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# The Fujiki class and positive degree maps

**Abstract:** We show that a map between complex-analytic manifolds, at least one of which is in the Fujiki class, is a biholomorphism under a natural condition on the second cohomologies. We use this to establish that, with mild restrictions, a certain relation of “domination” introduced by Gromov is in fact a partial order.

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## 1 Introduction

Gromov introduced a relation of *domination* between smooth closed manifolds of a fixed dimension by declaring  $M \geq N$  if there is a smooth map of positive degree from  $M$  to  $N$  (see Section 3 for details). This relation can be made more restrictive by demanding that the map be of degree one. The main aim of this paper is to describe this relation in the context of smooth projective varieties and Kähler manifolds, and show that this relation is in fact a partial order on certain natural classes of complex manifolds of a fixed dimension, e.g.,

1. on the class of smooth projective varieties of general type,
2. on the class of smooth projective varieties that are Kobayashi hyperbolic.

One can add to this list; see Theorem 3.5. Gromov introduced the relation “ $\geq$ ” in the context of constant negative sectional curvature. Theorem 3.5 may be regarded as an attempt to translate this into the context of negative curvature in complex algebraic geometry. To the best of our knowledge, this relation has not been dealt with in the context of complex-analytic manifolds or smooth varieties.

A first step towards the above project is to try to address the following loosely-worded (but related) question: Let  $X$  and  $Y$  be two compact complex manifolds and let  $f : Y \rightarrow X$  be a surjective holomorphic map. If one of these manifolds is Kähler, then can one deduce further information about the other manifold or the map  $f$ ? (In this article, it is implicitly assumed that a manifold is connected.)

It turns out that if  $Y$  is a Kähler manifold, then one can say quite a lot about  $X$ . We begin with the following definition:

**Definition 1.1** (Fujiki, [8]). A reduced compact complex space  $X$  is said to belong to the *Fujiki class*  $\mathcal{C}$  if it is a meromorphic image of a compact Kähler space.

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In particular, any manifold  $X$  as in the question above, with  $Y$  a Kähler manifold, is in the Fujiki class  $\mathcal{C}$  (and has all the properties known about the elements in this class). However, without any additional conditions, we cannot even infer that  $X$  is Kähler: any non-projective Moishezon manifold is in the class  $\mathcal{C}$  but is non-Kähler. In contrast, Varouchas has shown [13] that, with  $Y$ ,  $X$  and  $f$  as in the question above and  $Y$  Kähler, if every fiber of  $f$  has the same dimension, then  $X$  is Kähler. This tactic — i.e., imposing some condition on the map  $f$  in the above question — turns out to be quite useful. It gives us our first result, which is a key tool in proving Theorem 3.5.

We ought to mention that Varouchas [14, Section IV.3] has shown that a compact complex manifold belongs to Fujiki's class  $\mathcal{C}$  if and only if it is *bimeromorphic* to a compact Kähler manifold. It is the latter property that we shall use in our proof of the following result.

**Theorem 1.2.** *Let  $X$  and  $Y$  be compact connected complex manifolds satisfying*

$$\dim X = \dim Y \quad \text{and} \quad \dim H^2(X, \mathbb{Q}) = \dim H^2(Y, \mathbb{Q}).$$

*Let  $\varphi : Y \rightarrow X$  be a surjective holomorphic map of degree one. If at least one of  $X$  and  $Y$  is in the Fujiki class  $\mathcal{C}$ , then  $\varphi$  is a biholomorphism.*

Theorem 1.2 is a variation of a result of [5] which is recalled in Section 2. We shall use the Hironaka elimination of indeterminacies (cf. Lemma 2.1) as a tool. A straightforward consequence is the fact (known to experts) that the class of fundamental groups of compact manifolds in the Fujiki class coincide with the class of Kähler groups (see the last paragraph of [2]).

Theorem 1.2 is used in Section 3 to show that the Gromov relation introduced at the beginning of this paper is in fact a partial order under certain natural hypotheses. In the notation of Section 3, we have the following key application of Theorem 1.2, and the principal step to the main result of Section 3:

**Theorem 1.3.** *Let  $X$  and  $Y$  be compact connected complex manifolds with  $\dim X = \dim Y$  such that at least one of  $X$ ,  $Y$  belongs to the Fujiki class  $\mathcal{C}$ . Further, suppose that  $X \geq Y$  and  $Y \geq X$ . If  $X$ ,  $Y$  are not biholomorphic, then  $X$  admits a self-endorphism of degree greater than one.*

## 2 A criterion for biholomorphism: The proof of Theorem 1.2

We begin with the following lemma, which is a form of the Hironaka elimination of indeterminacies.

**Lemma 2.1.** *Let  $X$  be a compact connected complex manifold in the Fujiki class  $\mathcal{C}$ . Then there exists a pair  $(Y, f)$ , where  $Y$  is a compact connected Kähler manifold with  $\dim Y = \dim X$ , and*

$$f : Y \rightarrow X$$

*is a surjective holomorphic map of degree one.*

*Proof.* Let  $d$  be the complex dimension of  $X$ . A theorem of Varouchas in [14] says that  $X$  is bimeromorphic to a compact connected Kähler manifold of complex dimension  $d$  — see [3, p. 31, Theorem 10] for a short proof). Let

$$\phi : Z \dashrightarrow X$$

denote such a bimeromorphic map from a compact Kähler manifold  $Z$  of dimension  $d$ .

The *elimination of indeterminacies* says that there is a finite sequence of holomorphic maps

$$Z_n \xrightarrow{f_n} Z_{n-1} \xrightarrow{f_{n-1}} Z_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} Z_1 \xrightarrow{f_1} Z_0 = Z$$

such that each  $(Z_i, f_i)$ ,  $1 \leq i \leq n$ , is a blow-up of a smooth complex submanifold of  $Z_{i-1}$ , and the bimeromorphic map

$$\phi \circ f_1 \circ \cdots \circ f_n : Z_n \dashrightarrow X$$

extends to a holomorphic map

$$\tilde{\phi} : Z_n \longrightarrow X;$$

see [11] and [10] by Hironaka. We refer the reader to [1, p. 539, § 1.2.4] for some explanation of how the above process can be carried out — ensuring, especially, that each successive blow-up is along a smooth center — using [11]. (This process works in the analytic case as well as in the algebraic; the case of complex-analytic manifolds is addressed in the last two paragraphs of [1, § 1.2.4].) The blow-up of a smooth submanifold of a Kähler manifold is Kähler [6, p. 202, Théorème II.6]. Since  $Z_0$  is Kähler, we conclude that all  $Z_i$  are Kähler.

We set  $Y := Z_n$  and  $f := \tilde{\phi}$  to obtain the desired pair  $(Y, f)$ .  $\square$

Let  $X$  and  $Y$  be compact connected complex manifolds with  $\dim X = \dim Y$ , and let

$$\varphi : Y \longrightarrow X$$

be a surjective holomorphic map of degree one. In [5] the following was proved: if the underlying real manifolds for  $X$  and  $Y$  are diffeomorphic, and also

$$\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y),$$

then  $\varphi$  is a biholomorphism.

Theorem 1.2 is a variation of the above result.

*Proof of Theorem 1.2.* The pullback homomorphisms of cohomologies

$$\varphi_i^* : H^i(X, \mathbb{Q}) \longrightarrow H^i(Y, \mathbb{Q})$$

are injective for all  $i$ . Therefore, from the given condition that  $\dim H^2(X, \mathbb{Q}) = \dim H^2(Y, \mathbb{Q})$  it follows that the homomorphism

$$\varphi_2^* : H^2(X, \mathbb{Q}) \longrightarrow H^2(Y, \mathbb{Q}) \tag{2.1}$$

is an isomorphism.

The complex dimension of  $X$  will be denoted by  $d$ . Let

$$\bigwedge^d d\varphi \in H^0 \left( Y, \text{Hom} \left( \bigwedge^d T^{1,0} Y, \varphi^* \bigwedge^d T^{1,0} X \right) \right)$$

be the  $d$ -th exterior power of the differential of  $\varphi$ . The divisor for this homomorphism  $\bigwedge^d d\varphi$  will be denoted by  $D$ . The map  $\varphi$  is a biholomorphism if  $D$  is the zero divisor.

Since the degree of  $\varphi$  is one, the image  $\varphi(D)$  is of complex codimension at least two in  $X$ . Therefore, if

$$c_D \in H_{2d-2}(Y, \mathbb{Q})$$

is the class of  $D$ , then its image  $\varphi_*(c_D) \in H_{2d-2}(X, \mathbb{Q})$  is zero. This implies that the Poincaré duality pairing of  $c_D$  with  $\varphi^*(H^2(X, \mathbb{Q})) \subset H^2(Y, \mathbb{Q})$  vanishes identically. Since  $\varphi_2^*$  in (2.1) is surjective, we now conclude that

$$c_D = 0. \tag{2.2}$$

Let us first assume that  $Y$  lies in the Fujiki class  $\mathcal{C}$ . By Lemma 2.1, there is a compact connected Kähler manifold  $Z$  of dimension  $d$  and a surjective holomorphic map

$$f : Z \longrightarrow Y$$

of degree one. Consider the effective divisor

$$\tilde{D} := f^{-1}(D) \subset Z.$$

From (2.2) we know that the class of  $\tilde{D}$  in  $H_{2d-2}(Z, \mathbb{Q})$  vanishes. Since  $Z$  is Kähler, this implies that  $\tilde{D}$  is the zero divisor. Hence  $D$  is the zero divisor. Consequently,  $\varphi$  is a biholomorphism.

Now assume that  $X$  lies in the Fujiki class  $\mathcal{C}$ . Therefore,  $X$  is bimeromorphic to a compact connected Kähler manifold  $Z$  of dimension  $d$  [14]; also see [3, p. 31, Theorem 10]. Since  $\varphi$  is a bimeromorphic map from  $Y$  to  $X$ , it follows that  $Y$  is bimeromorphic to  $Z$ . Hence  $Y$  lies in the Fujiki class  $\mathcal{C}$ . We have already shown that  $\varphi$  is a biholomorphism if  $Y$  lies in the Fujiki class  $\mathcal{C}$ .  $\square$

### 3 Gromov Partial Order

In a lecture he gave at the Graduate Center CUNY in the spring of 1978, Gromov had introduced a notion of “domination” between smooth manifolds as follows [7, 12]:

Let  $X, Y$  be closed smooth  $n$ -manifolds. We say that  $X \geq Y$  if there is a smooth map of positive degree from  $X$  to  $Y$ .

Gromov introduced this notion in the context of real hyperbolic manifolds. It is not clear a priori whether “ $\geq$ ” is in fact a partial order or not. We transfer this question to the context of projective and Kähler manifolds and holomorphic maps between them. We rephrase this as follows:

**Question 3.1.** *Let  $X, Y$  be compact projective (respectively, Kähler) manifolds of complex dimension  $n$ . We say that  $X \geq Y$  if there is a holomorphic map of positive degree from  $X$  to  $Y$ . We say that  $X \geq_1 Y$  if there is a holomorphic map of degree one from  $X$  to  $Y$ .*

1) *If  $X \geq Y$  and  $Y \geq X$ , are  $X$  and  $Y$  biholomorphic?*

2) *If  $X \geq_1 Y$  and  $Y \geq_1 X$ , are  $X$  and  $Y$  biholomorphic?*

**Remark 3.2.** Given the conditions on the manifolds  $X, Y$  in the discussion above, any positive-degree holomorphic map from one of them to the other is automatically surjective. Thus, whenever we apply Theorem 1.2 to some positive-degree map in the proofs below, we will not remark upon its surjectivity.

As a consequence of Theorem 1.2 we have the following theorem, which is a step towards answering Question 3.1. Theorem 1.3, stated in the introduction forms a part of the following result, and Part (1) below provides an answer to Question 3.1(2).

**Theorem 3.3.** *Let  $X$  and  $Y$  be compact connected complex manifolds with  $\dim X = \dim Y$  such that at least one of  $X, Y$  belongs to the Fujiki class  $\mathcal{C}$ .*

1. *If  $X \geq_1 Y$  and  $Y \geq_1 X$ , then  $X$  and  $Y$  are biholomorphic.*

2. *Assume that  $X \geq Y$  and  $Y \geq X$ . If  $X, Y$  are not biholomorphic, then  $X$  admits a self-endomorphism of degree greater than one.*

*Proof.* We may take  $X$  to be in the Fujiki class  $\mathcal{C}$ . If  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are degree one maps, then  $g \circ f$  is a holomorphic automorphism by Theorem 1.2. Therefore,  $f$  is a biholomorphism, which proves Part (1).

We now consider Part (2). Since  $X \geq Y$  (respectively,  $Y \geq X$ ), we have  $b_2(X) \geq b_2(Y)$  (respectively,  $b_2(Y) \geq b_2(X)$ ) because, by the assumption of positivity of degree of the map from  $X$  to  $Y$  (respectively,  $Y$  to  $X$ ), the pullback homomorphism of cohomologies is injective. Therefore,  $b_2(X) = b_2(Y)$ . If  $X$  is not biholomorphic to  $Y$ , then by Theorem 1.2 we conclude that the degree of any surjective holomorphic map between  $X$  and  $Y$  is at least two. Now Part (2) now follows by taking composition of two such maps  $X \rightarrow Y$  and  $Y \rightarrow X$ .  $\square$

The following result summarizes standard facts about non-existence of non-trivial self-endomorphisms: see [4], and [9] along with the references therein.

**Result 3.4.** *Let  $X$  be a compact connected complex manifold, and let  $X$  satisfy one of the following:*

1.  $X$  is a projective manifold of general type.
2.  $X$  is Kobayashi hyperbolic.
3.  $X$  is a rational homogeneous manifold of Picard number 1 that is not biholomorphic to a complex projective space  $\mathbb{P}^n$ .
4.  $X$  is a smooth projective hypersurface of dimension greater than 1 and of degree greater than 2.

*Then, any self-endomorphism of  $X$  of positive degree is an automorphism.*

We combine Theorem 3.3 with Result 3.4 to deduce the following:

**Theorem 3.5.** *Let  $X$  and  $Y$  be compact connected projective manifolds with  $\dim X = \dim Y$ . Suppose that  $X \geq Y$  and  $Y \geq X$ . Also suppose that at least one of  $X, Y$  belongs to one of the four classes listed in Result 3.4. Then  $X$  and  $Y$  are biholomorphic.*

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