Supplementary material

I. CONVENTIONS

In the main text, we have introduced $c^{(k)}_{\Delta,\ell} = C_{\Delta,\ell}\mathcal{N}_{\Delta,\ell}\mathcal{R}^{(k)}_{\Delta,\ell}$, where $C_{\Delta,\ell}$ is the OPE coefficient squared as defined in [1] and

$$\mathcal{N}_{\Delta,\ell} = \frac{2^\ell(\Delta+\ell-1)\Gamma^2(\Delta+\ell-1)\Gamma(\Delta-h+1)}{\Gamma(\Delta-1)\Gamma^4\left(\frac{\Delta+\ell}{2}\right)\Gamma^2\left(\Delta_\phi-\frac{\Delta-\ell}{2}\right)\Gamma^2\left(\Delta_\phi-\frac{2h-\Delta-\ell}{2}\right)}\,, \mathcal{R}^{(k)}_{\Delta,\ell} = \frac{\Gamma^2\left(\frac{\Delta+\ell}{2}+\Delta_\phi-h\right)\left(1+\frac{\Delta-\ell}{2}-\Delta_\phi\right)_k^2}{k!\Gamma(\Delta-h+1+k)}\,.$$

A suitable form for the Mack polynomial can be found in [1]

$$P^{(s)}_{\Delta-h,\ell}(s,t) = \sum_{m=0}^{\ell}\sum_{n=0}^{\ell-m}\mu^{(\Delta,\ell)}_{n,m}\left(\frac{\Delta-\ell}{2}-s\right)_m\left(\Delta_\phi-t\right)_n\,, \tag{1}$$

where

$$\mu^{(\Delta,\ell)}_{n,m} = \frac{2^{-\ell}\ell!(-1)^{m+n}(h+\ell-1)_{-m}\left(\frac{\ell+\Delta}{2}-m\right)_m(\ell+\Delta-1)_{n-\ell}\left(\frac{\Delta-\ell}{2}+n\right)_{\ell-n}\left(\frac{\Delta-\ell}{2}+m+n\right)_{\ell-m-n}}{m!n!(\ell-m-n)!}$$
$${}_4F_3\left(-m,-h+\frac{\Delta-\ell}{2}+1,-h+\frac{\Delta-\ell}{2}+1,n+\Delta-1;\frac{\Delta+\ell}{2}-m,\frac{\Delta-\ell}{2}+n,-2h-\ell+\Delta+2;1\right)\,. \tag{2}$$

Our definition $P^{(s)}_{\Delta-h,\ell}(s,t)$ is different from [1] by a shift in t, namely $t\to t-\Delta_\phi$. For our purposes it is convenient to introduce

$$P_{\Delta,\ell}(s_1,s_2) = P^{(s)}_{\Delta-h,\ell}(s_1+\frac{2\Delta_\phi}{3},s_2+\frac{2\Delta_\phi}{3})\,.$$

II. 2D ISING MODEL

In our conventions, the Mellin amplitude for the 2d Ising model is given by ($\Delta_\phi = 1/8$)

$$\mathcal{M}^{(2d\text{ Ising})}(s_1,s_2) = \frac{\sqrt{\frac{2}{\pi}}\Gamma\left(-2s_1-\frac{1}{6}\right)\Gamma\left(-2s_2-\frac{1}{6}\right)\Gamma\left(-2s_3-\frac{1}{6}\right)}{\Gamma^2\left(\frac{1}{24}-s_1\right)\Gamma^2\left(\frac{1}{24}-s_2\right)\Gamma^2\left(\frac{1}{24}-s_3\right)}\,. \tag{3}$$

The dispersion relation Eq. (6) in main text gives the following crossing symmetric pole expansion of the amplitude:

$$\mathcal{M}^{(2d\ Ising)}(s_1,s_2) = \alpha_0^{(2d\ Ising)} + \sum_{k=0}^{\infty}\Bigg[\left(\frac{1}{\frac{k}{2}-s_1-\frac{1}{12}}+\frac{1}{\frac{k}{2}-s_2-\frac{1}{12}}+\frac{1}{\frac{k}{2}-s_3-\frac{1}{12}}-\frac{3}{\frac{k}{2}-\frac{1}{12}}\right)\times\frac{(-1)^{1-k}}{\sqrt{2\pi}k!\Gamma\left(\frac{1}{8}(1-4k)\right)^2}$$
$$\times\frac{\Gamma\left(\frac{1}{6}\left(-\frac{1}{2}(6k-1)\left(\sqrt{1-\frac{48a}{12a-6k+1}}-1\right)-1\right)\right)\Gamma\left(\frac{1}{12}\left(12k+(6k-1)\left(\sqrt{1-\frac{48a}{12a-6k+1}}-1\right)-4\right)\right)}{\Gamma^2\left(\frac{1}{24}\left(1-\left(\sqrt{1-\frac{48a}{12a-6k+1}}-1\right)(6k-1)\right)\right)\Gamma^2\left(\frac{1}{24}\left(12k+(6k-1)\left(\sqrt{1-\frac{48a}{12a-6k+1}}-1\right)-1\right)\right)}\Bigg]\,. \tag{4}$$

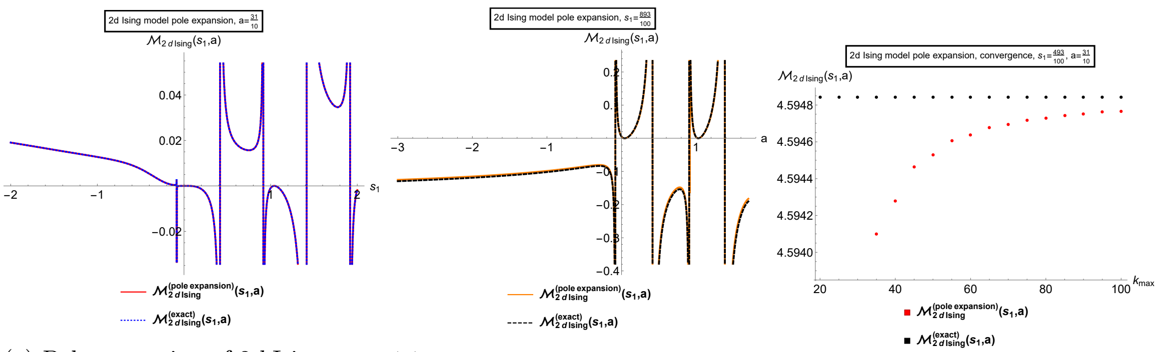
How good is this expansion? To answer this, we truncate the pole sum $\sum_{k=0}^{\infty}\to\sum_{k=0}^{k_{max}}$ and numerically compare with (3), see table (I) (in Mathematica we had to use \$MaxExtraPrecision = 1000 command for better numerical precision). We find that $\alpha_0^{(2d\ Ising)} = -1.48589\times10^{-6}$. As can be seen from the table, the representation is very good for a variety of random values (including complex ones) for s_1, s_2.

Important note: Note that there can be a small mismatch very close to the pole position, where the amplitude diverges. This slight mismatch near the poles is due to the truncation of k sum up to k_{max}.

CFT sum rules and 2d Ising model: We illustrate how CFT sum rules, that fix the spectrum, follow directly from the crossing symmetric pole expansion of the above amplitude. Note that we can write (4) as $\mathcal{M}_{2d\ Ising}(a,z) =$

s_1	s_2	Exact	$k_{\max} = 100$	$k_{\max} = 400$
4.6	0.1	-0.0032343	-0.0032459	-0.0032345
8.3	-1.6	-0.06009	-0.059382	-0.060007
8.2+2.1 i	-1.6-4.3 i	0.04502 -0.02430 i	0.0457-0.0239 i	0.04510 -0.02426 i
3.3+3.1 i	2.3 +6.7 i	0.038004 -0.0405 i	0.038049 -0.039 i	0.038007 -0.0403i
8.2 +9.7 i	16.3 +29.1 i	0.1001-0.111 i	0.1014-0.084 i	0.1005-0.107i
3.1 i	0.2 i	0.008639 -0.003339 i	0.0086 -0.00332 i	0.008636 -0.003337i
13+19 i	21+33 i	0.131 -0.144 i	0.134-0.099 i	0.132-0.136i

TABLE I: Numerical comparison between the $2d$ Ising model amplitude and the crossing symmetric pole expansion of the amplitude. For the pole expansion, we have truncated the k sum upto $k_{\max}$



(a) Pole expansion of $2d$ Ising model amplitude for fixed $a = 31/10$. k sum truncated at $k_{max} = 40$

(b) Pole expansion of $2d$ Ising model amplitude for fixed $s_1 = 893/100$. k sum truncated at $k_{max} = 40$

(c) Convergence with k_{max} for $a = 31/10,\ s_1 = 493/100$.

FIG. 1: Numerical comparisons and convergence of the pole sum (in k)

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$\sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{M}^{(k)}_{n-m,m}\,x^n\,a^m$. The expansion in eq. (4) in main text tells us that $\mathcal{M}^{(2d\ Ising)}_{n-m,m} = \sum_{k=0}^{\infty}\mathcal{M}^{(k)}_{n-m,m} = 0\,,$ when $m > n$. For numerical purpose, we truncate the k sum upto $k_{\max}$. As we increase $k_{\max}$, we get $\mathcal{M}^{(2d\ Ising)}_{n-m,m} \to 0$ when $m > n$. ***Further numerical checks with 2d Ising model:*** We will put $s_2 = s_2^{(+)}(s_1, a)$ in (3) and call it $\mathcal{M}^{(exact)}_{2d\ Ising}(s_1, a)$. In (4), we will put $s_2 = s_2^{(+)}(s_1, a)$ and call it $\mathcal{M}^{(pole\ expansion)}_{2d\ Ising}(s_1, a)$, with truncated sum $\sum_{k=0}^{\infty} \to \sum_{k=0}^{k_{max}}$. For $k_{max} = 40$ the numerical comparisons are presented in figure 1(a,b). Convergence with increasing k_{max} is presented in figure 1(c).

III. CONVERGENCE OF EQ (6) AND DOMAIN OF a

We derive the domain of a where eq. (6) in main text converges, following [2] (see their Eq. (2.6)). To check convergence, as in [2], we will focus on the large ℓ contribution to the sum over spectrum. In this contribution, it is known that operators with dimensions $\Delta = 2\Delta_\phi + \ell + 2n + \gamma(n,\ell)$ contribute with $\gamma(n,\ell) \sim \ell^{-2\tau^{(0)} - \frac{4\Delta_\phi}{3}}$. For $s_1 = \frac{\Delta_\phi}{3} + p$ the contribution of these operators to eq. (6) in main text is given by

$$\sum_{n=0}^{p}\sum_{\ell}\frac{c^{(p-n)}_{\Delta,\ell}}{\gamma(n,\ell)}P_{\Delta,\ell}(\tau_{p-n}, s_2'(\tau_{p-n}, a)) + (t,u)\text{-channel}\,. \tag{5}$$

The t,u-channels contribute at subleading order. From the large ℓ behaviour of $c^{(k)}_{\Delta,\ell}P_{\Delta,\ell}(\tau_k,s_2)\sim \ell^{-4\tau_k-\frac{4\Delta_\phi}{3}+2s_2-1}$, we get the leading term as

$$\sim \ell^{-2\tau^{(0)}+2s_2'(\tau^{(0)},a)-1}\,. \tag{6}$$

We consider a to be real and readily find that the ℓ sum converges if and only if $\tau^{(0)}>s_2'(\tau^{(0)},a)$, which, using the Reduce command in Mathematica, implies the conditions: for $\tau^{(0)}>0$, we have $-\frac{\tau^{(0)}}{3}\le a<\frac{2\tau^{(0)}}{3}$ and for $\tau^{(0)}<0$, we have $\tau^{(0)}<a<\frac{2\tau^{(0)}}{3}$. In order to implement the derivative Polyakov condition, one can take derivative of eq. (6) in main text w.r.t s_1 and doing a similar analysis for large ℓ, one gets

$$\sim \ell^{\frac{4\Delta_\phi}{3}+2s_2'(\tau^{(0)},a)-1}+(t,u)\text{-channel}\,. \tag{7}$$

The ℓ sum convergence if and only if $s_2'(\tau^{(0)},a)+\frac{2\Delta_\phi}{3}<0$ which leads to the conditions: for $\tau^{(0)}>0$, we have $0<\Delta_\phi<\frac{3\tau^{(0)}}{4}$ and $-\frac{\tau^{(0)}}{3}\le a<\frac{4\tau^{(0)}\Delta_\phi^2-6(\tau^{(0)})^2\Delta_\phi}{-6\tau^{(0)}\Delta_\phi+9(\tau^{(0)})^2+4\Delta_\phi^2}$, and for $\tau^{(0)}<0$, we have $\tau^{(0)}<a<\frac{4\tau^{(0)}\Delta_\phi^2-6(\tau^{(0)})^2\Delta_\phi}{-6\tau^{(0)}\Delta_\phi+9(\tau^{(0)})^2+4\Delta_\phi^2}$. Note that a is always negative in this case. Away from $s_1=\frac{\Delta_\phi}{3}+p$ we find the leading contribution at large ℓ to eq. (6) in main text to be

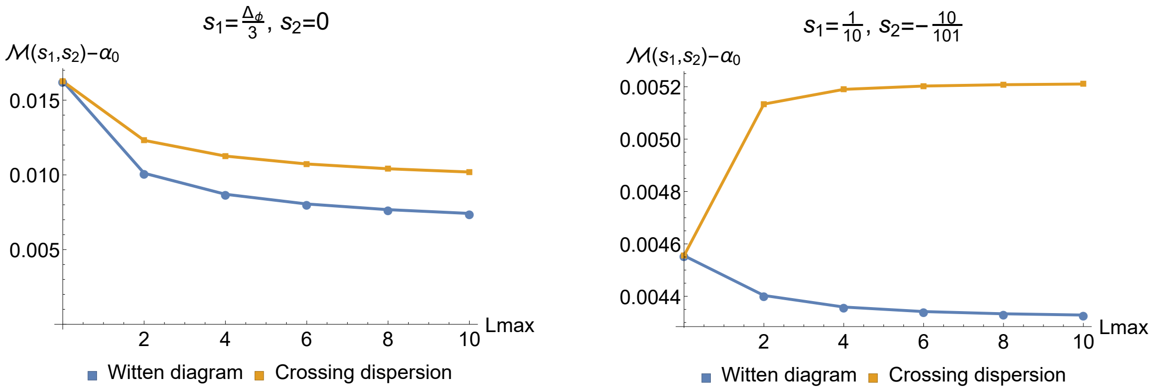
$$\sim \frac{\ell^{-4\tau^{(0)}-\frac{4\Delta_\phi}{3}+2s_2'(\tau^{(0)},a)-1}}{\tau_k-s_1}\,. \tag{8}$$

The ℓ sum convergence is guaranteed for $2\tau^{(0)}>s_2'(\tau^{(0)},a)$ (we assume $\Delta_\phi>0$): for $\tau^{(0)}>0$, we have $-\frac{\tau^{(0)}}{3}<a<\frac{6\tau^{(0)}}{7}$ and for $\tau^{(0)}<0$, we have $\tau^{(0)}<a<\frac{6\tau^{(0)}}{7}$. The bottom line is that we have demonstrated the existence of a range of a-values where the conformal partial wave expansion converges. The expansion in eq. (6) in main text and the expansion in terms of Witten diagrams as in eq. (11) in main text differ by the locality constraints which are the same as the conditions arising from imposing crossing symmetry in the fixed-t dispersion relation [2]. Assuming the latter set of conditions converge, the Witten diagram expansion eq. (11) in main text can be expected to be convergent.

IV. CONVERGENCE OF EQ. (6) AND EQ. (11): NUMERICAL CHECKS

Here we will perform only some preliminary numerical checks of convergence of eq. (6) and (11) (in main text) and will find evidence that the sums in both equations are convergent. In order to check convergence, we use the $3d$ Ising model spectrum [3] upto spin 10, namely $\Delta_\phi=0.518149$, $\Delta_{\ell=0}=1.41263, 3.82968, 6.8956, 7.2535$, $\Delta_{\ell=2}=3., 5.50915, 7.0758$, $\Delta_{\ell=4}=5.02267, 6.42065, 7.38568$, $\Delta_{\ell=6}=7.02849$, $\Delta_{\ell=8}=9.03102$, $\Delta_{10}=11.0324$. We truncate the spin sum upto L_{max}. The plots with $\mathcal{M}(s_1,s_2)-\alpha_0$ vs L_{max} is shown in the figure (2) and suggests that the sums in both the dispersion relation as well as the Witten diagrams converge.

We have also checked for convergence using data from the epsilon expansion and again find supporting evidence. For example using $\epsilon=1/10$ (with only double twist spectrum up to ϵ^2), we find $\mathcal{M}\left(s_1=\frac{1}{10},s_2=-\frac{10}{101}\right)-\alpha_0=0.000310909$ for $L_{max}=2$ and $\mathcal{M}\left(s_1=\frac{1}{10},s_2=-\frac{10}{101}\right)-\alpha_0=0.000310907$ for $L_{max}=6$ using eq. (11) in main text. If we use eq. (6) in main text, these numbers become $\mathcal{M}\left(s_1=\frac{1}{10},s_2=-\frac{10}{101}\right)-\alpha_0=0.000310936$ for $L_{max}=2$ and $\mathcal{M}\left(s_1=\frac{1}{10},s_2=-\frac{10}{101}\right)-\alpha_0=0.000310939$ for $L_{max}=6$. The slight discrepancy between using eq. (6) and (11) (in main text) is due to the fact that the locality constraints (eq. (5) in main text) are not fully satisfied due to considering only the lowest twist operators.



(a) $\mathcal{M}(s_1,s_2)-\alpha_0$ vs L_{max} for $s_1=\frac{\Delta_\phi}{3}, s_2=0$

(b) $\mathcal{M}(s_1,s_2)-\alpha_0$ vs L_{max} for $s_1=\frac{1}{10}, s_2=-\frac{10}{101}$

FIG. 2: Convergence of eq eq. (6) in main text and eq. (11) in main text

V. $\ell = 4$ CONTACT TERM: EXPLICIT EXPRESSION

Since $P_{\Delta,\ell=4}(s_1,s_2) = s_1^4 b_{4,0}^{(4)} + s_2^2\left(s_2^2 b_{0,4}^{(4)} + b_{0,2}^{(4)}\right) + s_1\left(s_2\left(s_2\left(2s_2 b_{0,4}^{(4)} + b_{1,2}^{(4)}\right) + b_{0,2}^{(4)}\right) + b_{1,0}^{(4)}\right) + s_1^2\left(s_2\left(s_2 b_{2,2}^{(4)} + b_{1,2}^{(4)}\right) + b_{2,0}^{(4)}\right) + s_1^3\left(s_2\left(b_{2,2}^{(4)} - b_{0,4}^{(4)}\right) + b_{3,0}^{(4)}\right) + b_{0,0}^{(4)}$, we find

$$\begin{aligned}\mathcal{D}_{\ell=4} = & \frac{a^2x^3}{y^4\tau_k\left(a-\tau_k\right)^2}\left(ay^2b_{0,2}^{(4)}\left(a-\tau_k\right) + ab_{0,4}^{(4)}\left(a^3x^3+\tau_k\left(y^2\tau_k\left(7a-2\tau_k\right)-a^2\left(x^3+6y^2\right)\right)\right) + \right.\\ & \left(a-\tau_k\right)\left(y^2\left(ab_{1,2}^{(4)}\left(3a-2\tau_k\right)+2b_{2,0}^{(4)}\left(a-\tau_k\right)+\left(3a-2\tau_k\right)\left(a\tau_k b_{2,2}^{(4)}+b_{3,0}^{(4)}\left(\tau_k-a\right)\right)\right)+\right.\\ & \left.\left. b_{4,0}^{(4)}\left(a-\tau_k\right)\left(2a^2x^3+y^2\tau_k\left(2\tau_k-3a\right)\right)\right)\right) .\end{aligned} \tag{9}$$

Throwing away the unphysical terms in $\mathcal{D}_{\ell=4}$, since they cancel once sum over spectrum is performed, one easily obtains $M^{(c)}_{\Delta,\ell=4,k}(s_1,s_2)$:

$$\begin{aligned}M^{(c)}_{\Delta,\ell=4,k}(s_1,s_2) = & \frac{1}{\tau_k^3}\left(-y^2b_{0,4}^{(4)} - y\tau_k\left(xb_{0,4}^{(4)}+b_{0,2}^{(4)}\right) + 2x\tau_k^4 b_{4,0}^{(4)} + \tau_k^3\left(2xb_{3,0}^{(4)} - 2yb_{0,4}^{(4)} + 2yb_{2,2}^{(4)} - 3yb_{4,0}^{(4)}\right) + \right.\\ & \left.\tau_k^2\left(2x\left(xb_{4,0}^{(4)}+b_{2,0}^{(4)}\right)+2yb_{1,2}^{(4)}-3yb_{3,0}^{(4)}\right)\right) .\end{aligned} \tag{10}$$

We have worked out similar expressions for all even spins up to $\ell = 10$.

VI. DERIVATION OF THE SUM RULES OF [4]

We can Taylor expand Eq.(12) (in main text) equation around $s_2 = 0$ i.e. $\mathfrak{F}_p(s_2) = \sum_{r=0}^{\infty} s_2^r\, \mathfrak{F}_p^{(r)}$. For example

$$\mathfrak{F}_p^{(1)} = \sum_{\substack{k=0\\ \Delta,\ell}}^{\infty} c_{\Delta,\ell}^{(k)}\left[\frac{2\left(\Delta_\phi+3p\right)^2 P_{\Delta,\ell;1}\left(\tau_k,0\right)}{9\tau_k^3-\tau_k\left(\Delta_\phi+3p\right)^2} + P_{\Delta,\ell}\left(\tau_k,0\right)\left(\frac{1}{\tau_k^2}-\frac{9}{\left(\Delta_\phi+3\left(\tau_k+p\right)\right)^2}\right)\right] , \tag{11}$$

$$\mathfrak{F}_p^{(2)} = \sum_{\substack{k=0\\ \Delta,\ell}}^{\infty} c_{\Delta,\ell}^{(k)} \Bigg[\frac{(\Delta_\phi+3p)^2 P_{\Delta,\ell;2}(\tau_k,0)}{9\tau_k^3 - \tau_k(\Delta_\phi+3p)^2} - P_{\Delta,\ell;1}(\tau_k,0)\left(\frac{9}{(\Delta_\phi+3(\tau_k+p))^2} - \frac{1}{\tau_k^2}\right) + P_{\Delta,\ell}(\tau_k,0)\left(\frac{1}{\left(\frac{\Delta_\phi}{3}+\tau_k+p\right)^3} + \frac{1}{\tau_k^3}\right)\Bigg]. \tag{12}$$

We will derive the sum rules presented in [4] by imposing the Polyakov conditions on eq. (6) in main text. We first write down their equation [4, eq (6.4) or eq (139) in arxiv version]

$$F(\gamma_{13}) = 2G\left(\left(\gamma_{13} - \frac{\Delta_\phi}{3}\right)^2\right) + \sum_{\substack{k=0\\ \Delta,\ell}}^{\infty} c_{\Delta,\ell}^{(k)} P_{\Delta,\ell}\left(\tau_k, \frac{\Delta_\phi}{3} - \gamma_{12}\right) \times \left(\frac{3}{3\gamma_{13}+2\Delta_\phi-3\tau_k} - \frac{1}{\gamma_{13}+\Delta_\phi-\tau_k} + \frac{3}{3\tau_k-4\Delta_\phi} + \frac{3}{5\Delta_\phi-3\tau_k}\right).$$

We taylor expand $F(\gamma_{13})$ around $\gamma_{13}=\Delta_\phi/3$, (i.e. $s_2 = 2\Delta_\phi/3$ in our notation) $F(\gamma_{13}) = \sum_{r=0}^{\infty}\left(\gamma_{13} - \frac{\Delta_\phi}{3}\right)^r F^{(r)}$. We find that $F^{(r)}$, $\mathfrak{F}_0^{(r)}$, $\mathcal{M}_{n-m,m}$ are related. For example

$$F^{(1)} = \mathfrak{F}_0^{(1)},\ F^{(3)} = \mathfrak{F}_0^{(3)},\ F^{(5)} - \mathfrak{F}_0^{(5)} = 2\mathcal{M}_{-2,3},\ F^{(7)} - \mathfrak{F}_0^{(7)} = 4\mathcal{M}_{-4,5} - \mathcal{M}_{-1,3} - \frac{27}{\Delta_\phi^2}\mathcal{M}_{-2,3} + \frac{27}{\Delta_\phi^3}\mathcal{M}_{-1,2},$$
$$F^{(9)} - \mathfrak{F}_0^{(9)} = 6\mathcal{M}_{-6,7} - 6\mathcal{M}_{-3,5} - \frac{90}{\Delta_\phi^2}\mathcal{M}_{-4,5} + \frac{9}{\Delta_\phi^2}\mathcal{M}_{-1,3} + \frac{162}{\Delta_\phi^3}\mathcal{M}_{-3,4} + \frac{81}{\Delta_\phi^4}\mathcal{M}_{-2,3} - \frac{243}{\Delta_\phi^5}\mathcal{M}_{-1,2}.$$

Similar relations hold for any odd r (we have compared only odd r since for even r, $F^{(r)}$ will contain $G^{(r)}(0)$ which is known in terms of the full amplitude, but not apriori (see comments near [4, eq (139)]).

VII. EPSILON EXPANSION OF $\mathcal{M}_{n-m,m}$, $m > n$

We will study the epsilon expansion of the locality constraints illustrating with the case of $\mathcal{M}_{-2,3}$. The operator dimensions $\Delta_{\ell,q}$ of the operators $\mathcal{O}_{\ell,q}$ (with twist $2+q$; $q \in \mathbb{Z}^{\geq 0}$) have an ϵ expansion $\Delta_{\ell,q} = 2-\epsilon+\ell+2q+\delta_1(q,\ell)\epsilon + \delta_2(q,\ell)\epsilon^2 + \dots$ and Δ_ϕ has an expansion $\Delta_\phi = 1 - \frac{1}{2}\epsilon + \delta_\phi^{(2)}\epsilon^2$.

$\mathbf{O(\epsilon^2)}$: The OPE coefficients of $\Delta_{\ell,0}$ starts at $O(\epsilon^0)$; $C_{\Delta_{\ell,0},\ell} = C_{0,\ell}^{(0)} + C_{0,\ell}^{(1)}\epsilon + C_{0,\ell}^{(2)}\epsilon^2 + \dots$. Then $\mathcal{M}_{-2,3}$ starts at $O(\epsilon^2)$. The only operators that contribute are $\Delta_{\ell,0}$ (higher twists start at $O(\epsilon^4)$, see below). At $O(\epsilon^2)$, we get contributions $A_2(0,\ell)C_{0,\ell}^{(0)}(\delta_1(0,\ell))^2\epsilon^2$, with $A_2(0,\ell)$ a known quantity (the exact form is cumbersome and is not needed). For example $A_2(0,2) = -1825.03$ and so on. The important point is that $A_2(0,\ell) < 0, \forall\ell$. In order to satisfy the constraint

$$\mathcal{M}_{-2,3} = \sum_{\ell=0}^{\infty} A_2(0,\ell)C_{0,\ell}^{(0)}(\delta_1(0,\ell))^2\epsilon^2 + O(\epsilon^3) = 0$$

condition at $O(\epsilon^2)$, we must have,

$$\delta_1(0,\ell) = 0. \tag{13}$$

In other words, the anomalous dimension of twist two operators should start at $O(\epsilon^2)$. Once the constraint is satisfied at $O(\epsilon^2)$ it is also automatically satisfied at $O(\epsilon^3)$, since the next nonzero contribution to $\mathcal{M}_{-2,3}$ begins at $O(\epsilon^4)$ when $\delta_1(0,\ell) = 0$.

$\mathbf{O(\epsilon^4)}$: At $O(\epsilon^4)$, all operators start contributing. The point to emphasize is that the OPE of operators with $q \neq 0$ starts at $O(\epsilon^2)$; $C_{\Delta_{\ell,q},\ell} = C_{q,\ell}^{(2)}\epsilon^2 + \dots$, ($q \neq 0$). It is therefore impractical to solve the constraints at $O(\epsilon^4)$. At this order, the contributions from the twist two operators ($q=0$) is $A_2(0,\ell)C_{0,\ell}^{(0)}\left(-2\delta_\phi^{(2)} + \delta_2(q,\ell)\right)^2\epsilon^4$ and those from twist four onwards ($q \neq 0$) is $A_4(q,\ell) C^{(2)}_{q,\ell} \left(\delta_1(q,\ell)\right)^2 \epsilon^4$ with $A_4(q,\ell)$ a known quantity (for example $A_4(1,2) = -10.22$ and so on). Some observations are as follows: $A_4(1,2) < 0,\ A_4(1,4) < 0,\ A_4(1,\ell) > 0$ for $\ell \geq 6$, and $A_4(2,2) < 0,\ A_4(2,\ell) > 0$ for $\ell \geq 4 \ldots$. Therefore at $O(\epsilon^4)$

$$\mathcal{M}_{-2,3} = \sum_{\ell=0}^{\infty} \left[A_2(0,\ell) C^{(0)}_{0,\ell} \left(-2\delta^{(2)}_{\phi} + \delta_2(q,\ell)\right)^2 + \sum_{q=1}^{\infty} A_4(q,\ell) C^{(2)}_{q,\ell} \left(\delta_1(q,\ell)\right)^2 \right] \epsilon^4 + O(\epsilon^5)\,.$$

Thus the contributions from the leading twist operators, gets cancelled by the higher spin contributions of the higher twist operators.

VIII. DERIVATION OF EQ (18)-(19)

We will follow the steps in [5], but now for our CFT amplitude. We need

$$H\left(\tau_k; s_1, s_2, s_3\right) = \frac{27a^2 z^3 \left(3a - 2\tau_k\right)}{-27a^3 z^3 + 27a^2 z^3 \tau_k + \left(z^3 - 1\right)^2 \left(\tau_k\right)^3}\,, \tag{14}$$

where $\frac{z^3}{(1-z^3)^2} = -\frac{x}{27a^2}$. We can now expand in power of x, and collect the powers of a to find, very similar to [5],

$$\begin{aligned} \mathcal{M}_{n-m,m} &\equiv \sum_{\Delta,\ell,k}^{\infty} c^{(k)}_{\Delta,\ell} \mathcal{B}^{(\Delta,\ell,k)}_{n,m}\,, n \geq 1\,. \\ \mathcal{B}^{(\Delta,\ell,k)}_{n,m} &= \sum_{j=0}^{\ell/2} \frac{1}{\tau_k^{2n+m+1}} \frac{p^{(j)}_{\ell}\left(\xi_0\right)}{j!} (4\xi_0)^j \times \frac{(3j-m-2n)(-n)_m}{(m-j)!(-n)_{j+1}}\,. \end{aligned} \tag{15}$$

Here $p^{(j)}_{\ell}\left(\xi_0\right) = \left.\frac{\partial^j P_{\Delta,\ell}\left(\tau_k, \frac{1}{2}\left(\sqrt{\xi}-1\right)\tau_k\right)}{\partial \xi^j}\right|_{\xi=\xi_0}$ and we have replaced $s_2'(\tau_k,a)$ by solving $\sqrt{\xi} = 1 + \frac{2\ s_2'(\tau_k,a)}{\tau_k}$ and $\xi_0 = 1$. For example $p^{(1)}_{\ell}\left(\xi_0\right) = \frac{\tau_k P_{\Delta,\ell;1}(\tau_k,0)}{4}, p^{(2)}_{\ell}\left(\xi_0\right) = \frac{\tau_k(\tau_k P_{\Delta,\ell;2}(\tau_k,0) - 2P_{\Delta,\ell;1}(\tau_k,0))}{16}$, and more generally [5], we find equation eq (19) in main text. Notice that for $m > n$, we have $\ell \geq 2n$, which follows on writing $(-n)_m/(-n)_{j+1} = \Gamma(-n+m)/\Gamma(-n+j+1)$ so that for this to not vanish we need $j \geq n$. Since the argument of the degree-ℓ polynomial is $\sqrt{\xi}$, we will need $\ell \geq 2n$ as noted in [5, 6].

IX. TWO SIDED BOUNDS, EFT IN ADS

In the large $\nu = \Delta - h,\ s_1$ limit ($s_1 = \delta + 4\Delta_\phi/3 = s - 2\Delta_\phi/3$), we have

$$P^{(s)}_{\nu,\ell}(s,t) = \frac{8^{-\ell}\ell! s^{\ell}}{(h-1)_\ell} \left[C^{(h-1)}_{\ell}(x) - \frac{(h-1)}{\nu^2} C^{(h)}_{\ell-2}(x) + O\left(\frac{1}{\nu^4}\right) \right] + O\left(s^{\ell-1}\right)\,. \tag{16}$$

We assume $s \gg \nu^2$. Following [7, 8] we get the s-channel discontinuity

$$\mathcal{A}^{(AdS)}(s,t) \approx \sum_{\ell} a_\ell(s) \left(C^{(h-1)}_{\ell}(x) - \frac{(h-1)}{\nu^2} C^{(h)}_{\ell-2}(x) \right)\,, \tag{17}$$

where

$$a_\ell(s) = \pi \sum_{\Delta} C_{\Delta,\ell} \mathcal{N}_{\Delta,\ell} \frac{\Gamma\left(2\Delta_\phi + \ell - h\right)}{2\Delta_\phi + \ell} \frac{\sin^2 \pi \left[\Delta_\phi - s\right]}{\sin^2 \pi \left[\Delta_\phi - \frac{\Delta}{2}\right]} \frac{8^{-\ell}\ell! s^{\ell}}{(h-1)_\ell} \delta\left(s - \frac{\Delta - \ell}{2} - q_\star\right)\,. \tag{18}$$

where $q_\star = \frac{\left(\frac{\Delta-\ell}{2}-\Delta_\phi\right)^2}{\ell+2\Delta_\phi}$. Assuming the lower limit of the Δ sum is very large ($\Delta-\ell\sim 2\Delta_\phi$ is the lower limit), the corresponding ν is given by $\nu_0 \approx 2mR$ since $m^2R^2=\Delta_\phi\,(\Delta_\phi-2h)$. Therefore for large R (upto $\frac{1}{R^2}$), we can write

$$\mathcal{A}^{(AdS)}(s,t) \approx \sum_\ell a_\ell(s)\left(C_\ell^{(h-1)}(x) - \frac{(h-1)}{\nu_0^2}C_{\ell-2}^{(h)}(x)\right). \tag{19}$$

In the limit $s \gg \nu^2$, other corrections will be sub-leading in s, R. We can now do a similar analysis as in [5]. We note that in that limit $\mathcal{B}_{n,0}^{(AdS,\ell)} = \frac{2}{\pi\delta^{2n}}\left(1-\frac{\alpha}{\nu_0^2}\right)$, which implies

$$\mathcal{B}_{n,0}^{(AdS,\ell)} = \left(1-\frac{\alpha}{\nu_0^2}\right)\mathcal{B}_{n,0}^{(Flat,\ell)} \text{ or } \mathcal{M}_{n,0}^{(AdS,\ell)} = \left(1-\frac{\alpha}{\nu_0^2}\right)\mathcal{M}_{n,0}^{(Flat,\ell)}. \tag{20}$$

Since $\delta_0 \gg 1$, we have $\mathcal{B}_{1,1}^{(AdS,\ell)} = \frac{4\ell(\ell+2\alpha)-3(2\alpha+1)}{\pi(2\alpha+1)\delta^3} - \frac{\alpha[4\ell(\ell+2\alpha+2)-3(2\alpha+2+1)]}{\pi(2\alpha+2+1)\delta^3\nu_0^2}$, which gives

$$\mathcal{B}_{1,1}^{(AdS,\ell)} \leq \left(1-\frac{\alpha(2\alpha+1)}{(2\alpha+3)\nu_0^2}\right)\mathcal{B}_{1,1}^{(Flat,\ell)} \text{ or } \mathcal{M}_{0,1}^{(AdS,\ell)} \leq \left(1-\frac{\alpha(2\alpha+1)}{(2\alpha+3)\nu_0^2}\right)\mathcal{M}_{0,1}^{(Flat,\ell)}. \tag{21}$$

From [9, 10] $\mathcal{M}_{0,1}^{(Flat)} < \frac{10\alpha+11}{(2\alpha+1)\delta_0}\mathcal{M}_{1,0}^{(Flat)}$, we get

$$\mathcal{M}_{0,1}^{(AdS)} < \left[1+\frac{2\alpha}{(2\alpha+3)\nu_0^2}\right]\frac{(10\alpha+11)}{(2\alpha+1)\delta_0}\mathcal{M}_{1,0}^{(AdS)} \text{ or } \mathcal{M}_{0,1}^{(AdS)} < \left[1+\frac{\alpha}{2(2\alpha+3)m^2R^2}\right]\frac{(10\alpha+11)}{(2\alpha+1)\delta_0}\mathcal{M}_{1,0}^{(AdS)}. \tag{22}$$

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