Supplementary Online Material

In this Supplementary material, for an EA RS code constructed from two $[n, n-d_m+1, d_m]$-classical RS codes with parity check matrices H_{b_1} and H_{b_2}, we obtain the set P_O for various cases of u and v, where $v = (b_1 + b_2 - 2) \bmod n$ and $u = v + 2(d_m - 1)$.

The ranges of i and j such that $(v + i + j) \bmod n = 0$ for various cases[1] based on u and v are provided in Table 1.

Table 1: Ranges of i and j satisfying $(v + i + j) \bmod n = 0$.

Case	**Range of u and v**	**Range of i and j**
Case 1	$u < n$	-
Case 2(a)	$n \leq u < 2n$ and $n - v < d_m$	$i \in \{1, \ldots, n-v-1\}$, $j = n-v-i_m$
Case 2(b)	$n \leq u < 2n$ and $n - v \geq d_m$	$i \in \{n-v-(d_m-1), \ldots, d_m-1\}$, $j = n-v-i_m$
Case 3	$2n \leq u < 3n$	$i \in \{1, \ldots, n-v-1\}$, $j = n-v-i_m$ $i \in \{2n-v-(d_m-1), \ldots, d_m-1\}$, $j = 2n-v-i_m$

We recall that the position index sets are

$$P_\mathrm{X} = \{n - b_1 - d_m + 3, \ldots, n - b_1 + 1\}, \tag{1}$$

$$P_\mathrm{Z} = \{b_2 + 1, \ldots, b_2 + d_m - 1\}, \tag{2}$$

$$P_\mathrm{O} = P_\mathrm{X} \cap P_\mathrm{Z}. \tag{3}$$

From equation (1), when an element s belongs to P_X, s satisfies the following inequality:

$$n - b_1 - d_m + 3 \leq s \leq n - b_1 + 1,$$

$$\Rightarrow n - (b_1 - 2 + (d_m - 1)) \leq s \leq n - (b_1 - 2 + 1).$$

Thus, the general form of an element s that belongs to P_{X} is

$$s = n - (b_1 - 2 + i), \text{ where } i \in \{1, \ldots, (d_m - 1)\}. \tag{4}$$

From equation (2), when an element t belongs to P_{Z}, t satisfies the following inequality:

$$b_2 + 1 \leq t \leq b_2 + d_m - 1,$$
$$\Rightarrow b_2 + 1 \leq t \leq b_2 + (d_m - 1).$$

Thus, the general form of an element t that belongs to P_{Z} is

$$b_2 + j, \quad \text{where } j \in \{1, \ldots, (d_m - 1)\}. \tag{5}$$

We note that the value of s and t in equations (4) and (5) are considered modulo n and belong to the range 1 to n.

We next consider an element r that belongs to P_{O} from equation (3). As $P_{\mathrm{O}} = P_{\mathrm{X}} \cap P_{\mathrm{Z}}$, the element r belongs to both P_{X} and P_{Z}. Thus, from equations (4) and (5), for some $i, j \in \{1, \ldots, (d_m - 1)\}$, we obtain

$$r = n - (b_1 - 2 + i) \text{ and } r = b_2 + j,$$
$$\Rightarrow r = n - (b_1 - 2 + i) = b_2 + j,$$
$$\Rightarrow (b_1 + b_2 - 2) + i + j = n. \tag{6}$$

As the general form of the elements of P_{X} and P_{Z} in equations (4) and (5) are considered modulo n, $r = (n - (b_1 - 2 + i))$ and $r = (b_2 + j)$ are also considered modulo n. As $v = (b_1 + b_2 - 2) \bmod n$, the equation (6) is same as the condition $(v + i + j) \bmod n = 0$ obtained in the paper to find the value of n_e. Thus, i and j in equation (6) belong to the ranges in Table 1 that was obtained using the condition $(v + i + j) \bmod n = 0$.

We note that the elements in the set P_{O} are modulo n and range from 1 to n. We next obtain the set P_{O} for the various cases in Table 1 by obtaining the range of r based on the ranges of i

and j as follows:

Case 1:

As $n_e = 0$, there are no entangled qudits, and also there are no overlaps from Table 1. Thus, $P_{\mathrm{O}} = \emptyset$.

Case 2(a):

From Table 1, the range of i and j is from 1 to $(n - v - 1)$. In equation (4), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of i to be from 1 to $(n - v - 1)$, we obtain that r satisfies the following inequality:

$$
\begin{aligned}
& n - (b_1 - 2 + (n - v - 1)) \leq r \leq n - b_1 + 1, \\
& n - (b_1 - 2 + (n - (b_1 + b_2 - 2) - 1)) \leq r \leq n - b_1 + 1, \\
\Rightarrow & b_2 + 1 \leq r \leq n - b_1 + 1.
\end{aligned} \tag{7}
$$

In equation (5), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of j to be from 1 to $(n - v - 1)$, we obtain that r satisfies the following inequality:

$$
\begin{aligned}
& b_2 + 1 \leq r \leq b_2 + (n - v - 1), \\
& b_2 + 1 \leq r \leq b_2 + (n - (b_1 + b_2 - 2) - 1), \\
\Rightarrow & b_2 + 1 \leq r \leq n - b_1 + 1.
\end{aligned} \tag{8}
$$

From equations (7) and (8), the element r that belongs to both the sets P_{X} and P_{Z} also belongs to the range from $(b_2 + 1)$ to $(n - b_1 + 1)$. Thus, $P_{\mathrm{O}} = \{b_2 + 1, \ldots, n - b_1 + 1\}$.

Case 2(b):

From Table 1, the range of i and j is from $(n - v - (d_m - 1))$ to $(d_m - 1)$. In equation (4), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of i to be from $(n - v - (d_m - 1))$ to

$(d_m - 1)$, we obtain that r satisfies the following inequality:

$$n-(b_1-2+(d_m-1)) \leq r \leq n-(b_1-2+(n-v-(d_m-1))),$$
$$n-(b_1-2+(d_m-1)) \leq r \leq n-(b_1-2+(n-(b_1+b_2-2)-(d_m-1))),$$
$$\Rightarrow n - d_m - b_1 + 3 \leq r \leq b_2 + d_m - 1. \tag{9}$$

In equation (5), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of j to be from $(n - v - (d_m - 1))$ to $(d_m - 1)$, we obtain that r satisfies the following inequality:

$$b_2 + (n - v - (d_m - 1)) \leq r \leq b_2 + (d_m - 1),$$
$$b_2 + (n-(b_1 + b_2-2) - (d_m-1)) \leq r \leq b_2 + (d_m-1),$$
$$\Rightarrow n - d_m - b_1 + 3 \leq r \leq b_2 + d_m - 1. \tag{10}$$

From equations (9) and (10), the element r that belongs to both the sets P_{X} and P_{Z} also belongs to the range from $(n-d_m-b_1+3)$ to (b_2+d_m-1). Thus, $P_{\mathrm{O}} = \{n-d_m-b_1+3, \ldots, b_2+d_m-1\}$.

Case 3:

From Table 1, there are two ranges of i and j as follows:

1) The first range of i and j is from 1 to $n - v - 1$. In equation (4), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of i to be from 1 to $n - v - 1$, we obtain that r satisfies the following inequality:

$$n-(b_1-2+n-v-1) \leq r \leq n-b_1+1,$$
$$n-(b_1-2+n-(b_1+b_2-2)-1) \leq r \leq n-b_1+1,$$
$$\Rightarrow b_2 + 1 \leq r \leq n - b_1 + 1. \tag{11}$$

In equation (5), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of j to be from 1 to

$n - v - 1$, we obtain that r satisfies the following inequality:

$$\begin{aligned} & b_2 + 1 \leq r \leq b_2 + n - v - 1, \\ & b_2 + 1 \leq r \leq b_2 + n - (b_1 + b_2 - 2) - 1, \\ \Rightarrow & b_2 + 1 \leq r \leq n - b_1 + 1. \end{aligned} \tag{12}$$

From equations (11) and (12), the element r that belongs to both sets P_{X} and P_{Z} also belongs to the range from $(b_2 + 1)$ to $(n - b_1 + 1)$. Thus,

$$\{b_2 + 1, \ldots, n - b_1 + 1\} \subset P_{\mathrm{O}}. \tag{13}$$

2) The second range of i and j is from $(2n - v - (d_m - 1))$ to $(d_m - 1)$. In equation (4), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of i to be from $(2n - v - (d_m - 1))$ to $(d_m - 1)$, we obtain that r satisfies the following inequality:

$$\begin{aligned} & n-(b_1-2+(d_m-1)) \leq r \leq n-(b_1-2+(2n-v-(d_m-1))), \\ & n-(b_1-2+(d_m-1)) \leq r \leq n-(b_1-2+2n-(b_1+b_2-2)-(d_m-1)), \\ \Rightarrow & n - d_m - b_1 + 3 \leq r \leq b_2 + d_m - 1, \end{aligned} \tag{14}$$

In equation (5), considering $v = (b_1 + b_2 - 2) \bmod n$ and the range of j to be from $(2n - v - (d_m - 1))$ to $(d_m - 1)$, we obtain that r satisfies the following inequality:

$$\begin{aligned} & b_2+(2n-v-(d_m-1)) \leq r \leq b_2+(d_m-1), \\ & b_2+2n-(b_1+b_2-2)-(d_m-1) \leq r \leq b_2+d_m-1, \\ \Rightarrow & 2n - d_m - b_1 + 3 \leq r \leq b_2 + d_m - 1, \\ \Rightarrow & n - d_m - b_1 + 3 \leq r \leq b_2 + d_m - 1, \end{aligned} \tag{15}$$

where $(n - d_m - b_1 + 3)$ is obtained from $(2n - d_m - b_1 + 3)$ by considering modulo n as the element $r \in P_{\mathrm{O}}$ is considered modulo n.

From equation (14) and (15), the element r that belongs to both the sets P_{X} and P_{Z} also belongs to the range from $(n - d_m - b_1 + 3)$ to $(b_2 + d_m - 1)$. Thus,

$$\{n - d_m - b_1 + 3, \ldots, b_2 + d_m - 1\} \subset P_{\text{O}}. \tag{16}$$

Thus, for Case 3, from equations (13) and (16), the set $P_{\text{O}} = \{b_2 + 1, \ldots, n - b_1 + 1, n - d_m - b_1 + 3, \ldots, b_2 + d_m - 1\}$.

References and Notes

[1] Nadkarni, P. J., Garani, S. S.: Entanglement-Assisted Quantum Reed-Solomon Codes. IEEE Information Theory Applications Workshop, San Diego (2019).