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# Covering modules by proper submodules

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#### ABSTRACT

A classical problem in the literature seeks the minimal number of proper subgroups whose union is a given finite group. A different question, with applications to error-correcting codes and graph colorings, involves covering vector spaces over finite fields by (minimally many) proper subspaces. In this note we cover *R*-modules by proper submodules for commutative rings *R*, thereby subsuming and recovering both cases above. Specifically, we study the smallest cardinal number  $\aleph$ , possibly infinite, such that a given *R*-module is a union of  $\aleph$ -many proper submodules. (1) We completely characterize when  $\aleph$  is a finite cardinal; this parallels for modules a 1954 result of Neumann. (2) We also compute the covering (cardinal) numbers of finitely generated modules over quasi-local rings and PIDs, recovering past results for vector spaces and abelian groups respectively. (3) As a variant, we compute the covering number of an arbitrary direct sum of cyclic monoids. Our proofs are self-contained.

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# 1. Introduction

The covering problem is a well-known and classical question in group theory:

Given a non-cyclic group G, find the minimal (cardinal) number of proper subgroups, termed the covering number  $\sigma(G)$ , whose union is G.

This problem has a long history in the literature. It is immediate that two proper subgroups can never cover *G*; in 1926, Scorza [32] showed that  $\sigma(G) = 3$  if and only if *G* has a quotient isomorphic to the Klein-4 group  $(\mathbb{Z}/2\mathbb{Z})^2$ . This question and related variants have since been the subject of a vast number of papers; a small sampling from the 20th century is [11, 12, 15, 20, 27, 30, 31, 36–38]. The subject also continues to attract much attention more recently, see e.g. [1–6, 8, 10, 17, 23, 25, 33–35] and the references therein.

One of the early results on (finite) coverings of groups is by Neumann (1954), who worked in slightly greater generality—for unions of *cosets* of subgroups:

**Theorem 1.1** (Neumann [27]). Suppose a group G is the union of finitely many cosets

 $g_1G_1,...,g_nG_n$ , where  $g_i \in G, G_i \leq G$ ,

where we write  $H \leq G$  to denote that H is a subgroup of G.

- (1) Then the index  $[G:G_i]$  is finite for some *i*.
- (2) Now suppose no proper sub-union equals G. Then  $[G : G_i]$  is finite for all *i*. In particular, if moreover  $g_i = e \forall i$ , then  $[G : G_1 \cap \cdots \cap G_n]$  is finite.

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(3) Consequently, G can be written as a union of (i.e., admits a covering by) finitely many proper subgroups, if and only if G has a finite non-cyclic quotient.

On a tangential note, we remind the reader that covering the specific group  $G = \mathbb{Z}$  by finitely many cosets (with distinct moduli) was a subject well-studied by Erdős; for instance, some of his conjectures (including with Graham) were settled in 2007—in a stronger form— by Filaseta–Ford–Konyagin–Pomerance–Yu [18].

Returning to coverings of general groups, Rao-Reid used Neumann's Theorem 1.1, together with ideas of Rao-Rao [29], to prove:

**Theorem 1.2** (Rao–Reid [30]). Suppose an abelian group G admits an irredundant covering by finitely many (proper) subgroups.

- (1) Every such covering is induced by lifting a covering of a finite quotient of G.
- (2) The covering number  $\sigma(G) = p + 1$ , where  $p \ge 2$  is the smallest prime such that G/pG is not cyclic.

It is natural to try and extend such results to other settings. For instance, coverings of rings have been studied (including very recently) in [7, 9, 13, 24, 28, 39]. Yet another motivation comes from the short note [22], where the first named author found a sharp bound for the number of proper subspaces of a fixed codimension  $d \ge 1$  needed to cover a vector space. We present here the d=1 result—which is folklore, and is all that is required for the purposes of this paper:

**Proposition 1.3.** Suppose V is a vector space of dimension at least 2 over a field  $\mathbb{F}$ , and define  $\nu(\mathbb{F}, V)$  to be the cardinal number

$$\nu(\mathbb{F}, V) := \begin{cases} |\mathbb{Z}|, & \text{if } \mathbb{F}, \dim V \text{ are both infinite,} \\ |\mathbb{F}| + 1, & \text{otherwise.} \end{cases}$$
(1.4)

Now V can be written as a union of  $\aleph$ -many proper vector subspaces, if and only if  $\aleph \ge \nu(\mathbb{F}, V)$  as cardinal numbers. In other words,  $\sigma(V) = \nu(\mathbb{F}, V)$ .

(Here,  $\sigma(V)$  of course denotes the covering number of V in the category of vector spaces.) As we show in Corollary 3.5, the above result extends to cover free modules  $\mathbb{R}^n$ ,  $n \ge 2$  over a local ring  $(\mathbb{R}, \mathfrak{m})$ : we prove  $\sigma(\mathbb{R}^n) = |\mathbb{R}/\mathfrak{m}| + 1$  as cardinal numbers, irrespective of whether or not  $\mathbb{R}/\mathfrak{m}$  is finite. For now, we provide a quick proof of Proposition 1.3 when  $\mathbb{F}$  is finite. That  $\aleph \ge$  $|\mathbb{F}| + 1$  works, follows by lifting the cover by lines of the plane  $\mathbb{F}^2$ , i.e., projective space, to V. Conversely, if  $\{V_i : i \in \mathcal{I}\}$  is a finite and minimal/irredundant cover of V, we may suppose each  $V_i$  has codimension one, whence  $\bigcap_i V_i$  has finite codimension in V. Working modulo  $\bigcap_i V_i$  now reduces the case to dim $V < \infty$ , so  $|V| < \infty$  and  $|V_i| = |V|/|\mathbb{F}|$ . It follows that  $|\mathcal{I}| = \sigma(V) > |\mathbb{F}|$ .

Covering vector spaces by proper subspaces, and variants of this problem, find applications to error-correcting codes and block designs over finite fields (see the references in *loc. cit.*), as well as to graph colorings (see e.g. [35]). Other variants involving covering a vector space by (translates/shifts) of subspaces can be found in [14, 21].

### 1.1. Contributions

The goal of this note is to unify and extend the study of coverings of abelian groups and of vector spaces by working with R-modules, where R will always denote a unital commutative ring. We list here a few of the contributions.

(1) We completely characterize the modules over an arbitrary ring R, that admit a finite covering. For  $R = \mathbb{Z}$ , we recover the result for finite coverings of abelian groups; our proof is self-contained and works for arbitrary R.

(2) We next determine the covering number in a large number of cases, including: (a) all finitely generated modules over quasi-local rings; (b) all divisible modules over a PID; (c) direct sums of cyclic modules over a local ring (which subsumes the case of vector spaces).

#### 2. Finite coverings of modules, and coverings of finitely generated modules

In this section, we extend the results for abelian groups above, first to finite coverings of (arbitrary) R-modules M for arbitrary unital commutative rings R. This is quickly followed by the case of coverings of finitely generated modules over quasi-local rings (i.e., ones with only finitely many maximal ideals). Our proofs are self-contained.

Our first result completely characterizes—for an arbitrary unital commutative ring R—when an (arbitrary) R-module M admits a finite covering; note this extends Theorem 1.2. In particular, our proof (of the nontrivial, second part) differs from that in [30], as this latter proof for groups does not extend to R-modules.

We begin with two easy observations. First, if M is a finite set and an R-module, and  $\mathfrak{m}M\subsetneq M$ for a maximal ideal  $\mathfrak{m} \subset R$ , then  $R/\mathfrak{m}$  is a finite field (since  $M/\mathfrak{m}M$  is a finite-dimensional vector space over  $R/\mathfrak{m}$ ). From this it follows that  $\mathfrak{m}$  contains a (unique) prime integer  $p_{\mathfrak{m}} \in \mathbb{Z}$ , i.e.,  $\mathfrak{m} \cap$  $R_0 = p_\mathfrak{m}R_0$ , where  $R_0$  is the unital subring of R generated by  $1_R$ . Second, from this it follows that

$$M = \bigoplus_{p \in \mathbb{Z} \text{ prime}} M_{(p)} \tag{2.1}$$

as *R*-modules, where  $M_{(p)}$  is the *p*-(torsion) subgroup of *M* under addition. We now have:

**Theorem 2.2.** Suppose R is a unital commutative ring and M an R module.

(1) Let S be the set of maximal ideals  $\mathfrak{m}$  for which  $\dim_{R/\mathfrak{m}} M/\mathfrak{m}M \ge 2$ , and define

$$\nu'(R,M) := \min_{\mathfrak{m}\in S} |R/\mathfrak{m}| + 1,$$
 (2.3)

as cardinal numbers. If S is nonempty, then  $\sigma(M) \leq \nu'(R, M)$ .

- (2) M admits a finite covering by proper submodules, if and only if M has a finite non-cyclic quotient R-module. Moreover, every such irredundant finite covering is induced by lifting a covering of a finite quotient of M.
- (3) If M admits a finite covering by proper R-submodules, then the covering number  $\sigma(M) = \nu'(R, M)$ .

In other words, the definition of the cardinal number  $\nu(\mathbb{F}, V) = \nu'(\mathbb{F}, V)$  in Equation (1.4) can be extended from *finite-dimensional*  $\mathbb{F}$ -vector spaces V, to R-modules that admit a finite covering:  $\sigma(M) < \infty \Rightarrow \sigma(M) = \nu'(R, M)$ . Note, this cardinal equals  $|R/\mathfrak{m}_0| + 1$  with  $R/\mathfrak{m}_0$  a finite field.

To prove Theorem 2.2, we first provide a short proof—using amenability—of Neumann's Theorem 1.1 for abelian groups.

**Proposition 2.4.** Suppose an amenable group G is an irredundant union of finitely many proper subgroups  $G_1, ..., G_n$ . Then  $[G : G_1 \cap \cdots \cap G_n]$  is finite (whence so is each  $[G : G_i]$ ).

Consequently, an abelian group G can be written as a union of (i.e., admits a covering by) finitely many proper subgroups, if and only if G has a finite non-cyclic quotient.

*Proof.* Since G is amenable, denote the relevant finitely additive left-invariant probability measure on G by  $\mu$ . By relabeling, there exists  $0 \le t \le n$  such that  $G_1, ..., G_t$  have finite index in G, while  $G_{t+1}, ..., G_n$  do not. Note that  $\mu(G_i) = 0$  for i > t, so we must have t > 0, whence the index

$$m := \left[G : \cap_{i=1}^t G_i\right] \in (1,\infty).$$

Now  $G \setminus \bigcup_{i=1}^{t} G_i$  is a disjoint union of cosets of  $\bigcap_{i=1}^{t} G_i$ , so if t < n, then we compute:

$$1-\frac{1}{m}=\mu(G\setminus \cup_{i=1}^t G_i)\leq \mu(G_{t+1}\cup\cdots\cup G_n)\leq \sum_{i=t+1}^n\mu(G_i)=0.$$

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This contradiction shows the first part. The second part is standard: if G is abelian and admits a finite covering, then so does its finite quotient  $G / \bigcap_{i=1}^{n} G_i$  from above, whence this cannot be a cyclic group. The converse is immediate by lifting a cover by the cyclic proper subgroups, of G/H (a finite, non-cyclic quotient group).

The next argument is useful in multiple situations, hence is isolated into a standalone result. In it and in the sequel,  $\sigma(M)$  will always denote the covering number of an *R*-module *M* by proper *R*-submodules.

**Proposition 2.5.** Fix an integer  $n \ge 1$ , finitely many unital commutative rings  $R_1, ..., R_n$ , and an  $R_k$ -module  $M^{(k)}$  for all k, such that  $M^{(k_0)}$  admits a covering by proper  $R_{k_0}$ -submodules for at least one  $k_0 \in [1, n]$ . Letting  $\mathcal{K}_0$  denote the set of all such  $k_0$ , and setting  $R := \bigoplus_{k=1}^n R_k, M := \bigoplus_{k=1}^n M^{(k)}$ , we have  $\sigma(M) = \min_{k_0 \in \mathcal{K}_0} \sigma(M^{(k_0)})$  as cardinal numbers.

This minimum cardinal number exists (as do all such minima below) e.g. by [26].

*Proof.* That  $\sigma(M) \leq \min_{k_0 \in \mathcal{K}_0} \sigma(M^{(k_0)})$  is immediate: let  $k' \in \mathcal{K}_0$  attain this minimum, and lift a cover of  $M^{(k')}$  to M.

Conversely, suppose  $M = \bigcup_{i \in \mathcal{I}} M_i$  is a covering with  $|\mathcal{I}| = \sigma(M)$ . We first get rid of the "cyclic" factors. Namely, suppose without loss of generality that  $M^{(k)}$  is cyclic as an  $R_k$ -module, say  $M^{(k)} = R_k v_k$ , for k = 1, ..., t. Note that t < n since  $\mathcal{K}_0$  is nonempty. Now given  $M = \bigcup_{i \in \mathcal{I}} M_i$ , we claim that the module  $M/(M^{(1)} \oplus \cdots \oplus M^{(t)})$  over the ring  $R' := \times_{k>t} R_k$  is covered by  $M_i/(M^{(1)} \oplus \cdots \oplus M^{(t)})$  for all  $i \in \mathcal{I}$  such that  $v_1, ..., v_t \in M_i$ . (Call this subset  $\mathcal{I}_0 \subset \mathcal{I}$ .) Indeed, given  $(v_k)_{k>t} \in M/(M^{(1)} \oplus \cdots \oplus M^{(t)})$ , we must have  $\mathbf{v} := (v_1, ..., v_t, (v_k)_{k>t}) \in M_i$  for some  $i \in \mathcal{I}$ . But then  $v_k = 1_{R_k} \mathbf{v} \in M_i$  for  $1 \le k \le t$ , whence  $i \in \mathcal{I}_0$ ; and  $\bigoplus_{k>t} 1_{R_k} \cdot \mathbf{v} = (v_k)_{k>t} \in M_i$  as well, showing the claim.

Thus, we may assume now that every  $M^{(k)}$  admits a covering by proper  $R_k$ -submodules. Given a module N over  $R = \times_{k=1}^n R_k$ , write  $N = \bigoplus_{k=1}^n N^{(k)}$ , with  $N^{(k)} = R_k N$ . Thus for each *i*, we have  $M_i = \bigoplus_{k=1}^n M_i^{(k)}$  is proper, whence there exists  $1 \le k_i \le n$  with  $M_i^{(k_i)} \ne M^{(k_i)}$ . We may then replace  $M_i$  by

$$M'_i := M_i^{(k_i)} \oplus \oplus_{k \neq k_i} M^{(k)},$$

and still obtain a covering of M by  $\sigma(M)$ -many proper R-submodules  $M'_i$ .

Finally, let  $\mathfrak{I}_k := \{i \in \mathfrak{I} : k_i = k\}$  for  $1 \leq k \leq n$ . We claim there exists  $1 \leq k \leq n$  such that  $\{M_i^{(k)} : i \in \mathfrak{I}_k\}$  is a covering of  $M^{(k)}$ ; notice that this would imply that  $\sigma(M) \geq \sigma(M^{(k)})$ , and conclude the proof. This claim is shown by contradiction: if no such k exists, then for every k there exists  $m_k \in M^{(k)}$  that does not belong to  $M_i^{(k)}$  for any  $i \in \mathfrak{I}_k$ . But then  $\bigoplus_{k=1}^n m_k \notin M'_i$  for all  $i \in \mathfrak{I}$ .

With these preliminary results at hand, we can now completely characterize when an R-module M has a finite covering.

### Proof of Theorem 2.2.

(1) This is immediate: if  $\dim_{R/\mathfrak{m}} M/\mathfrak{m} M \ge 2$ , then by Proposition 1.3, the quotient module (over the field  $R/\mathfrak{m}$ )

$$M \rightarrow M/\mathfrak{m} M \rightarrow (R/\mathfrak{m})^2$$

is a union of  $|R/\mathfrak{m}| + 1$  proper submodules, which can then be lifted to a covering of *M*. Now take the minimum over  $\mathfrak{m} \in S$ .

- (2) First suppose M admits a finite covering, say a minimal/irredundant one, by proper submodules  $M_1, ..., M_n$ . Then  $M/\bigcap_i M_i$  is finite by Proposition 2.4, and is not cyclic, else the cyclic generator would be contained in some  $M_{i_0}/\bigcap_i M_i$ . Moreover, the covering of M by the  $M_{i_0}$  is the lift of the covering of  $M/\bigcap_i M_i$  by the  $M_{i_0}/\bigcap_i M_i$ . Conversely, if M/N is finite and non-cyclic, then M/N is the union of its (finitely many) cyclic submodules, and this lifts to a finite covering of M.
- (3) One inequality follows from (2.3). To show the reverse inequality, given a module M admitting a finite covering, we begin with two reductions. (a) From the preceding part, we may assume M is finite. (b) We reduce to the case of finite rings, by working over  $R_0 := R/\operatorname{Ann}_R(M)$  instead of R. Indeed,  $R_0$  embeds into  $\operatorname{End}(M)$  (via  $r \mapsto (m \mapsto rm)$ ), which is a finite set.

Now we proceed. Given an irredundant covering  $M = M_1 \cup \cdots \cup M_n$ , each  $M_i$  has a nonzero simple quotient, say  $R_0/\mathfrak{m}_i$ , whence  $M_i/\mathfrak{m}_iM_i \neq 0$ . Let the ideal  $I \subset R_0$  be the product of the distinct ideals among  $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ , which we will index henceforth by  $\mathfrak{m}'_k$ . Working over the quasi-local ring

$$R_0/I = R_0/\cap_k \mathfrak{m}'_k \simeq \times_k (R_0/\mathfrak{m}'_k), \qquad (2.6)$$

the images  $M_i/(IM \cap M_i)$  are proper submodules of M/IM (by Nakayama's lemma) which also provide a covering. This shows that  $\sigma(M) \ge \sigma(M/IM)$ , and we are reduced to the case where the ring  $R_0$  is a product of the fields  $\mathbb{F}^{(k)} := R_0/\mathfrak{m}'_k$ . Now write  $M = \bigoplus_k M^{(k)}$  with  $M^{(k)} := \mathbb{F}^{(k)}M$ ; if  $\dim_{\mathbb{F}^{(k)}}M^{(k)} \le 1 \forall k$ , then  $M = R_0/I$  is cyclic and cannot be covered by proper submodules, a contradiction. Thus there exists k such that  $\dim_{\mathbb{F}^{(k)}}M^{(k)} \ge 2$ , whence the vector space  $M^{(k)}$  admits a covering by proper  $\mathbb{F}^{(k)}$ -subspaces. The proof is now completed by applying Proposition 2.5.

# 2.1. Two settings involving covering finitely generated modules

We now come to the case of possibly infinite minimal/irredundant coverings of *R*-modules. Note that this case does not arise when considering abelian groups, since in that case  $R = \mathbb{Z}$  has no maximal ideal m with infinite residue field m. Thus, the results and methods in this section necessarily lie outside the group covering setting.

**Definition 2.7.** Let *R* be a unital commutative ring, and *M* an *R*-module admitting a covering by proper *R*-submodules. Define the **covering number**  $\sigma(M)$  to be the smallest cardinal number  $\aleph$  such that *M* is covered by  $\aleph$ -many proper submodules, but no fewer—where "fewer" means any cardinal that injects into  $\aleph$  but is not in bijection with it. (We use here the trichotomy of cardinal numbers, which is a consequence of the Axiom of Choice; also,  $\sigma(M)$  exists e.g. by [26].)

In this subsection, we study coverings of finitely generated modules in two settings. The first involves R being a quasi-local ring—i.e., a commutative unital ring with only finitely many maximal ideals. In this case, the same formula as in Theorem 2.2(2) applies:

**Theorem 2.8.** Let R denote a quasi-local ring, and M be a finitely generated R-module. Then

$$\sigma(M) = \nu'(R, M) = \min_{\mathfrak{m} \in S} |R/\mathfrak{m}| + 1$$

as cardinal numbers, where  $\nu'$ , S were defined in (2.3). (If S is empty, then M is cyclic.)

Thus, akin to Equation (1.4) and also the remarks following Theorem 2.2, once again the cardinal number  $\nu'(R, M)$  turns out to be the covering number.

As the arguments were essentially written out above, we merely sketch this proof.

*Proof.* That  $\sigma(M) \leq \min_{\mathfrak{m} \in S} |R/\mathfrak{m}| + 1$  (when at least one such  $\mathfrak{m}$  exists) is as in the proof of Theorem 2.2(2). To show the reverse inequality, let  $M = \bigcup_{i \in J} M_i$  be a covering by proper

submodules, where  $|\mathcal{I}|$  is in bijection with  $\sigma(M)$ . If  $\mathfrak{m}'_k$  denote the finitely many maximal ideals in R, let  $I = \bigcap_k \mathfrak{m}'_k$  and work modulo I. Now the arguments following (2.6) apply verbatim to complete the proof.

In our second setting here, the ring *R* no longer needs to be quasi-local, but contains a field  $\mathbb{F}$  (with unit  $1_R$ ).

**Theorem 2.9.** Suppose  $R \supset \mathbb{F} \ni 1_R = 1_{\mathbb{F}}$  as above, and M is a finitely generated non-cyclic R-module. Then  $\sigma(M) \ge |\mathbb{F}| + 1$ . If moreover  $\mathbb{F}$  is an infinite field and R is countably generated over  $\mathbb{F}$ , then  $\sigma(M) = |\mathbb{F}| + 1$ .

The point of interest here is that the (lower and upper) bounds are universal for all finitely generated *R*-modules.

*Proof.* Take the smallest integer  $n \ge 2$  such that M has n generators. Now  $\mathbb{R}^n \to M$ , so any covering of M lifts to one of  $\mathbb{R}^n$ , whence  $\sigma(M) \ge \sigma(\mathbb{R}^n)$ . Thus we reduce to  $M = \mathbb{R}^n$  with  $n \ge 2$ . Now suppose  $\mathbb{R}^n = \bigcup_{i \in \mathbb{J}} M_i$  is an irredundant covering, so that each  $M_i \ne \mathbb{R}^n$  and  $|\mathbb{J}| = \sigma(\mathbb{R}^n)$ . Since  $\mathbb{R} \cdot \mathbb{F}^n = \mathbb{R}^n$ , it follows that  $M_i \cap \mathbb{F}^n$  is a proper vector subspace of  $\mathbb{F}^n$ . But then,

$$|\mathbb{F}| + 1 = \sigma(\mathbb{F}^n) \le |\mathfrak{I}| = \sigma(R^n) \le \sigma(M),$$

where the first equality is by Proposition 1.3. This shows the first assertion; the second will follow from the claim that  $\sigma(M) \leq |\mathbb{F}| + 1$ . To show the claim, first note that  $|R| = |\mathbb{F}|$ , since every finitely generated  $\mathbb{F}$ -algebra has size  $|\mathbb{F}|$ . Now since M is finitely generated,  $|M| = |\mathbb{F}|$ ; and since M is not cyclic, it is the union of its cyclic submodules, whence  $\sigma(M) \leq |\mathbb{F}|$ .  $\Box$ 

In the setting of the preceding result, notice that every maximal ideal  $\mathfrak{m}$  is an  $\mathbb{F}$ -vector subspace of R, whence every residue field  $R/\mathfrak{m}$  has the same size as  $\mathbb{F}$ . Given this and the earlier results in this section, we end with the following natural question:

In what generality for the ring R, is it true that for all finitely generated non-cyclic R-modules M, we have  $\sigma(M) = \nu'(R, M) = \min_{\mathfrak{m} \in S} |R/\mathfrak{m}| + 1$  as in (2.3)?

# 3. Divisible groups and modules; direct sums of cyclic monoids

Having understood covering numbers of abelian groups with finite coverings, in particular finitely generated abelian groups, it is natural to turn to other groups. In particular, every divisible abelian group (such as  $\mathbb{Q}$ ) is not a finite union but is a countable union of proper subgroups. (The more general result—over PIDs—is mentioned below.) The same result holds for groups of the form  $\bigoplus_p(\mathbb{Z}/p\mathbb{Z})$ , whenever the sum runs over an infinite set of pairwise distinct prime integers. Indeed, that it is a countable union of proper subgroups is as in the next paragraph, and it is not a finite union by e.g. Theorem 1.2. This is an example of a  $\mathbb{Z}$ -module which is not cyclic, but whose quotient modulo every maximal ideal is.

In a similar vein, for a field  $\mathbb{F}$  it is easy to see (the assertion in Proposition 1.3) that an infinite-dimensional vector space over an infinite field is a countable but not finite union of proper subspaces: simply write the basis as the union of a countable nested sequence of nonempty proper subsets, and take their spans.

**Remark 3.1.** There are also negative examples—see e.g. [40, Example 2.7] for the example of a local (in fact valuation) ring  $(R, \mathfrak{m})$ , with  $M = \mathfrak{m}$  not a countable union of proper submodules.

Yet another setting (which subsumes the case of abelian groups but not vector spaces) is that of modules over a PID R. The case of finitely generated torsion R-modules is easy to discuss (see later in this section); for now, we mention the case of divisible modules.

**Proposition 3.2.** If M is a nonzero divisible module over a PID R (which is assumed to not be a field), then  $\sigma(M) = |\mathbb{Z}|$ . More generally,  $\sigma(M) \leq |\mathbb{Z}|$  whenever M is not reduced.

**Proof.** No divisible module M has a finite (in fact nonzero) quotient  $M/\mathfrak{m}M$ , hence is not a finite union of proper submodules by Theorem 2.2. On the other hand, by standard results—see e.g. [16, Exercises, §4.7] and [19]—every R-module is the direct sum of a divisible (equivalently, injective) module and a reduced module; and every divisible R-module is the direct sum of copies of its field of fractions  $\mathbb{F}$  and the "Prüfer p-modules"

$$M_p = R[p^{\infty}] := R[1/p]/R \subset \mathbb{F}/R, \quad p \text{ prime } \in \operatorname{Specm}(R).$$

Now it is easy to show that the only *R*-submodules of  $M_p$  are  $R[1/p^n]/R$ , whence the assertion follows for  $M = M_p$ . Similarly, for  $M = \mathbb{F}$  (the quotient field of *R*),  $\mathbb{F}$  is the nested union of submodules  $M'_n$ , where we fix a prime  $0 \neq p \in R$  and let

$$M'_n := \{a/b : a, b \in R, (a, b) = 1, p^n \not\mid b\}, \quad n \ge 1.$$

Since at least one module from among  $\mathbb{F}$ ,  $M_p$  occurs in M, we have  $\sigma(M) = |\mathbb{Z}|$  for M divisible. If instead M is merely non-reduced, then M contains  $\mathbb{F}$  or  $M_p$  as a direct summand, whence  $\sigma(M) \leq |\mathbb{Z}|$ .

We conclude this part by discussing how far the above results take us, in a restricted setting which in the above special cases (abelian groups, vector spaces, or modules over PIDs) is a prominent case already studied above. Namely, instead of working with finitely generated *R*-modules, we instead consider direct sums of cyclic modules. In this case, the "stopping point" seems to be the free module  $M = R^2$  for general rings *R*.

We now make this more precise, by writing down a sequence of observations, and follow this by deducing several corollaries for coverings of modules. We also apply these observations to minimally cover direct sums of cyclic monoids. Begin with a nonempty direct sum of *nonzero* cyclic modules

$$M = \oplus_{k \in \mathcal{K}} M_k$$

over an arbitrary commutative unital ring R.

- (1) Define S and  $\nu'(R, M)$  as in Theorem 2.2(1). Then S equals the collection of maximal ideals m that contain the annihilators of  $M_k$  for at least two  $k \in \mathcal{K}$ .
- (2) If S is empty and  $\mathcal{K}$  is finite, then M is cyclic. Indeed, if  $M_k \cong R/I_k$  for all k, then each  $I_k$  is contained in exactly one maximal ideal, so the  $I_k$  are coprime, and  $M \cong R/\bigcap_{k \in \mathcal{K}} I_k$  by the Chinese remainder theorem.
- (3) If S is empty and  $\mathcal{K}$  is infinite, then M is covered by a countable sequence of nested submodules (as discussed prior to Remark 3.1). Since M is not a finite union of proper submodules by Theorem 2.2, we have  $\sigma(M) = |\mathbb{Z}|$ .
- (4) Otherwise, S is nonempty. If  $\nu'(R, M) < \infty$ , then Theorem 2.2 applies, and  $\sigma(M) = \nu'(R, M)$ .
- (5) Otherwise, S is nonempty and  $\nu'(R, M)$  is an infinite cardinal. If  $|\mathcal{K}| = \infty$ , then  $\sigma(M)$  is not finite (by Theorem 2.2), so  $\sigma(M) = |\mathbb{Z}|$  (as discussed in a previous case).
- (6) Otherwise, S is nonempty, |𝔅| < ∞, and ν'(R, M) is infinite. If R has finitely many maximal ideals—i.e., |Specm(R)| < ∞, then σ(M) = ν'(R, M) by Theorem 2.8.</p>
- (7) (This is the "outstanding" case.) Otherwise, S is nonempty, |𝔅| < ∞, and ν'(R, M) and |Specm(R)| are both infinite cardinals. As mentioned above, the torsion-free component of M is the "stopping point" when one tries to understand minimal coverings.</p>

As a consequence of the above discussion, several special cases of rings (and sums of cyclic modules over them) fall out as immediate corollaries. We present here a sampling, starting with

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the final point above, where the "torsion-free component" of M is trivial, and we work over a Dedekind domain:

**Corollary 3.3.** Suppose M is a finitely generated torsion module over a Dedekind domain. Then  $\sigma(M) = \nu'(R, M)$  if S is nonempty, where  $\nu'(R, M)$  and S are as in Theorem 2.2(1); and if S is empty then M is cyclic.

*Proof.* It is well-known here that M is a finite direct sum of cyclic torsion R-modules, say  $M_1, ..., M_n$ . Thus, we reduce the situation to covering the finitely generated module M/IM over the quasi-local ring R/I, with  $I = \bigcap_{k=1}^n \operatorname{Ann}_R(M_k)$ . Now apply Theorem 2.8.

The next corollary is immediate by Theorem 2.2, and resolves the above "outstanding" case under alternate additional assumptions on R.

**Corollary 3.4.** Suppose the assumptions in the final, outstanding case in the preceding discussion apply:  $M = \bigoplus_{k \in \mathcal{K}} M_k$  is a direct sum of nonzero cyclic R-modules, where S is nonempty,  $|\mathcal{K}| < \infty$ , and  $\nu'(R, M)$  and |Specm(R)| are both infinite cardinals. If moreover R is such that every residue field of R/m is at most countable, then  $\sigma(M) = |\mathbb{Z}|$ , the smallest infinite cardinal.

As a third sample, we extend Proposition 1.3 to the case of local rings:

**Corollary 3.5.** Suppose  $(R, \mathfrak{m})$  is a local ring, and  $M = \bigoplus_{k \in \mathcal{K}} M_k$  is a direct sum of cyclic nonzero R-modules, with  $|\mathcal{K}| \ge 2$ . Then  $\sigma(M) = |\mathbb{Z}|$  if  $R/\mathfrak{m}$  and  $\mathcal{K}$  are both infinite, else  $\sigma(M) = \nu'(R, M) = |R/\mathfrak{m}| + 1$ .

This too is shown by following the steps in the above discussion. There is a similar result (and proof) for R a direct product of finitely many local rings.

# 3.1. Covering sums of cyclic monoids

We end with a final result, which is an application of the discussion immediately above. We use the above results to answer a related variant: that of covering monoids—specifically, direct sums of cyclic monoids—in the spirit of finitely generated abelian groups or vector spaces. Namely:

Given a direct sum M of cyclic monoids, how many proper sub-monoids are required to cover M?

**Theorem 3.6** (Sums of cyclic monoids). Suppose M is a direct sum of cyclic monoids. Then exactly one of the following holds:

- (1) M is a cyclic monoid, in which case it has no covering by proper sub-monoids.
- (2) *M* is an abelian group but not a cyclic monoid. In this case, either the set *S* defined in Theorem 2.2(1) is empty and  $\sigma(M) = |\mathbb{Z}|$ ; or *S* is nonempty, in which case  $3 \le \sigma(M) < \infty$  and  $\sigma(M) = \nu'(\mathbb{Z}, M)$  is obtained from Theorem 2.2 (or Theorem 1.2).

(3) In all other cases, 
$$\sigma(M) = 2$$
.

In contrast, recall that no group is a union of two proper subgroups.

*Proof.* Suppose M is not a cyclic monoid. If M is a group and a direct sum of cyclic monoids, then each factor is of the form  $\mathbb{Z}/n_k\mathbb{Z}$  with  $n_k \geq 2$ , for  $k \in \mathcal{K}$ , say. Now the above discussion, following the proof of Proposition 3.2, shows the assertion (2) in this corollary. Finally, if M is not a group and not a cyclic monoid, then one can write  $M = M_1 \oplus M_2$ , with  $M_1 = \langle f_1 \rangle$  a cyclic monoid that is not a group, and  $M_2$  a nontrivial monoid. Then M is the union of  $M_2$  and the monoid  $(M \setminus M_2) \sqcup \{0\}$ ; in fact this is a partition (in that the two sub-monoids intersect only at 0).

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