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A generalization of Fiedler's lemma and the spectra of H -join of graphs[☆]M. Saravanan^{a,*}, S.P. Murugan^b, G. Arunkumar^c^a Government Arts and Science College (Erstwhile Madurai Kamaraj University Constituent College), Sattur, India^b Indian Institute of Science Education and Research, Mohali, India^c Indian Institute of Science, Bangalore, India

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ABSTRACT

A new generalization of Fiedler's lemma is obtained by introducing the concept of the main function of a matrix. As applications, the universal spectra of the H -join of any graphs (possibly non-regular) and the adjacency spectra of the H -generalized join constrained by (arbitrary) vertex subsets are obtained. The adjacency spectra of the generalized corona of graphs is deduced from the spectra of the H -join of graphs. Also, the construction of infinitely many pairs of non-regular universal cospectral graphs is provided.

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1. Introduction

All the graphs considered in this paper are finite and simple. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix $A(G)$. The set of all eigenvalues of G is called the spectrum of G , denoted by $\text{spec}(G)$. For more on graphs and their eigenvalues we refer [9,10]. Let H be a graph with vertex set $\{v_1, v_2, \dots, v_k\}$ and let $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a family of graphs. In [4], the H -join operation of the graphs $G_1, G_2, \dots, G_k$, denoted by $\bigvee_H \mathcal{F}$, is obtained by replacing the vertex v_i of H by the graph G_i for $1 \leq i \leq k$ and every vertex of G_i is made adjacent with every vertex of G_j , whenever v_i is adjacent to v_j in H . Precisely, $\bigvee_H \mathcal{F}$ is the graph with vertex

set $V(\bigvee_H \mathcal{F}) = \bigcup_{i=1}^k V(G_i)$ and edge set $E(\bigvee_H \mathcal{F}) = (\bigcup_{i=1}^k E(G_i)) \cup (\bigcup_{v_i v_j \in E(H)} \{xy : x \in V(G_i), y \in V(G_j)\})$. In addition to H and $\mathcal{F}$, by considering a family of vertex subsets $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ where $S_i \subset V(G_i)$ for each $1 \leq i \leq k$, a generalization of H -join operation, known as H -generalized join operation constrained by vertex subsets, $\bigvee_{H, \mathcal{S}} \mathcal{F}$

is introduced in [5] as follows: $V(\bigvee_{H, \mathcal{S}} \mathcal{F}) = \bigcup_{i=1}^k V(G_i)$ and $E(\bigvee_{H, \mathcal{S}} \mathcal{F}) = (\bigcup_{i=1}^k E(G_i)) \cup (\bigcup_{v_i v_j \in E(H)} \{xy : x \in S_i, y \in S_j\})$. For instance consider the examples in Section 8. If

we take $S_i = V(G_i)$ for each $1 \leq i \leq k$, then the H -generalized join operation $\bigvee_{H, \mathcal{S}} \mathcal{F}$ coincides with the H -join operation of the graphs $G_1, G_2, \dots, G_k$. In [21], the H -join operation of the graphs was initially introduced as generalized composition by Schwenk, denoted by $H[G_1, G_2, \dots, G_k]$. Also, the same operation is studied in some other names as generalized lexicographic product and joined union in [19,22,23]. When all G_i 's are equal to the same graph G , it is called the lexicographic product [15], denoted by $H[G]$.

The following lemma [12, Lemma 2.2] is proved by M. Fiedler and effectively used in the study of finding sufficient conditions for k arbitrary real numbers to be eigenvalues of a non-negative $k \times k$ symmetric matrix.

Lemma 1. [12] *Let A be a symmetric $m \times m$ matrix with eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_m$ and B be a symmetric $n \times n$ matrix with eigenvalues $\beta_1, \beta_2, \dots, \beta_n$. Let u be an eigenvector of A corresponding to α_1 and v be an eigenvector of B corresponding to β_1 such that $\|u\| = \|v\| = 1$. Then for any constant ρ the matrix*

$$C = \begin{bmatrix} A & \rho uv^t \\ \rho vu^t & B \end{bmatrix}$$

has eigenvalues $\alpha_2, \dots, \alpha_m, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2$ where γ_1 and γ_2 are the eigenvalues of the matrix

$$\widehat{C} = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}.$$

In [3,4], the above lemma is called Fiedler's lemma. It is easy to see that, this lemma can be used to find the spectrum of H -join of regular graphs when $H = K_2$. In [3, Lemma 2], a generalization of the Fiedler's Lemma is obtained by Cardoso et al. so that, it can be applied in the H -join of regular graphs when $H = P_k$, the path graph on k vertices. Then in [4, Theorem 3] Cardoso et al. obtained another improved generalization of Fiedler's lemma as follows, which can be applied in the H -join of regular graphs for any H .

Theorem 1. [4] Let M_i be a symmetric matrix of order n_i and u_i be an eigenvector of M_i corresponding to the eigenvalue α_i , such that $\|u_i\| = 1$ for $1 \leq i \leq k$. Let $\rho_{i,j}$ be a collection of arbitrary scalars such that $\rho_{i,j} = \rho_{j,i}$ for $1 \leq i < j \leq k$. Considering

$$\mathbf{M} = (M_1, M_2, \dots, M_k), \mathbf{u} = (u_1, u_2, \dots, u_k)$$

as k -tuples, and

$$\rho = (\rho_{12}, \dots, \rho_{1,k}, \rho_{23}, \dots, \rho_{2,k}, \dots, \rho_{k-1,k})$$

as $\frac{k(k-1)}{2}$ -tuple, the following matrices are defined.

$$A(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} M_1 & \rho_{1,2}u_1u_2^t & \cdots & \rho_{1,k}u_1u_k^t \\ \rho_{2,1}u_2u_1^t & M_2 & \cdots & \rho_{2,k}u_2u_k^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}u_ku_1^t & \rho_{k,2}u_ku_2^t & \cdots & M_k \end{bmatrix} \quad \text{and}$$

$$\widehat{A}(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} \alpha_1 & \rho_{1,2} & \cdots & \rho_{1,k} \\ \rho_{2,1} & \alpha_2 & \cdots & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1} & \rho_{k,2} & \cdots & \alpha_k \end{bmatrix}.$$

$$\text{Then } \text{spec}(A(\mathbf{M}, \mathbf{u}, \rho)) = \left(\bigcup_{i=1}^k (\text{spec}(M_i) \setminus \{\alpha_i\}) \right) \cup \text{spec}(\widehat{A}(\mathbf{M}, \mathbf{u}, \rho)).$$

In this paper, we obtain a new generalization of the Fiedler's lemma, in terms of characteristic polynomials as follows. Let $n = \sum_{i=1}^k n_i$. Then

$$\det(\lambda I_n - A(\mathbf{M}, \mathbf{u}, \rho)) = \left(\prod_{i=1}^k \phi_i(\lambda) \Gamma_i(\lambda) \right) \det(\widetilde{A}(\mathbf{M}, \mathbf{u}, \rho))$$

where $\phi_i(\lambda) = \det(\lambda I_{n_i} - M_i)$, $\Gamma_i(\lambda) = u^t(\lambda I_{n_i} - M_i)^{-1}u$ and

$$\tilde{A}(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} \frac{1}{\Gamma_1(\lambda)} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2(\lambda)} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k(\lambda)} \end{bmatrix}.$$

The main difference is we are not restricting $\mathbf{u}$ as the k -tuple of eigenvectors and $\mathbf{M}$ as the k -tuple of symmetric matrices. But, we consider $\mathbf{u}$ as the k -tuple of any complex vectors and $\mathbf{M}$ as the k -tuple of any complex square matrices of appropriate size. The precious statement is given in Theorem 2. We call the function $\Gamma_i(\lambda)$ used above, as the main function of the matrix M_i . This concept of the main function of a matrix, discussed in Section 2, helps us to accomplish the task. Also as an application of our result, we obtain the characteristic polynomial of H -join of graphs when the graphs G_i 's are any graphs (possibly non-regular). In [21] it is remarked by Schwenk, that “In general, it does not appear likely that the characteristic polynomial of the generalized composition can always be expressed in terms of the characteristic polynomials of $H, G_1, G_2, \dots, G_k$ ”. In our paper, we prove that it is possible to express the characteristic polynomial of H -join operation of graphs (i.e. generalized composition) in terms of the characteristic polynomials and main functions of $G_1, G_2, \dots, G_k$, and another function obtained from the adjacency matrix of H . Moreover for the H -join operation of any graphs, we obtain the characteristic polynomial and the spectrum of its universal adjacency matrix.

The universal adjacency matrix of a graph G is defined as follows: Let $A(G)$, I , J , and $D(G)$ be the adjacency matrix of G , the identity matrix, the all-one matrix, and the degree matrix of G , respectively. Any matrix of the form $U(G) = \alpha A + \beta I + \gamma J + \delta D$ where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \neq 0$ is called the universal adjacency matrix of G . Many interesting and important matrices associated to a graph can be obtained as special cases of $U(G)$. For example, from the universal adjacency matrix $U(G)$, we get adjacency matrix $A(G)$, Laplacian matrix $L(G) = D(G) - A(G)$, signless Laplacian matrix $Q(G) = D(G) + A(G)$, and Seidel matrix $S(G) = J - I - 2A(G)$ by taking appropriate values for α, β, γ , and δ .

In [14], the Laplacian spectra of H -join of any graphs is obtained. In [8] the characteristic polynomial of the matrix $A(G) - tD(G)$ is obtained for H -join of regular graphs. In [23] the characteristic polynomial of the adjacency matrix of the lexicographic product of any graphs is obtained. Recently in [16] the universal adjacency spectra of the disjoint union of regular graphs is obtained. Also in [5] the eigenvalues of H -generalized join operation $\bigvee_{H,S} \mathcal{F}$ are obtained, when the subsets S_i 's are (κ, τ) -regular.

In this paper, we obtain the characteristic polynomial and eigenvalues of the universal adjacency matrix of H -join of any graphs $G_1, G_2, \dots, G_k$. Then we obtain the characteristic polynomial and eigenvalues of the adjacency matrix of H -generalized join of graphs $G_1, G_2, \dots, G_k$, where the subsets $S_i(G) \subset V(G_i)$ are arbitrary for $1 \leq i \leq k$. Also, we deduce the characteristic polynomial of the generalized corona of graphs by visualizing corona as an H -join of graphs. Hence the results obtained (mostly for regular graphs) in [4,5,8,13,16,23,24], are generalized here for any graphs.

Throughout this paper, we denote the identity matrix of order n by I_n , the all-one matrix of order n by J_n and the all-one vector of size $n \times 1$ by $\mathbf{1}_n$.

2. The main function of a matrix and a generalization of Fiedler's lemma

Consider a graph G on n vertices with adjacency matrix $A(G)$. Suppose $A(G)$ has spectral decomposition $A(G) = \sum_{i=1}^k \theta_i E_{\theta_i}$, where θ_i 's are distinct eigenvalues of G and E_{θ_i} is the orthogonal projection on the eigenspace of θ_i , $\mathcal{E}(\theta_i) = \ker(A(G) - \theta_i I_n)$. An eigenvalue θ_i is called a main eigenvalue if the corresponding eigenspace $\mathcal{E}(\theta_i)$ is not orthogonal to $\mathbf{1}_n$. The cosines of the angles between $\mathbf{1}_n$ and the eigenspaces of A are known as main angles of G , given by $\beta_i = \frac{1}{\sqrt{n}} \|E_{\theta_i} \mathbf{1}_n\|$, for $1 \leq i \leq k$. So θ_i is a main eigenvalue if and only if $\beta_i \neq 0$. For more on the main angles and main eigenvalues, we refer [20] and references therein.

Consider the field of rational functions $\mathbb{C}(\lambda)$. The $\det(\lambda I - A)$ is a non-zero element of $\mathbb{C}(\lambda)$ and hence the matrix $\lambda I - A$ is invertible over $\mathbb{C}(\lambda)$. In [17], the function $\mathbf{1}_n^t (\lambda I_n - A(G))^{-1} \mathbf{1}_n$ is introduced in the name of coronal of G and is used to find the characteristic polynomial of the corona of two graphs. Since $E_{\theta_i}^2 = E_{\theta_i}$, it is easy to see that

$$\mathbf{1}_n^t (\lambda I_n - A(G))^{-1} \mathbf{1}_n = \sum_{i=1}^k \frac{\mathbf{1}_n^t E_{\theta_i} \mathbf{1}_n}{\lambda - \theta_i} = \sum_{i=1}^k \frac{\|E_{\theta_i} \mathbf{1}_n\|^2}{\lambda - \theta_i} = \sum_{i=1}^k \frac{n\beta_i^2}{\lambda - \theta_i}, \quad (2.1)$$

in which only non-vanishing terms are those terms corresponding to main eigenvalues.

Also in [17], the authors remarked that graphs with different eigenvalues can have the same coronals, meanwhile cospectral graphs can have different coronals. This is because the coronal of a graph depends not only on the eigenvalues but also on the main angles of the graph. In this paper, because of this relationship among main eigenvalues, main angles, and coronal of the graph G , we prefer to call $\mathbf{1}_n^t (\lambda I_n - A(G))^{-1} \mathbf{1}_n$, the main function of the graph G , and denote as $\Gamma_G(\lambda)$. Moreover for any vectors u and v , and a matrix M of the same dimension, we introduce the following notions.

Definition 1. Let M be an $n \times n$ complex matrix, and let u and v be $n \times 1$ complex vectors. The main function associated to the matrix M corresponding to the vectors u and v , denoted by $\Gamma_M(u, v)$, is defined to be $\Gamma_M(u, v; \lambda) = v^t (\lambda I - M)^{-1} u \in \mathbb{C}(\lambda)$. When $u = v$, we denote $\Gamma_M(u, v; \lambda)$ by $\Gamma_M(u; \lambda)$.

Definition 2. Let M be an $n \times n$ normal matrix over $\mathbb{C}$ and let u be an $n \times 1$ complex vector. An eigenvalue λ of M is called as u -main eigenvalue if the corresponding eigenspace $\mathcal{E}_M(\lambda)$ is not orthogonal to the vector u . In the case of $u = \mathbf{1}_n$, the all-one vector, we don't specify the vector and call eigenvalue λ of M as the main eigenvalue of M .

Motivated by [15, Theorem 8.13.3], we prove the following lemma.

Lemma 2. Let M be a complex normal matrix of order n and let u be any $n \times 1$ vector. Then the poles of $u^t(\lambda I - M)^{-1}u$ are the u -main eigenvalues of M and are simple.

Proof. Let $\{\theta_1, \theta_2, \dots, \theta_k\}$ be the distinct eigenvalues and let $\{\theta_1, \theta_2, \dots, \theta_m\}$ be the set of u -main eigenvalues of M . Suppose the spectral decomposition of M is $M = \sum_{i=1}^k \theta_i E_{\theta_i}$, where E_{θ_i} is the orthogonal projection on the eigenspace of θ_i . Then $(\lambda I - M)^{-1} = \sum_{i=1}^k \frac{E_{\theta_i}}{\lambda - \theta_i}$, and $\Gamma_M(u; \lambda) = u^t(\lambda I - M)^{-1}u = \sum_{i=1}^k \frac{u^t E_{\theta_i} u}{\lambda - \theta_i}$. Now, $u^t E_{\theta_i} u \neq 0$ if and only if θ_i is a u -main eigenvalue of M . So, $\Gamma_M(u; \lambda) = \sum_{i=1}^m \frac{u^t E_{\theta_i} u}{\lambda - \theta_i}$ and the result follows. $\square$

Lemma 3. Let M be a matrix of order n with an eigenvector u corresponding to the eigenvalue μ . Then $\Gamma_M(u; \lambda) = \frac{\|u\|^2}{\lambda - \mu}$.

Proof. Now $(\lambda I_n - M)u = (\lambda - \mu)u$. Applying $(\lambda I_n - M)^{-1}$ both sides, we can get $u = (\lambda I_n - M)^{-1}(\lambda - \mu)u$, which implies $\frac{u^t u}{\lambda - \mu} = u^t(\lambda I_n - M)^{-1}u = \frac{\|u\|^2}{\lambda - \mu}$. $\square$

Lemma 4. Let M be an $n \times n$ normal matrix over $\mathbb{C}$. Let u be an $n \times 1$ complex vector and $p(M)$ be a polynomial in M with complex coefficients. Then an eigenvalue μ is a u -main eigenvalue of M if and only if $p(\mu)$ is a u -main eigenvalue of $p(M)$.

Proof. For any eigenvalue of M and corresponding eigenvalue of $p(M)$ the eigenvectors are the same. So the eigenspaces are the same and hence the result follows. $\square$

Now we can state our main result, a new generalization of Fiedler's lemma.

Theorem 2. Let M_i be a complex matrix of order n_i , and let u_i and v_i be arbitrary complex vectors of size $n_i \times 1$ for $1 \leq i \leq k$. Let $n = \sum_{i=1}^k n_i$. Let $\rho_{i,j}$ be arbitrary complex numbers for $1 \leq i, j \leq k$ and $i \neq j$. For each $1 \leq i \leq k$, let $\phi_i(\lambda) = \det(\lambda I_{n_i} - M_i)$ be the characteristic polynomial of the matrix M_i and $\Gamma_i(\lambda) = \Gamma_{M_i}(u_i, v_i; \lambda) = v_i^t(\lambda I - M_i)^{-1}u_i$. Let $\mathbf{M}$ be the k -tuple $(M_1, M_2, \dots, M_k)$, $\mathbf{u}$ be the $2k$ -tuple $(u_1, v_1, u_2, v_2, \dots, u_k, v_k)$ and ρ be the $k(k-1)$ -tuple $(\rho_{1,2}, \rho_{1,2}, \dots, \rho_{1,k}, \rho_{2,1}, \rho_{2,3}, \dots, \rho_{2,k}, \dots, \rho_{k,1}, \rho_{k,2}, \dots, \rho_{k-1,k})$. Considering $\mathbf{M}$, $\mathbf{u}$ and ρ , the following matrices are defined:

$$A(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} M_1 & \rho_{1,2}u_1v_2^t & \cdots & \rho_{1,k}u_1v_k^t \\ \rho_{2,1}u_2v_1^t & M_2 & \cdots & \rho_{2,k}u_2v_k^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}u_kv_1^t & \rho_{k,2}u_kv_2^t & \cdots & M_k \end{bmatrix}$$

$$\text{and } \tilde{A}(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} \frac{1}{\Gamma_1(\lambda)} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2(\lambda)} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k(\lambda)} \end{bmatrix}.$$

Then the characteristic polynomial of $A(\mathbf{M}, \mathbf{u}, \rho)$ is given as

$$\det(\lambda I_n - A(\mathbf{M}, \mathbf{u}, \rho)) = \left(\prod_{i=1}^k \phi_i(\lambda) \Gamma_i(\lambda) \right) \det(\tilde{A}(\mathbf{M}, \mathbf{u}, \rho)). \quad (2.2)$$

Proof of this theorem is given in Section 3.2. At first, we deduce Theorem 1 in terms of characteristic polynomials as a corollary of Theorem 2.

Corollary 1. Consider the notations defined in Theorem 2. Suppose $u_i = v_i$ is an eigenvector of M_i corresponding to an eigenvalue α_i with $\|u_i\| = 1$, then the characteristic polynomial of $A(\mathbf{M}, \mathbf{u}, \rho)$ is

$$\phi(A(\mathbf{M}, \mathbf{u}, \rho)) = \frac{\phi_1}{\lambda - \alpha_1} \frac{\phi_2}{\lambda - \alpha_2} \cdots \frac{\phi_k}{\lambda - \alpha_k} \det(\tilde{A}(\mathbf{M}, \mathbf{u}, \rho))$$

where $\tilde{A}(\mathbf{M}, \mathbf{u}, \rho) = \begin{bmatrix} \lambda - \alpha_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \lambda - \alpha_2 & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \lambda - \alpha_k \end{bmatrix}.$

Proof. Since $\|u_i\| = 1$, by Lemma 3 we get $\Gamma_i = \frac{1}{\lambda - \alpha_i}$. Now the proof follows from Theorem 2. $\square$

In [8, Theorem 2.3] another generalization of Fiedler's lemma, similar to Theorem 1, is given for the matrices with fixed row sum and the result is used to find the generalized characteristic polynomial of H -join of regular graphs. We observe that any such matrix has the all-one vector as an eigenvector. By taking u_i to be the all-one vector of appropriate size, we can deduce [8, Theorem 2.3] from Theorem 2.

3. Proof of the main result

In this section, we prove Theorem 2. We start with the following essential lemmas.

3.1. Some important lemmas

Lemma 5. [9] Let A, B, C and D be matrices such that $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. If D is invertible, then $\det(M) = \det(D) \det(A - BD^{-1}C)$.

Lemma 6. [1,11] Let A be an $n \times n$ invertible matrix, and let u and v be any two $n \times 1$ vectors such that $1 + v^t A^{-1}u \neq 0$. Then

$$(1) \quad \det(A + uv^t) = (1 + v^t A^{-1}u) \det(A).$$

$$(2) \quad (A + uv^t)^{-1} = A^{-1} - \frac{A^{-1}uv^tA^{-1}}{1 + v^tA^{-1}u}.$$

Lemma 7. Let A be an $n \times n$ complex matrix, and let u and v be any $n \times 1$ complex vectors. Also, let $\Gamma = v^t(\lambda I - A)^{-1}u$. Then

$$(1) \quad \det(\lambda I - A + \alpha uv^t) = (1 + \alpha \Gamma) \det(\lambda I - A) = (1 + \alpha \Gamma) \phi_A(\lambda)$$

$$(2) \quad v^t(\lambda I - A + \alpha uv^t)u = \frac{\Gamma}{1 + \alpha \Gamma}$$

Proof. The proof of (1) follows directly from Lemma 6(1), as $\det(\lambda I - A + \alpha uv^t) = (1 + \alpha v^t(\lambda I - A)^{-1}u) \det(\lambda I - A)$. So we prove (2).

By Lemma 6(2),

$$(\lambda I - A + \alpha uv^t)^{-1} = (\lambda I - A)^{-1} - \alpha \frac{(\lambda I - A)^{-1}uv^t(\lambda I - A)^{-1}}{1 + \alpha v^t(\lambda I - A)^{-1}u}$$

which implies,

$$v^t(\lambda I - A + \alpha uv^t)^{-1}u = \Gamma - \alpha \frac{\Gamma^2}{1 + \alpha \Gamma} = \frac{\Gamma}{1 + \alpha \Gamma} \quad \square$$

3.2. Proof of Theorem 2

Proof. We prove the result by using induction on k . For convenience, we take $\Gamma_i = \Gamma_i(\lambda)$. The base case $k = 1$ is clear. We prove the result also for $k = 2$ for the sake of understanding. Now, by Lemma 5, we have

$$\begin{aligned} \begin{vmatrix} \lambda I_{n_1} - M_1 & -\rho_{1,2}u_1v_2^t \\ -\rho_{2,1}u_2v_1^t & \lambda I_{n_2} - M_2 \end{vmatrix} &= \det(\lambda I_{n_2} - M_2) \det(\lambda I_{n_1} - M_1 - \rho_{1,2}\rho_{2,1}\Gamma_2u_1v_1^t) \\ &= \phi_1\phi_2(1 - \rho_{1,2}\rho_{2,1}\Gamma_2\Gamma_1), \text{ by Lemma 7(1)} \\ &= \phi_1\phi_2 \begin{vmatrix} 1 & -\rho_{1,2}\Gamma_1 \\ -\rho_{2,1}\Gamma_2 & 1 \end{vmatrix} \end{aligned}$$

This proves the result for the case $k = 2$. We assume the result is true for $k - 1$. Consider the matrix

$$\lambda I_n - A(\mathbf{M}, \mathbf{u}, \rho) = \begin{bmatrix} \lambda I_{n_1} - M_1 & -\rho_{1,2}u_1v_2^t & \cdots & -\rho_{1,k}u_1v_k^t \\ -\rho_{2,1}u_2v_1^t & \lambda I_{n_2} - M_2 & \cdots & -\rho_{2,k}u_2v_k^t \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1}u_kv_1^t & -\rho_{k,2}u_kv_2^t & \cdots & \lambda I_{n_k} - M_k \end{bmatrix}$$

Now, by Lemma 5, we have

$$\det(\lambda I_n - A(\mathbf{M}, \mathbf{u}, \rho)) = \det(\lambda I_{n_k} - M_k) \det(S) \quad (3.1)$$

where

$$S = \begin{bmatrix} \lambda I_{n_1} - M_1 & -\rho_{1,2} u_1 v_2^t & \cdots & -\rho_{1,k-1} u_1 v_{k-1}^t \\ -\rho_{2,1} u_2 v_1^t & \lambda I_{n_2} - M_2 & \cdots & -\rho_{2,k-1} u_2 v_{k-1}^t \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k-1,1} u_{k-1} v_1^t & -\rho_{k-1,2} u_{k-1} v_2^t & \cdots & \lambda I_{n_{k-1}} - M_{k-1} \end{bmatrix} \\ - \begin{bmatrix} -\rho_{1,k} u_1 v_k^t \\ -\rho_{2,k} u_2 v_k^t \\ \vdots \\ -\rho_{k-1,k} u_{k-1} v_k^t \end{bmatrix} (\lambda I_{n_k} - M_k)^{-1} \begin{bmatrix} -\rho_{k,1} u_k v_1^t & -\rho_{k,2} u_k v_2^t & \cdots & -\rho_{k,k-1} u_k v_{k-1}^t \end{bmatrix} \\ = [s_{ij}]$$

$$\text{given by } s_{ij} = \begin{cases} \lambda I_{n_i} - M_i - \Gamma_k \rho_{i,k} \rho_{k,i} u_i v_i^t & \text{if } i = j \\ -(\rho_{i,j} + \Gamma_k \rho_{i,k} \rho_{k,j}) u_i v_j^t & \text{if } i \neq j \end{cases}.$$

$$\text{By Lemma 7, } \det(\lambda I_{n_i} - M_i - \Gamma_k \rho_{i,k} \rho_{k,i} u_i v_i^t) = \det(\lambda I_{n_i} - M_i) (1 - (\Gamma_k \rho_{i,k} \rho_{k,i}) \Gamma_i) \\ \text{and } v_i^t (\lambda I_{n_i} - M_i - \Gamma_k \rho_{i,k} \rho_{k,i} u_i v_i^t)^{-1} u_i = \frac{\Gamma_i}{1 - (\Gamma_k \rho_{i,k} \rho_{k,i}) \Gamma_i}.$$

Now by the induction hypothesis on S we get

$$\det(S) = \left(\prod_{i=1}^{k-1} \det(\lambda I_{n_i} - M_i - \Gamma_k \rho_{i,k} \rho_{k,i} u_i v_i^t) \frac{\Gamma_i}{1 - (\Gamma_k \rho_{i,k} \rho_{k,i}) \Gamma_i} \right) \det(\tilde{S}) \\ = \left(\prod_{i=1}^{k-1} \det(\lambda I_{n_i} - M_i) \Gamma_i \right) \det(\tilde{S})$$

$$\text{where } \tilde{S} = [\tilde{s}_{ij}] \text{ given by } \tilde{s}_{ij} = \begin{cases} \frac{1 - \rho_{i,k} \rho_{k,i} \Gamma_k \Gamma_i}{\Gamma_i} & \text{if } i = j \\ -(\rho_{i,j} + \Gamma_k \rho_{i,k} \rho_{k,j}) & \text{if } i \neq j \end{cases}.$$

Therefore

$$\det(S) = \phi_1 \phi_2 \cdots \phi_{k-1} \\ \times \begin{vmatrix} 1 - (\rho_{1,k} \rho_{k,1} \Gamma_k \Gamma_1) & -(\rho_{1,2} + \Gamma_k \rho_{1,k} \rho_{k,2}) \Gamma_1 & \cdots & -(\rho_{1,k-1} + \Gamma_k \rho_{1,k} \rho_{k,k-1}) \Gamma_1 \\ -(\rho_{2,1} + \Gamma_k \rho_{2,k} \rho_{k,1}) \Gamma_2 & 1 - (\rho_{2,k} \rho_{k,2} \Gamma_k \Gamma_2) & \cdots & -(\rho_{2,k-1} + \Gamma_k \rho_{2,k} \rho_{k,k-1}) \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ -(\rho_{k-1,1} + \Gamma_k \rho_{k-1,k} \rho_{k,1}) \Gamma_{k-1} & -(\rho_{k-1,2} + \Gamma_k \rho_{k-1,k} \rho_{k,2}) \Gamma_{k-1} & \cdots & 1 - (\rho_{k-1,k} \rho_{k,k-1} \Gamma_k \Gamma_{k-1}) \end{vmatrix} \\ = \phi_1 \phi_2 \cdots \phi_{k-1} \times \left(\begin{vmatrix} 1 & -\rho_{1,2} \Gamma_1 & \cdots & -\rho_{1,k-1} \Gamma_1 \\ -\rho_{2,1} \Gamma_2 & 1 & \cdots & -\rho_{2,k-1} \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k-1,1} \Gamma_{k-1} & -\rho_{k-1,2} \Gamma_{k-1} & \cdots & 1 \end{vmatrix} \right. \\ \left. - \begin{vmatrix} -\rho_{1,k} \rho_{k,1} \Gamma_k \Gamma_1 & -\rho_{1,k} \rho_{k,2} \Gamma_k \Gamma_1 & \cdots & -\rho_{1,k} \rho_{k,k-1} \Gamma_k \Gamma_1 \\ -\rho_{2,k} \rho_{k,1} \Gamma_k \Gamma_2 & -\rho_{2,k} \rho_{k,2} \Gamma_k \Gamma_2 & \cdots & -\rho_{2,k} \rho_{k,k-1} \Gamma_k \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k-1,k} \rho_{k,1} \Gamma_k \Gamma_{k-1} & -\rho_{k-1,k} \rho_{k,2} \Gamma_k \Gamma_{k-1} & \cdots & -\rho_{k-1,k} \rho_{k,k-1} \Gamma_k \Gamma_{k-1} \end{vmatrix} \right)$$

$$= \phi_1 \phi_2 \cdots \phi_{k-1} \times \begin{vmatrix} 1 & -\rho_{1,2}\Gamma_1 & \cdots & -\rho_{1,k-1}\Gamma_1 & -\rho_{1,k}\Gamma_1 \\ -\rho_{2,1}\Gamma_2 & 1 & \cdots & -\rho_{2,k-1}\Gamma_2 & -\rho_{2,k}\Gamma_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\rho_{k-1,1}\Gamma_{k-1} & -\rho_{k-1,2}\Gamma_{k-1} & \cdots & 1 & -\rho_{k-1,k}\Gamma_{k-1} \\ -\rho_{k,1}\Gamma_k & -\rho_{k,2}\Gamma_k & \cdots & -\rho_{k,k-1}\Gamma_k & 1 \end{vmatrix},$$

again by Lemma 5.

By substituting this $\det(S)$ value in Equation (3.1), we get the required result for k . This completes the proof of Theorem 2. $\square$

Suppose the matrices M_i 's are normal and $\{\theta_1, \theta_2, \dots, \theta_{m_i}\}$ is the set of distinct u_i -main eigenvalues of M_i , for $1 \leq i \leq k$. Then as discussed in the proof of Lemma 2, we can write

$$\Gamma_i = \frac{f_i}{g_i} \text{ where } g_i = \prod_{j=1}^{m_i} (\lambda - \theta_j). \quad (3.2)$$

Hence by the Theorem 2,

$$\det(\lambda I - A(\mathbf{M}, \mathbf{u}, \rho)) = \left(\frac{\phi_1}{g_1}\right) \cdots \left(\frac{\phi_k}{g_k}\right) \Phi(\lambda) \quad (3.3)$$

where $\Phi(\lambda) = \begin{vmatrix} g_1(\lambda) & -\rho_{1,2}f_1(\lambda) & \cdots & -\rho_{1,k}f_1(\lambda) \\ -\rho_{2,1}f_2(\lambda) & g_2(\lambda) & \cdots & -\rho_{2,k}f_2(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1}f_k(\lambda) & -\rho_{k,2}f_k(\lambda) & \cdots & g_k(\lambda) \end{vmatrix}$. So we can describe the spectrum of $A(\mathbf{M}, \mathbf{u}, \rho)$ as follows.

Theorem 3. Consider the notations defined above. Suppose the matrices M_i 's are normal, then

- Every eigenvalue, which is not a u_i -main eigenvalue of M_i , say λ with multiplicity $m(\lambda)$ is an eigenvalue of $A(\mathbf{M}, \mathbf{u}, \rho)$ with multiplicity $m(\lambda)$.
- Every u_i -main eigenvalue of M_i , say λ with multiplicity $m(\lambda)$ is an eigenvalue of $A(\mathbf{M}, \mathbf{u}, \rho)$ with multiplicity $m(\lambda) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

Proof. By Lemma 2 the poles of Γ_i are u_i -main eigenvalues and they are simple. Now the proof easily follows from Equation (3.3). $\square$

4. Universal spectra of the H -join of graphs

In this section, by applying Theorem 2, we obtain the results on characteristic polynomial and spectrum of the universal adjacency matrix of H -join of graphs.

Consider a graph H on k vertices and a family of graphs $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$. Let $G = \bigvee_H \mathcal{F}$ be the H -join of graphs in $\mathcal{F}$, and let n_i , A_i and D_i be the number of vertices, the adjacency matrix and the degree matrix of the graph G_i respectively, for $1 \leq i \leq k$. Also let $\rho_{i,j}$ be the scalars defined by $\rho_{i,j} = \rho_{j,i} = 1$ if $ij \in E(H)$ and 0 otherwise, for $1 \leq i, j \leq k$ and $i \neq j$. Once and for all we fix the partition of $V(G)$ as $V(G_1) \cup V(G_2) \cup \dots \cup V(G_k)$, so that the adjacency matrix of the graph G is given by

$$A(G) = \begin{bmatrix} A_1 & \rho_{1,2} \mathbf{1}_{n_1} \mathbf{1}_{n_2}^t & \cdots & \rho_{1,k} \mathbf{1}_{n_1} \mathbf{1}_{n_k}^t \\ \rho_{2,1} \mathbf{1}_{n_2} \mathbf{1}_{n_1}^t & A_2 & \cdots & \rho_{2,k} \mathbf{1}_{n_2} \mathbf{1}_{n_k}^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1} \mathbf{1}_{n_k} \mathbf{1}_{n_1}^t & \rho_{k,2} \mathbf{1}_{n_k} \mathbf{1}_{n_2}^t & \cdots & A_k \end{bmatrix}. \quad (4.1)$$

4.1. Universal spectra of the H -join of graphs

The proof of the following Observation is immediate from the definition of the H -join of graphs.

Observation 1. Let H be a graph with vertex set $\{v_1, \dots, v_k\}$ and $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a family of k graphs such that $V(G_i) = \{v_1^{(i)}, \dots, v_{n_i}^{(i)}\}$ for $1 \leq i \leq k$. Then the degree of the vertex $v_j^{(i)}$ in $G = \bigvee_H \mathcal{F}$ is given by

$$\deg_G(v_j^{(i)}) = \deg_{G_i}(v_j^{(i)}) + w_i, 1 \leq i \leq k, 1 \leq j \leq n_i$$

where $w_i = \sum_{v_l \in N_H(v_i)} n_l$.

Let $U(G) = \alpha A(G) + \beta I_n + \gamma J_n + \delta D(G)$ with $\alpha \neq 0$ be the universal adjacency matrix of the graph G , where $n = \sum_{i=1}^k n_i$. Let $U_i = \alpha A_i + \beta I_{n_i} + \gamma J_{n_i} + \delta D_i$ be the universal adjacency matrix of the graph G_i , for $1 \leq i \leq k$. Therefore by the Equation (4.1) the universal adjacency matrix of G can be written as follows:

$$U(G) = \begin{bmatrix} U_1 + \delta w_1 I_{n_1} & (\rho_{1,2}\alpha + \gamma) \mathbf{1}_{n_1} \mathbf{1}_{n_2}^t & \cdots & (\rho_{1,k}\alpha + \gamma) \mathbf{1}_{n_1} \mathbf{1}_{n_k}^t \\ (\rho_{2,1}\alpha + \gamma) \mathbf{1}_{n_2} \mathbf{1}_{n_1}^t & U_2 + \delta w_2 I_{n_2} & \cdots & (\rho_{2,k}\alpha + \gamma) \mathbf{1}_{n_2} \mathbf{1}_{n_k}^t \\ \vdots & \vdots & \ddots & \vdots \\ (\rho_{k,1}\alpha + \gamma) \mathbf{1}_{n_k} \mathbf{1}_{n_1}^t & (\rho_{k,2}\alpha + \gamma) \mathbf{1}_{n_k} \mathbf{1}_{n_2}^t & \cdots & U_k + \delta w_k I_{n_k} \end{bmatrix} \quad (4.2)$$

In the following theorem, we obtain the characteristic polynomial of universal adjacency matrix $U(G)$.

Theorem 4. Let H be a graph on k vertices and $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a family of any graphs. Consider the graph $G = \bigvee_H \mathcal{F}$. Let $\phi_i(\lambda)$ be the characteristic polynomial of U_i and $\Gamma_i(\lambda) = \Gamma_{U_i}(\mathbf{1}_{n_i}; \lambda)$. Then we have the following.

i) The characteristic polynomial of the universal adjacency matrix $U(G)$ given in Equation (4.2) is

$$\phi_{U(G)}(\lambda) = \left(\prod_{i=1}^k \phi_i(\lambda - \delta w_i) \Gamma_i(\lambda - \delta w_i) \right) \det(\tilde{U}(G))$$

$$\text{where } \tilde{U}(G) = \begin{bmatrix} \frac{1}{\Gamma_1(\lambda - \delta w_1)} & -(\rho_{1,2}\alpha + \gamma) & \cdots & -(\rho_{1,k}\alpha + \gamma) \\ -(\rho_{2,1}\alpha + \gamma) & \frac{1}{\Gamma_2(\lambda - \delta w_2)} & \cdots & -(\rho_{2,k}\alpha + \gamma) \\ \vdots & \vdots & \ddots & \vdots \\ -(\rho_{k,1}\alpha + \gamma) & -(\rho_{k,2}\alpha + \gamma) & \cdots & \frac{1}{\Gamma_k(\lambda - \delta w_k)} \end{bmatrix} \quad (4.3)$$

ii) Analogous to the Equations (3.2) and (3.3), we define f_i, g_i , and $\Phi(\lambda)$ corresponding to the main eigenvalues of U_i for $1 \leq i \leq k$. Then the universal spectrum of G is given as below.

- For every eigenvalue μ of U_i with multiplicity $m(\mu)$, which is not a main eigenvalue, $\mu + \delta w_i$ is a universal eigenvalue of G with multiplicity $m(\mu)$.
- For every main eigenvalue μ of U_i with multiplicity $m(\mu)$, $\mu + \delta w_i$ is a universal eigenvalue of G with multiplicity $m(\mu) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

Proof. For each $1 \leq i \leq k$, let $P_i = U_i + \delta w_i I_{n_i}$. Then we have the following relations,

$$\phi_{P_i}(\lambda) = \det(\lambda I_{n_i} - (U_i + \delta w_i I_{n_i})) = \phi_{U_i}(\lambda - \delta w_i) \text{ and}$$

$$\Gamma_{P_i}(\lambda) = \mathbf{1}_{n_i}^t (\lambda I_{n_i} - (U_i + \delta w_i I_{n_i}))^{-1} \mathbf{1}_{n_i} = \Gamma_{U_i}(\lambda - \delta w_i).$$

Let $\hat{\rho}_{i,j} = \hat{\rho}_{j,i} = \rho_{i,j}\alpha + \gamma$ for $1 \leq i < j \leq k$. Considering the triplet $(\mathbf{M}, \mathbf{u}, \hat{\rho})$, given by

$$\mathbf{M} = (P_1, P_2, \dots, P_k), \mathbf{u} = (\mathbf{1}_{n_1}, \mathbf{1}_{n_2}, \dots, \mathbf{1}_{n_k}) \text{ and}$$

$$\hat{\rho} = (\hat{\rho}_{1,2}, \dots, \hat{\rho}_{1,k}, \hat{\rho}_{2,3}, \dots, \hat{\rho}_{2,k}, \dots, \hat{\rho}_{k-1,k})$$

we can write the matrices in the Equations (4.2) and (4.3) as $U(G) = A(\mathbf{M}, \mathbf{u}, \hat{\rho})$ and $\tilde{U}(G) = \tilde{A}(\mathbf{M}, \mathbf{u}, \hat{\rho})$. Now using Theorem 2 the proof of (i) follows. By Lemma 4, μ is not a main eigenvalue of U_i if and only if $\mu + \delta w_i$ is not a main eigenvalue of P_i . Now by Theorem 3 the proof of (ii) follows. $\square$

Corollary 2. Consider the notations defined in Theorem 4. Suppose the graph G_i is r_i -regular for $1 \leq i \leq k$. Then $p_i = \alpha r_i + \beta + \gamma n_i + \delta(r_i + w_i)$ is an eigenvalue of $P_i = U_i + \delta w_i I_{n_i}$ and

$$\text{spec}(U(G)) = \left(\bigcup_{i=1}^k (\text{spec}(P_i) \setminus \{p_i\}) \right) \cup \text{spec}(\tilde{U}(G))$$

$$\text{where } \widetilde{U}'(G) = \begin{bmatrix} \frac{p_1}{\sqrt{n_2 n_1} \widehat{\rho}_{2,1}} & \sqrt{n_1 n_2} \widehat{\rho}_{1,2} & \cdots & \frac{\sqrt{n_1 n_k} \widehat{\rho}_{1,k}}{\sqrt{n_2 n_k} \widehat{\rho}_{2,k}} \\ \vdots & p_2 & \ddots & \vdots \\ \sqrt{n_k n_1} \widehat{\rho}_{k,1} & \sqrt{n_k n_2} \widehat{\rho}_{k,2} & \cdots & p_k \end{bmatrix}.$$

Proof. Clearly $\mathbf{1}_{n_i}$ is an eigenvector of P_i corresponding to the eigenvalue $p_i = \alpha r_i + \beta + \gamma n_i + \delta(r_i + w_i)$. Let $\phi_i = \det(\lambda I_{n_i} - P_i)$ and $\Gamma_i = \Gamma_{P_i}(\mathbf{1}_{n_i}; \lambda)$. Now, by Lemma 3, we have $\Gamma_i = \frac{n_i}{\lambda - p_i}$ and so by Theorem 4, we get

$$\phi(U(G)) = \frac{\phi_1}{\lambda - p_1} \frac{\phi_2}{\lambda - p_2} \cdots \frac{\phi_k}{\lambda - p_k} n_1 n_2 \cdots n_k \det(\widetilde{U}(G)).$$

Distributing n_i inside the determinant of $\widetilde{U}(G)$, as $\sqrt{n_i}$ into the i^{th} row and $\sqrt{n_i}$ into the i^{th} column, we can write

$$\begin{aligned} n_1 n_2 \cdots n_k \det(\widetilde{U}(G)) &= \det \begin{bmatrix} (\lambda - p_1) & -\sqrt{n_1 n_2} \widehat{\rho}_{1,2} & \cdots & -\sqrt{n_1 n_k} \widehat{\rho}_{1,k} \\ -\sqrt{n_2 n_1} \widehat{\rho}_{2,1} & (\lambda - p_2) & \cdots & -\sqrt{n_2 n_k} \widehat{\rho}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{n_k n_1} \widehat{\rho}_{k,1} & -\sqrt{n_k n_2} \widehat{\rho}_{k,2} & \cdots & (\lambda - p_k) \end{bmatrix} \\ &= \det(\lambda I_n - \widetilde{U}'(G)). \end{aligned}$$

Now the proof follows. $\square$

Corollary 3. Consider the notations defined in Theorem 4. Assume that $\alpha + \delta = 0$. Then $p_i = \beta + \gamma n_i + \delta w_i$ is an eigenvalue of $P_i = U_i + \delta w_i I_{n_i}$ and

$$\text{spec}(U(G)) = \left(\bigcup_{i=1}^k (\text{spec}(P_i) \setminus \{p_i\}) \right) \cup \text{spec}(\widetilde{U}'(G))$$

$$\text{where } \widetilde{U}'(G) = \begin{bmatrix} \frac{p_1}{\sqrt{n_2 n_1} \widehat{\rho}_{2,1}} & \sqrt{n_1 n_2} \widehat{\rho}_{1,2} & \cdots & \frac{\sqrt{n_1 n_k} \widehat{\rho}_{1,k}}{\sqrt{n_2 n_k} \widehat{\rho}_{2,k}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_k n_1} \widehat{\rho}_{k,1} & \sqrt{n_k n_2} \widehat{\rho}_{k,2} & \cdots & p_k \end{bmatrix}.$$

Proof. It is easy to see that

$$(\alpha A_i + \beta I_{n_i} + \gamma J_{n_i} + \delta D_i) \mathbf{1}_{n_i} = (\alpha + \delta) \begin{bmatrix} \deg_{G_i}(v_1^{(i)}) \\ \deg_{G_i}(v_2^{(i)}) \\ \vdots \\ \deg_{G_i}(v_{n_i}^{(i)}) \end{bmatrix} + (\beta + \gamma n_i) \mathbf{1}_{n_i}.$$

So $\mathbf{1}_{n_i}$ is an eigenvector of P_i corresponding to the eigenvalue $p_i = \beta + \gamma n_i + \delta w_i$.

Then by the same argument as in the previous corollary, the proof follows. $\square$

Remark 1. In [4, Theorem 8], the authors obtained the Laplacian spectra of H -join of any graphs. For a given graph G , the Laplacian matrix $L(G)$, is obtained from the universal adjacency matrix $U(G)$, by taking $(\alpha, \beta, \gamma, \delta) = (-1, 0, 0, 1)$. So, we can deduce [4, Theorem 8] from Corollary 3.

Let H be a graph on k vertices and G' be any graph. We recall that the Lexicographic product of graphs H and G' , denoted by $H[G']$, is obtained as the H -join of graphs in $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$, where $G_i = G'$ for $1 \leq i \leq k$. In [23], the authors obtained the characteristic polynomial of $H[G']$ and investigated the spectrum in various cases. Now we generalize [23, Theorem 2.4] by obtaining the characteristic polynomial of the universal adjacency matrix of $H[G']$ when $\delta = 0$.

Theorem 5. Let H be a graph on k vertices and G' be a graph on n' vertices. Consider the graph $G = H[G']$, the lexicographic product of H and G' . Suppose $\text{spec}(H) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Then the characteristic polynomial of the universal adjacency matrix of $U(G)$ when $\delta = 0$, is

$$\phi_{U(G)}(\lambda) = \phi^k(\lambda) \left(\prod_{i=1}^k (1 - \lambda_i \Gamma(\lambda)) \right),$$

where $\phi(\lambda)$ is the characteristic polynomial of $U(G')$ and $\Gamma(\lambda) = \Gamma_{U(G')}(\mathbf{1}_{n'}; \lambda)$ when $\delta = 0$.

Proof. By Theorem 4,

$$\begin{aligned} \phi_{U(G)}(\lambda) &= \phi^k(\lambda) \Gamma^k(\lambda) \det\left(\frac{1}{\Gamma(\lambda)} I_k - A(H)\right) \\ &= \phi^k(\lambda) \Gamma^k(\lambda) \left(\prod_{i=1}^k \left(\frac{1}{\Gamma(\lambda)} - \lambda_i \right) \right) \\ &= \phi^k(\lambda) \left(\prod_{i=1}^k (1 - \lambda_i \Gamma(\lambda)) \right). \quad \square \end{aligned}$$

4.2. The generalized characteristic polynomial of the H -join of graphs

The generalized characteristic polynomial of a graph G is introduced in [9], as the bivariate polynomial defined by $\phi_G(\lambda, t) = \det(\lambda I - (A(G) - tD(G)))$ where $A(G)$ and $D(G)$ are the adjacency and the degree matrix associated to the graph G . As mentioned earlier, in [8, Theorem 3.1] the authors obtained a generalization of Fiedler's lemma, for the matrices with fixed row sum and as an application, they obtained the generalized characteristic polynomial of H -join of regular graphs. In the following theorem, we obtain the generalized characteristic polynomial of H -join of any graphs.

Theorem 6. Let H be any graph and $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a family of any k graphs. Consider the graph $G = \bigvee_H \mathcal{F}$. Let $M(G) = A(G) - tD(G)$ and $M_i = A_i - tD_i$ for

$1 \leq i \leq k$. Let ϕ_i be the characteristic polynomial of M_i and $\Gamma_i = \Gamma_{M_i}(\mathbf{1}_{n_i}; \lambda)$. Then

i) The generalized characteristic polynomial of the graph G is

$$\phi_{M(G)}(\lambda) = \left(\prod_{i=1}^k \phi_i(\lambda + tw_i) \Gamma_i(\lambda + tw_i) \right) \det(\widetilde{M}(G))$$

$$\text{where } \widetilde{M}(G) = \begin{bmatrix} \frac{1}{\Gamma_1(\lambda + tw_1)} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2(\lambda + tw_2)} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k(\lambda + tw_k)} \end{bmatrix}.$$

ii) Analogous to the Equations (3.2) and (3.3), we define f_i, g_i , and $\Phi(\lambda)$ corresponding to the main eigenvalues of M_i for $1 \leq i \leq k$. Then the spectrum of $M(G)$ is given below.

- For every eigenvalue μ of M_i with multiplicity $m(\mu)$, which is not a main eigenvalue, $\mu - tw_i$ is an eigenvalue of $M(G)$ with multiplicity $m(\mu)$.
- For every main eigenvalue μ of M_i with multiplicity $m(\mu)$, $\mu - tw_i$ is an eigenvalue of $M(G)$ with multiplicity $m(\mu) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

Proof. The proof directly follows from Theorem 4, by taking $(\alpha, \beta, \gamma, \delta) = (1, 0, 0, -t)$ in the universal adjacency matrix $U(G)$. $\square$

Remark 2. Suppose the graph G_i is r_i -regular for each $1 \leq i \leq k$. Then $p_i = r_i - t(r_i + w_i)$ is an eigenvalue of $P_i = M_i - tw_i I_{n_i}$. Similar to the proof of the Corollary 2 of Theorem 4, here also we obtain the following result on spectrum of $M(G)$.

$$\text{spec}(M(G)) = \left(\bigcup_{i=1}^k (\text{spec}(P_i) \setminus \{p_i\}) \right) \cup \text{spec}(\widetilde{M}'(G))$$

$$\text{where } \widetilde{M}'(G) = \begin{bmatrix} p_1 & \sqrt{n_1 n_2} \rho_{1,2} & \cdots & \sqrt{n_1 n_k} \rho_{1,k} \\ \sqrt{n_2 n_1} \rho_{2,1} & p_2 & \cdots & \sqrt{n_2 n_k} \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_k n_1} \rho_{k,1} & \sqrt{n_k n_2} \rho_{k,2} & \cdots & p_k \end{bmatrix}.$$

This result is one of the main results of [8].

Remark 3. In [8, Corollary 3.2], from the generalized characteristic polynomial, the characteristic polynomial of the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix, and the normalized Laplacian matrix are deduced for H -join of regular graphs. Similarly, we can also deduce them for H -join of any graphs from Theorem 6.

Also, we can deduce the characteristic polynomial of the Seidel adjacency matrix of H -join of any graphs by taking $(\alpha, \beta, \gamma, \delta) = (-2, -1, 1, 0)$ in the universal adjacency matrix $U(G)$ in Theorem 4.

Lemma 8. [17] Let G be a complete bipartite graph $K_{p,q}$ on $p + q = n$ vertices. Then $\Gamma_{A(G)}(\mathbf{1}_n; \lambda) = \frac{n\lambda + 2pq}{\lambda^2 - pq}$.

Corollary 4. Let H be a graph on k vertices and $\mathcal{F} = \{G_1, G_2, \dots, G_k\}$ be a family of k graphs such that $G_i = K_{p_i, q_i}$ for $1 \leq i \leq k$. Consider the graph $G = \bigvee_H \mathcal{F}$. Let $n = \sum_{i=1}^k (p_i + q_i)$ be the number of vertices of G . Then the spectrum of G is given as below.

- 0 is an eigenvalue with multiplicity $n - 2k$.
- The remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$, where

$$\Phi(\lambda) = \det \begin{bmatrix} \lambda^2 - p_1 q_1 & -\rho_{1,2}(n_1 \lambda + 2p_1 q_1) & \cdots & -\rho_{1,k}(n_1 \lambda + 2p_1 q_1) \\ -\rho_{2,1}(n_2 \lambda + 2p_2 q_2) & \lambda^2 - p_2 q_2 & \cdots & -\rho_{2,k}(n_2 \lambda + 2p_2 q_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1}(n_k \lambda + 2p_k q_k) & -\rho_{k,2}(n_k \lambda + 2p_k q_k) & \cdots & \lambda^2 - p_k q_k \end{bmatrix}.$$

Proof. Let $\phi_i(\lambda) = \det(\lambda I_{n_i} - A(G_i))$ and $\Gamma_i(\lambda) = \Gamma_{A(G_i)}(\mathbf{1}_{n_i}; \lambda)$ where $n_i = p_i + q_i$. By Lemma 8, $\Gamma_i(\lambda) = \frac{n_i \lambda + 2p_i q_i}{\lambda^2 - p_i q_i}$. By taking $t = 0$ in Theorem 6, the characteristic polynomial of $A(G)$ is given as

$$\phi_{A(G)}(\lambda) = \left(\prod_{i=1}^k \frac{\phi_i(\lambda)}{\lambda^2 - p_i q_i} \right) \det(\tilde{A}(G))$$

$$\text{where } \tilde{A}(G) = \begin{bmatrix} \lambda^2 - p_1 q_1 & -\rho_{1,2}(n_1 \lambda + 2p_1 q_1) & \cdots & -\rho_{1,k}(n_1 \lambda + 2p_1 q_1) \\ -\rho_{2,1}(n_2 \lambda + 2p_2 q_2) & \lambda^2 - p_2 q_2 & \cdots & -\rho_{2,k}(n_2 \lambda + 2p_2 q_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1}(n_k \lambda + 2p_k q_k) & -\rho_{k,2}(n_k \lambda + 2p_k q_k) & \cdots & \lambda^2 - p_k q_k \end{bmatrix}.$$

Then the proof follows from the fact that $\phi_i(\lambda) = \lambda^{n_i-2}(\lambda^2 - p_i q_i)$ for $1 \leq i \leq k$. $\square$

Remark 4. The above result can be generalized by replacing the family $\mathcal{F}$ of complete bipartite graphs with any family of graphs having exactly two main eigenvalues.

5. Cospectral graphs

Two graphs are said to be cospectral if their (adjacency) spectrum are equal. In general, for any matrix say $M(G)$ for a given graph G , two graphs are said to be M -cospectral if their M -spectrum is equal. In [2], the author questioned the existence of

non-regular graphs, which are cospectral with respect to the adjacency, the Laplacian, the signless Laplacian, and the normalized Laplacian spectrum simultaneously. In [8, Theorem 3.7], the authors affirmatively answered the question by the construction of such graphs using the H -join of regular graphs. In the following theorem, we prove that those graphs are U -cospectral too. In particular, those graphs are cospectral with respect to the Seidel spectrum also.

Lemma 9. *If G_1 and G_2 are cospectral r -regular graphs on n vertices, then $\text{spec}(A(G_1) + J_n) = \text{spec}(A(G_2) + J_n)$.*

Proof. $\mathbf{1}_n$ is an eigenvector of $A(G_1) + J_n$ and $A(G_2) + J_n$ corresponding to the eigenvalue $r + n$. Every eigenvector different from $\mathbf{1}_n$ of G_1 and G_2 lies in the kernel of J_n . Hence the result follows. $\square$

Theorem 7. *Let $\mathcal{F} = \mathcal{F}_1 = \{G_1, G_2, \dots, G_k\}$ and $\mathcal{F}_2 = \{G'_1, G'_2, \dots, G'_k\}$ be two families of graphs.*

- (i) *If G_i and G'_i are cospectral regular graphs on n_i vertices for $1 \leq i \leq k$, and H is an arbitrary graph on k vertices, then $\bigvee_H \mathcal{F}_1$ and $\bigvee_H \mathcal{F}_2$ are U -cospectral.*
- (ii) *If H_1 and H_2 are cospectral r_1 -regular graphs on k vertices and every G_i is r_2 -regular on m vertices for $1 \leq i \leq k$, then $\bigvee_{H_1} \mathcal{F}$ and $\bigvee_{H_2} \mathcal{F}$ are U -cospectral.*
- (iii) *If H_1 and H_2 are cospectral r_1 -regular graphs on k vertices and, G_i and G'_i are cospectral r_2 -regular graphs on m vertices for $1 \leq i \leq k$ then $\bigvee_{H_1} \mathcal{F}_1$ and $\bigvee_{H_2} \mathcal{F}_2$ are U -cospectral.*

Proof. Let $G = \bigvee_H \mathcal{F}_1$ and $G' = \bigvee_H \mathcal{F}_2$. Let U_i and U'_i be the universal adjacency matrix of G_i and G'_i respectively. Let $P_i = U_i + \delta w_i I_{n_i}$ and $P'_i = U'_i + \delta w_i I_{n_i}$ where w_i is defined as in Observation 1. Then by Lemma 9, $\text{spec}(P_i) = \text{spec}(P'_i)$ for $1 \leq i \leq k$. Now by Corollary 2, $\text{spec}(U(G)) = \text{spec}(U(G'))$, since $\widetilde{U}'(G) = \widetilde{U}'(G')$. Thus (i) is proved.

Let $G = \bigvee_{H_1} \mathcal{F}$ and $G' = \bigvee_{H_2} \mathcal{F}$. Let U_i be the universal adjacency matrix of G_i . Since H_1 and H_2 are r_1 -regular, $w_i = r_1 m$ and $p_i = \alpha r_2 + \beta + \gamma m + \delta(r_2 + r_1 m)$ for $1 \leq i \leq k$. By Lemma 9, $\text{spec}(\alpha A(H_1) + \gamma(J_k - I_k)) = \text{spec}(\alpha A(H_2) + \gamma(J_k - I_k))$. Now by Corollary 2, $\text{spec}(U(G)) = \text{spec}(U(G'))$, since $\widetilde{U}'(G)$ and $\widetilde{U}'(G')$ are cospectral. Thus (ii) is proved.

To prove (iii), let $G = \bigvee_{H_1} \mathcal{F}_1$ and $G' = \bigvee_{H_2} \mathcal{F}_2$. By the same arguments as in (i) and (ii), we get P_i and P'_i are cospectral and, $\widetilde{U}'(G)$ and $\widetilde{U}'(G')$ are cospectral. Thus the proof of (iii) follows. $\square$

6. Spectra of the H -generalized join of graphs

In this section, we obtain the characteristic polynomial of H -generalized join of graphs $\bigvee_{H,S} \mathcal{F}$ introduced in [5].

Definition 3. Let G be any graph. A vertex subset S of a graph G is said to be (κ, τ) -regular if S induces a κ -regular graph in G and every vertex outside of S has τ neighbours in S . When G is a regular graph, for convenience $S = V(G)$ is considered as $(\kappa, 0)$ -regular.

Definition 4. Let G be any graph with vertex set $\{v_1, v_2, \dots, v_n\}$. For any subset $S \subset V(G)$, the characteristic vector of S , denoted by χ_S , is defined as the 0-1 vector such that i^{th} place of χ_S is 1 if and only if the vertex $v_i \in S$.

Lemma 10. [6,18] Let G be an r -regular graph on n vertices with a (κ, τ) -regular set S . Then $\kappa - \tau \in \text{spec}(G)$ and $\mathbf{v} = (r - \kappa)\chi_S - \tau\chi_{\bar{S}}$ is an eigenvector corresponding to $\kappa - \tau$, where $\bar{S} = V(G) \setminus S$.

Lemma 11. [7] Let G be a graph with a (κ, τ) -regular set S , where $\tau > 0$, and $\lambda \in \sigma(A(G))$. Then, λ is not a main eigenvalue of G if and only if $\lambda = \kappa - \tau$ or $\chi_S \in (\mathcal{E}_G(\lambda))^\perp$.

Fix a (κ, τ) -regular subset S of $V(G)$. An eigenvalue $\lambda \in \sigma(G)$ is said to be a *special eigenvalue* if $\lambda \neq \kappa - \tau$ and λ is not a main eigenvalue. Then by Lemma 11, if λ is a special eigenvalue of G then λ is not a χ_S -main eigenvalue. In [5] the authors obtained all eigenvalues of $\bigvee_{H,S} \mathcal{F}$ when G_i is regular and the subsets $S_i \in \mathcal{S}$ are such that $S_i = V(G_i)$

for $1 \leq i \leq k$, in which case $\bigvee_{H,S} \mathcal{F}$ coincides with the H -join of regular graphs $\bigvee_H \mathcal{F}$. In other cases, it is proved that every special eigenvalue corresponding to (κ_i, τ_i) -regular subset S_i is an eigenvalue of $\bigvee_{H,S} \mathcal{F}$ and thus the partial spectrum is obtained for $\bigvee_{H,S} \mathcal{F}$.

In the following theorem, we obtain the characteristic polynomial of $\bigvee_{H,S} \mathcal{F}$ for any family of subsets $\mathcal{S}$ and obtain the complete set of eigenvalues.

Theorem 8. Consider a graph H of order k and a family of graphs $\mathcal{F} = \{G_1, \dots, G_k\}$. Consider also a family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_k\}$, such that $S_i \subset V(G_i)$ for $1 \leq i \leq k$. Let $G = \bigvee_{H,S} \mathcal{F}$. Let n_i and A_i be the number of vertices and the adjacency matrix of the graph G_i respectively for $1 \leq i \leq k$. For $1 \leq i, j \leq k$, let $\rho_{i,j}$ be the scalars defined by $\rho_{i,j} = 1$ if $ij \in E(H)$ and 0 otherwise. Then we have the following.

i) The characteristic polynomial of G is

$$\phi_G(\lambda) = \left(\prod_{i=1}^k \phi_i(\lambda) \Gamma_i(\lambda) \right) \det(\tilde{A}(G))$$

$$\text{where } \tilde{A}(G) = \begin{bmatrix} \frac{1}{\Gamma_1} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k} \end{bmatrix}, \quad (6.1)$$

where $\phi_i(\lambda) = \det(\lambda I_{n_i} - A(G_i))$ and $\Gamma_i(\lambda) = \Gamma_{A_i}(\chi_{S_i}; \lambda)$.

ii) Analogous to the Equations (3.2) and (3.3), we define f_i, g_i and $\Phi(\lambda)$ corresponding to the χ_{S_i} -main eigenvalues of G_i for $1 \leq i \leq k$. Then the spectrum of G is given as below.

- Every eigenvalue μ of A_i with multiplicity $m(\mu)$, which is not χ_{S_i} -main eigenvalue, is an eigenvalue of G with multiplicity $m(\mu)$.
- Every χ_{S_i} -main eigenvalue μ of A_i with multiplicity $m(\mu)$, is an eigenvalue of G with multiplicity $m(\mu) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

Proof. By the definition of $\bigvee_{(H,S)} \mathcal{F}$, the adjacency matrix of G is given as

$$A(G) = \begin{bmatrix} A_1 & \rho_{1,2} \chi_{S_1} \chi_{S_2}^t & \cdots & \rho_{1,k} \chi_{S_1} \chi_{S_k}^t \\ \rho_{2,1} \chi_{S_2} \chi_{S_1}^t & A_2 & \cdots & \rho_{2,k} \chi_{S_2} \chi_{S_k}^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1} \chi_{S_k} \chi_{S_1}^t & \rho_{k,2} \chi_{S_k} \chi_{S_2}^t & \cdots & A_k \end{bmatrix}.$$

Then by direct application of Theorem 2 and Theorem 3 on $A(G)$, the proofs of (i) and (ii) follow immediately. $\square$

Lemma 12. Let G be an r -regular graph on n vertices. Let S be a (κ, τ) -regular set of G . Suppose $\kappa - \tau$ is a simple eigenvalue of G . Then $\Gamma_{A(G)}(\chi_S; \lambda) = \frac{|S|^2}{n(\lambda - r)} + \frac{|S|^2(r - \kappa)^2}{c(\lambda - (\kappa - \tau))}$, where $c = (r - \kappa)^2|S| + \tau^2(n - |S|)$.

Proof. By Lemma 10, $\kappa - \tau$ is an eigenvalue of G with corresponding eigenvector $\mathbf{v} = (r - \kappa)\chi_S - \tau\chi_{\bar{S}}$. Every eigenvector corresponding to an eigenvalue λ such that $\kappa - \tau \neq \lambda \neq r$, is orthogonal to the vector χ_S . Thus the only χ_S -main eigenvalues of G are $\kappa - \tau$ and r . Let the orthogonal projection on the eigenspace corresponding to the eigenvalue r be E_r and to the eigenvalue $\kappa - \tau$ be $E_{\kappa - \tau}$. Then $E_r = \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^t$ and $E_{\kappa - \tau} = \frac{1}{c} \mathbf{v} \mathbf{v}^t$,

where $c = (r - \kappa)^2|S| + \tau^2|\bar{S}|$. Therefore, $\Gamma_{A(G)}(\chi_S; \lambda) = \frac{\chi_S^t E_r \chi_S}{\lambda - r} + \frac{\chi_S^t E_{\kappa - \tau} \chi_S}{\lambda - (\kappa - \tau)} = \frac{|S|^2}{n(\lambda - r)} + \frac{|S|^2(r - \kappa)^2}{c(\lambda - (\kappa - \tau))}$. $\square$

Corollary 5. Consider a graph H of order k and a family of graphs $\mathcal{F} = \{G_1, \dots, G_k\}$, where G_i is r_i -regular graph on n_i vertices for $1 \leq i \leq k$. Consider also a family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_k\}$, such that $S_i \in V(G_i)$ is a (κ_i, τ_i) -regular set and, $\kappa_i - \tau_i$ is a simple eigenvalue of G_i for $1 \leq i \leq k$. Then

$$\Gamma_i(\lambda) = \Gamma_{A(G_i)}(\chi_{S_i}; \lambda) = \frac{|S_i|^2}{n(\lambda - r_i)} + \frac{|S_i|^2(r_i - \kappa_i)^2}{c_i(\lambda - (\kappa_i - \tau_i))},$$

where $c_i = (r_i - \kappa_i)^2|S_i| + \tau_i^2(n_i - |S_i|)$. Let $g_i = (\lambda - r_i)(\lambda - (\kappa_i - \tau_i))$ and $f_i = \Gamma_i(\lambda)g_i$. Let $G = \bigvee_{H, \mathcal{S}} \mathcal{F}$. The spectrum of G is given as

$$\text{spec}(G) = \left(\bigcup_{i=1}^k (\text{spec}(G_i) \setminus \{r_i, \kappa_i - \tau_i\}) \right) \cup \left(\bigcup_{i=1}^k \{x_1^{(i)}, x_2^{(i)}\} \right),$$

where $x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_1^{(k)}$ and $x_2^{(k)}$ are the roots of the polynomial

$$\Phi(\lambda) = \det \begin{bmatrix} g_1 & -\rho_{1,2}f_1 & \cdots & -\rho_{1,k}f_1 \\ -\rho_{2,1}f_2 & g_2 & \cdots & -\rho_{2,k}f_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1}f_k & -\rho_{k,2}f_k & \cdots & g_k \end{bmatrix}.$$

Proof. The expression of $\Gamma_i(\lambda)$ follows from Lemma 12. The remaining proof is obtained from Theorem 8, by the similar arguments used in the proof of Corollary 4. $\square$

7. Spectra of the generalized corona of graphs

In [13, Theorem 3.1], the generalized corona product is defined as below and its characteristic polynomial is obtained. In this section, we deduce this result as a corollary of Theorem 2. This is done by viewing the corona product as the H -join of suitably chosen graphs.

Definition 5. Let H' be a graph on k vertices. Let $G_1, G_2, \dots, G_k$ be graphs of order $n_1, n_2, \dots, n_k$ respectively. The generalized corona product of H' with $G_1, G_2, \dots, G_k$, denoted by $H' \circ \Lambda_{i=1}^k G_i$, is obtained by taking one copy of graphs $H', G_1, G_2, \dots, G_k$, and joining the i^{th} vertex of H' to every vertex of G_i .

When $G_i = G'$ for all i , the graph $H' \circ \Lambda_{i=1}^k G_i$ is called simply corona of H' and G' , denoted by $H' \circ G'$.

Theorem 9. Let H' be a graph with vertex set $V(H') = \{v_1, v_2, \dots, v_k\}$. Let $G_1, G_2, \dots, G_k$ be any graphs. $\rho_{i,j} = 1$ if $v_i v_j \in E(H')$ and 0 otherwise. The characteristic polynomial of the generalized corona product $G = H' \circ \Lambda_{i=1}^k G_i$ is given by

$$\phi_G(\lambda) = \left(\prod_{i=1}^k \phi_{G_i}(\lambda) \right) \det(\tilde{A}(H'))$$

$$\text{where } \tilde{A}(H') = \begin{bmatrix} \lambda - \Gamma_{G_1}(\lambda) & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \lambda - \Gamma_{G_2}(\lambda) & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \lambda - \Gamma_{G_k}(\lambda) \end{bmatrix}.$$

Proof. Let $H = H' \circ K_1$. Let v_{k+i} be the new vertex in H attached with the vertex v_i in the copy of H' , for $1 \leq i \leq k$. Let $\mathcal{F} = \{K_1, K_1, \dots, K_1, G_1, G_2, \dots, G_k\}$. Then we get the following visualization of generalized corona as H -join of graphs in $\mathcal{F}$.

$$(H' \circ \Lambda_{i=1}^k G_i) = \left(\bigvee_H \mathcal{F} \right)$$

That is, each v_i is replaced by K_1 and v_{k+i} is replaced by G_i in H , to form the H -join.

Now $A(H) = \begin{bmatrix} A(H') & I_k \\ I_k & 0_k \end{bmatrix}$. Since $\phi_{K_1}(\lambda) = \lambda$ and $\Gamma_{K_1}(\lambda) = \frac{1}{\lambda}$, by letting $\alpha = 1$, $\beta = \gamma = \delta = 0$ in Theorem 4, we get

$$\phi_G(\lambda) = \left(\prod_{i=1}^k (\phi_{K_1}(\lambda) \phi_{G_i}(\lambda) \Gamma_{K_1}(\lambda) \Gamma_{G_i}(\lambda)) \right) \det(\tilde{A}(H))$$

which implies

$$\phi_G(\lambda) = \left(\prod_{i=1}^k (\phi_{G_i}(\lambda) \Gamma_{G_i}(\lambda)) \right) \det(\tilde{A}(H)) \quad (7.1)$$

where $\tilde{A}(H) = \begin{bmatrix} \lambda I_k - A(H') & -I_k \\ -I_k & \text{diag}(\frac{1}{\Gamma_{G_1}(\lambda)}, \frac{1}{\Gamma_{G_2}(\lambda)}, \dots, \frac{1}{\Gamma_{G_k}(\lambda)}) \end{bmatrix}$. Now by Lemma 5, $\det(\tilde{A}(H))$ is given as

$$\det \begin{bmatrix} \frac{1}{\Gamma_{G_1}(\lambda)} & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma_{G_2}(\lambda)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\Gamma_{G_k}(\lambda)} \end{bmatrix} \det \left(\lambda I_k - A(H') - \begin{bmatrix} \Gamma_{G_1}(\lambda) & 0 & \cdots & 0 \\ 0 & \Gamma_{G_2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{G_k}(\lambda) \end{bmatrix} \right)$$

$$= \left(\prod_{i=1}^k \frac{1}{\Gamma_{G_i}(\lambda)} \right) \det \left(\begin{bmatrix} \lambda - \Gamma_{G_1}(\lambda) & 0 & \cdots & 0 \\ 0 & \lambda - \Gamma_{G_2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - \Gamma_{G_k}(\lambda) \end{bmatrix} - A(H') \right)$$

Now by substituting $\det(\tilde{A}(H))$ in Equation (7.1) we get the required result. $\square$

Remark 5. Similarly, we can get the other variants of the spectra of the generalized corona of any graphs. The same work can be done on other variants of corona also by suitable choice of H .

8. Examples

To illustrate our results, we compute the characteristic polynomials of two particular examples on H -join of graphs and H -generalized join of graphs constrained by vertex subsets. Similarly, we can apply our other results also.

Example 1. Consider the graphs $H = P_3, G_1 = P_3, G_2 = K_{1,3}$ and $G_3 = K_2$ as follows.

Let $\mathcal{F} = \{G_1, G_2, G_3\}$. Then the H -join graph $G = \bigvee_H \mathcal{F}$ is given as

We see that $\phi_1(\lambda) = \lambda^3 - 2\lambda$, $\phi_2(\lambda) = \lambda^4 - 3\lambda^2$, $\phi_3(\lambda) = \lambda^3 - \lambda$, $\Gamma_1(\lambda) = \frac{3\lambda + 4}{\lambda^2 - 2}$, $\Gamma_2(\lambda) = \frac{4\lambda + 6}{\lambda^2 - 3}$ and $\Gamma_3(\lambda) = \frac{3\lambda - 1}{\lambda^2 - \lambda}$. The characteristic polynomial of G is $\lambda^3(\lambda^3 + 4\lambda^2 - \lambda - 6)(\lambda^3 - 5\lambda^2 - 8\lambda + 2)(\lambda + 1)$ which is equal to

$$\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)\Gamma_1(\lambda)\Gamma_2(\lambda)\Gamma_3(\lambda) \det \begin{bmatrix} \frac{1}{\Gamma_1(\lambda)} & -1 & 0 \\ -1 & \frac{1}{\Gamma_2(\lambda)} & -1 \\ 0 & -1 & \frac{1}{\Gamma_3(\lambda)} \end{bmatrix}.$$

Example 2. Consider H and $\mathcal{F}$ as in Example 1. Let $S_1 = \{v_1^{(1)}, v_2^{(1)}\}$, $S_2 = \{v_1^{(2)}, v_2^{(2)}, v_4^{(2)}\}$ and $S_3 = \{v_2^{(3)}, v_3^{(3)}\}$. Then the H -generalized join graph $G = \bigvee_{H,S} \mathcal{F}$

is given as

Here, $\phi_1(\lambda) = \lambda^3 - 2\lambda$, $\phi_2(\lambda) = \lambda^4 - 3\lambda^2$ and $\phi_3(\lambda) = \lambda^3 - \lambda$. Based on the choices of S_1, S_2 and S_3 we get $\Gamma_1(\chi_{S_1}; \lambda) = \frac{2\lambda^2 + 2\lambda - 1}{\lambda^3 - 2\lambda}$, $\Gamma_2(\chi_{S_2}; \lambda) = \frac{3\lambda}{\lambda^2 - 3}$ and $\Gamma_3(\chi_{S_3}; \lambda) = \frac{2\lambda^2 - 1}{\lambda^3 - \lambda}$. The characteristic polynomial of G is $\lambda^4(\lambda^6 - 18\lambda^4 - 6\lambda^3 + 35\lambda^2 + 6\lambda - 15)$ which is equal to

$$\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)\Gamma_1(\chi_{S_1}; \lambda)\Gamma_2(\chi_{S_2}; \lambda)\Gamma_3(\chi_{S_3}; \lambda) \det \begin{bmatrix} \frac{1}{\Gamma_1(\chi_{S_1}; \lambda)} & -1 & 0 \\ -1 & \frac{1}{\Gamma_2(\chi_{S_2}; \lambda)} & -1 \\ 0 & -1 & \frac{1}{\Gamma_3(\chi_{S_3}; \lambda)} \end{bmatrix}.$$

Declaration of competing interest

There is no competing interest.

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