptic operator and unique continuation*. Ann. Inst. Fourier (Grenoble) **44** (1994), no. 1, 129-166.
- [32] N. Garofalo & D. Vassilev, *Strong unique continuation properties of generalized Baouendi-Grushin operators.*, Comm. Partial Differential Equations **32** (2007), no. 4-6, 643-663.
- [33] L. Hörmander, *Uniqueness theorems for second order elliptic differential equations*. Comm. Partial Differential Equations **8** (1983), no. 1, 21-64.
- [34] D. Jerison, *Carleman inequalities for the Dirac and Laplace operators and unique continuation*. Adv. in Math. **62** (1986), no. 2, 118-134.
- [35] D. Jerison & C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrodinger operators*, Ann. of Math. (2) **121** (1985), no. 3, 463-494.
- [36] H. Koch, A. Petrosyan & W. Shi, *Higher regularity of the free boundary in the elliptic Signorini problem*, Nonlinear Anal. **126** (2015), 3-44.
- [37] H. Koch & D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math. **54** (2001), no. 3, 339-360.
- [38] V. Z. Meshkov, *On the possible rate of decrease at infinity of the solutions of second-order partial differential equations*, Mat. Sb., **182:3** (1991), 364-383; Math. USSR-Sb. **72** (1992), no. 2, 343-361.
- [39] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*. Arch. Rational Mech. Anal. **54** (1974), 105-117.
- [40] A. Plis, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **11** (1963), 95-100.
- [41] R. Regbaoui, *Strong uniqueness for second order differential operators*. (English summary) J. Differential Equations **141** (1997), no. 2, 201-217.
- [42] A. Rüländ, *Unique Continuation for sublinear elliptic equations based on Carleman estimates*. J. Differential Equations **265** (2018), no. 11, 6009-6035.
- [43] A. Rüländ, *On quantitative unique continuation properties of fractional Schrödinger equations: doubling, vanishing order and nodal domain estimates*, Trans. Amer. Math. Soc. **369** (2017), no. 4, 2311-2362.
- [44] N. Soave & S. Terracini, *The nodal set of solutions to some elliptic problems: sublinear equations, and unstable two-phase membrane problem*, Adv. Math. **334** (2018), 243-299.
- [45] N. Soave & T. Weth, *The unique continuation property of sublinear equations*. SIAM J. Math. Anal. **50** (2018), no. 4, 3919-3938.
- [46] C. D. Sogge, *Oscillatory integrals and spherical harmonics*. Duke Math. J. **53** (1986), no. 1, 43-65.
- [47] G. Tortone, *The nodal set of solutions to some nonlocal sublinear problems*, [arXiv:2004.04652](https://arxiv.org/abs/2004.04652)
- [48] L. Wang, *Hölder estimates for subelliptic operators*. (English summary), J. Funct. Anal. **199** (2003), no. 1, 228-242.
- [49] J. Zhu, *Quantitative uniqueness for elliptic equations*. Amer. J. Math. **138** (2016), no. 3, 733-762.

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