model. In particular, the resulting signal subspace fills up the whole M-dimensional space, where M is the array size, and the resulting covariance matrix is not guaranteed to be nonnegative definite. As a result of the spreading of the signal subspace, subspace-based methods, like MUSIC, induce bias in the DOA estimates. We further show that complete decorrelation is not possible with this method even if we make the array size infinitely large.

II. Problem Statement

Assume that two narrow-band correlated (partially or fully) sources impinge on a linear array consisting of M equispaced isotropic sensors, and the directions of arrival (DOA's) of these sources are θ_d and θ_s with respect to the array normal. Let the signals emitted by these sources be denoted by $d(t)$ and $s(t)$, respectively, and the sensor noise $v(t)$ be assumed uncorrelated from sensor to sensor and independent of the signals. Assuming that the signals and the noise are zero mean and stationary random processes, we can express the asymptotic array covariance matrix as

$$\Phi = \Phi_{dd} + \Phi_{ss} + \Phi_{ds} + \Phi_{sd} + \Phi_{vv} \tag{1}$$

where

$$\Phi_{ii} = \sigma_i^2 \boldsymbol{a}(\theta_i)\boldsymbol{a}^+(\theta_i), \qquad i = d, s \tag{2}$$

$$\Phi_{ds} = \rho\sigma_d\sigma_s \boldsymbol{a}(\theta_d)\boldsymbol{a}^+(\theta_s) = \Phi_{sd}^+ \tag{3}$$

and

$$\Phi_{vv} = \sigma_v^2 I. \tag{4}$$

Here, σ_d^2, σ_s^2 and σ_v^2 denote the signal and noise powers, ρ denotes the coefficient of correlation between $d(t)$ and $s(t)$ and $^+$ denotes Hermitian transpose. The direction vector $\boldsymbol{a}(\theta_i)$ is given by $\boldsymbol{a}(\theta_i) = [1, \exp[-j\omega_0\Delta/c \sin\theta_i], \ldots, \exp[-j(M-1)\omega_0\Delta/c \sin\theta_i]]^T$, where Δ is the interelement spacing, ω_0 is the center frequency of the sources, and c is the velocity of propagation of the plane waves.

Now consider the noise-free covariance matrix $(\Phi - \Phi_{vv})$. We know that the column space of this matrix is the signal subspace which is two-dimensional in the present case if $\rho \neq 1$. Further, this matrix is not Toeplitz because of the presence of cross-correlation matrices Φ_{ds} and Φ_{sd}, and it reduces to Toeplitz form when these matrices vanish, which would be the case if the impinging signals are uncorrelated.

In redundancy averaging, we average all the elements along a diagonal and replace each element in the diagonal by its average, and we repeat this for all the diagonals. In this correspondence, we address the following problems: 1) what would be the eigenstructure of the resulting matrix after redundancy averaging? 2) will the cross-correlation matrices Φ_{ds} and Φ_{sd} vanish after redundancy averaging?

III. Analysis of Redundancy Averaging Method

Consider the noise-free covariance matrix $(\Phi - \Phi_{vv})$. From the assumed signal and array models, the mnth element of this matrix is given by

$$\begin{aligned}[\Phi - \Phi_{vv}]_{mn} &= \sigma_d^2 \exp[j\omega_0(n-m)T_d] + \sigma_s^2 \exp[j\omega_0(n-m)T_s] \\ &\quad + \rho\sigma_s\sigma_d \exp\{j\omega_0[(n-1)T_s - (m-1)T_d]\} \\ &\quad + \rho^*\sigma_d\sigma_s \exp\{-j\omega_0[(m-1)T_s - (n-1)T_d]\}\end{aligned} \tag{5}$$

where $T_d = (\Delta/c)\sin\theta_d$ and $T_s = (\Delta/c)\sin\theta_s$

Let the Toeplitz matrix obtained after performing the redundancy averaging on $[\Phi - \Phi_{vv}]$ be denoted by Φ_T. We can then show that

$$\Phi_T = \begin{bmatrix} \beta(0) & \beta(1) & \ddots & \beta(M-1) \\ \beta^*(1) & \beta(0) & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \beta^*(M-1) & \ddots & \ddots & \beta(0) \end{bmatrix} \tag{6}$$

where $\beta(l)$ is given by

$$\begin{aligned}\beta(l) &= \sigma_d^2 e^{j\omega_0 l T_d} + \sigma_s^2 e^{j\omega_0 l T_s} \\ &\quad + \frac{\alpha}{(M-l)} \exp\left\{j\frac{\omega_0}{2}[(M+1)T_s - (M+1-2l)T_d]\right\} \\ &\quad \cdot \sum_{k=1}^{M-l} \exp[-j\omega_0(T_s - T_d)k] \\ &\qquad 0 \leq l \leq (M-1)\end{aligned} \tag{7}$$

with

$$\begin{aligned}\alpha &= \sigma_s\sigma_d(\rho^* \exp[-j\omega_0(M-1)(T_s - T_d)/2] \\ &\quad + \rho \exp[j\omega_0(M-1)(T_s - T_d)/2).\end{aligned} \tag{8}$$

We note from (7) that when α goes to zero, $\beta(1)$ reduces to the result that we would obtain in the uncorrelated case. Now, consider the expression for α (cf. (8)). It can be easily verified that α goes to zero when

$$\begin{aligned}&= \frac{}{\omega_o(T_s - T_d)} \left\lfloor \angle\rho \pm (2n+1)\frac{}{2} \right\rfloor + 1 \\ &\qquad n = 0, 1, \cdots\end{aligned} \tag{9}$$

indicating that for certain combinations of ρ, θ_d, θ_s, and M, redundancy averaging yields complete decorrelation of the impinging signals; a similar observation was also made by Godara [6]. We, however, note that for a given signal scenario and the array size, there may be at most one value of n for which (9) is satisfied. In other words, for a given array size and the correlation coefficient, the combinations of DOA's that satisfy (9) form a discrete set with countably infinite number of points. It then follows that the combinations of DOA's which do not satisfy (9) form a continuous set excluding these discrete points implying that in almost all the cases, this condition will not be met. Thus, in practice, this method does not achieve complete decorrelation of the impinging signals. We now show how the resulting signal subspace is inconsistent with that of the underlying signal model.

Note that the column space of Φ_T is the resulting signal subspace and its dimension is equal to the rank of Φ_T. By performing elementary row operations[1] on the matrix Φ_T, we can get an upper triangular matrix as follows:

$$\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1M} \\ 0 & d_{22} & \cdots & d_{2M} \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & d_{MM} \end{bmatrix} \tag{10}$$

[1]Elementary row operations preserve the rank of a matrix

where

$$d_{11} = \beta(0)$$

$$d_{22} = \frac{|\beta(0)|^2 - |\beta(1)|^2}{\beta(0)}$$

$$d_{33} = \frac{(|\beta(0)|^2 - |\beta(2)|^2)(|\beta(0)|^2 - |\beta(1)|^2) - (\beta^*(1)\beta(0) - \beta^*(2)\beta(1))(\beta(0)\beta(1) - \beta^*(1)\beta(2))}{\beta(0)(|\beta(0)|^2 - |\beta(1)|^2)} \tag{11}$$

and so on.

Recall that the rank of a triangular matrix is equal to or more than the number of nonzero entries on the main diagonal [7]. We now investigate the situations under which all the diagonal terms are nonzero.

Consider the expression for $\beta(l)$ (cf. (7)). For any set of preassigned values of M, $\omega_0 T_s$, ρ, σ_s^2 and σ_d^2, $\beta(l)$ is a rational polynomial in $e^{j\omega_0 T_d}$. Substituting $\mu = e^{j\omega_0 T_d}$ and α (cf. (8)) in (7), and simplifying, we get

$$\beta(l) = \left[\sum_{r=0}^{M-1-l} K_1(r)\mu^r + \sum_{r=M-1}^{2(M-1)-l} K_2(r)\mu^r\right] \Big/ \mu^{(M-1-l)} \tag{12}$$

where $K_1(r)$ and $K_2(r)$ are given by

$$K_1(r) = \begin{cases} \dfrac{\rho\sigma_s\sigma_d}{(M-l)} \exp\left[-j(r-M+1)\omega_0 T_s\right], & 0 \le r \le M-2-l \\ \left(\sigma_s^2 + \dfrac{\rho\sigma_s\sigma_d}{(M-l)}\right) e^{jl\omega_0 T_s}, & r = M-1-l \end{cases} \tag{13}$$

$$K_2(r) = \begin{cases} \left(\sigma_d^2 + \dfrac{\rho^*\sigma_s\sigma_d}{(M-l)}\right) & r = M-1 \\ \dfrac{\rho^*\sigma_s\sigma_d}{(M-l)} \exp\left[-j(r-M+1)\omega_0 T_s\right], & M \le r \le 2(M-1)-l. \end{cases} \tag{14}$$

We observe the following from (11) and (12): i) The order of numerator polynomial of the diagonal term, d_{nn}, n = 1, 2, . . . , M, is, $2(M-1)2^{(n-1)}$. ii) The poles of d_{nn} are the zeros of $d_{(n-1)(n-1)}$, $d_{(n-2)(n-2)}$, . . . and d_{11}.

Thus, the number of zeros of d_{11} to d_{nn} is at most $\Sigma_{n=1}^{M} 2(M-1)2^{(n-1)} = 2(M-1)(2^M-1)$. In other words, the number of values of μ at which one or several of the diagonal terms become zero is finite. What this means is the following.

For given values of the array size M, the correlation coefficient ρ, signal powers σ_d^2 and σ_s^2, and the **DOA** of one source θ_s, the number of values of θ_d for which the matrix Φ_T may not attain full rank is finite implying that for all other values of θ_d, which constitute a continuous set (excluding a finite number of discrete points), it attains full rank. Thus, the resulting signal subspace (column span of Φ_T) fills up the whole M-dimensional space in almost all the cases. Consequently, the space spanned by the eigenvectors corresponding to the $(M-D)$ least eigenvalues of $(\Phi_T + \sigma_r^2 I)$ does not represent the true noise subspace. Hence, a MUSIC-like algorithm, when applied to this covariance matrix, gives biased **DOA** estimates.

To illustrate the effect of redundancy averaging on the eigenstructure of the resulting matrix, we considered a scenario with two equipower ($\sigma_s^2 = \sigma_d^2 = 10$) narrow-band coherent ($\rho = 1$) sources located at 0° and 10°, and a linear array with 4 isotropic sensors placed half-wavelength apart.

Table I shows the ordered eigenvalues of the noise-free covariance matrix under different situations. For the purposes of completeness, we also included the eigenvalues that result when the sources are uncorrelated. Note that the redundancy averaging stretches the dimensionality of the signal subspace to 4 and the resulting matrix is indefinite; the possibility of the second phenomenon was also noted in [5], but in the context of finite data.

The MUSIC algorithm (spectral version) was applied to the above example, taking the sensor noise power as unity ($\sigma_r^2 = 1$) and the spatial spectrum obtained with both methods, one using spatial smoothing with 2 subarrays and the other using redundancy averaging. This is plotted in Fig. 1. Note the bias in the **DOA** estimates obtained with the redundancy averaging method.

We now investigate whether perfect decorrelation is guaranteed if the array size is made infinitely large. In this context, we use Frobenius norm (F-norm) of the cross-correlation part of Φ_T as a measure of the correlation. The motivation for the use of this measure is as follows.

Recall that the cross-correlation matrices, Φ_{ds} and Φ_{sd} (cf. (3)), vanish when the correlation coefficient ρ is zero, and the redundancy averaging affects only these matrices since the autocorrelation matrices, Φ_{dd} and Φ_{ss} (cf. (2)), are already in Toeplitz form. Further, we can show that

$$\|\Phi_{ds} + \Phi_{sd}\|_F^2 = |\rho|^2 \left(4\sigma_d^2\sigma_s^2 \sum_{n=1}^{M}\sum_{m=1}^{M} \left|\cos\left(\frac{\omega_0}{2}(n+m-2) \cdot (T_s - T_d) + \angle\rho\right)\right|^2\right) \tag{15}$$

(with $\|X\|_F^2$ denoting the squared F-norm of the matrix X), which clearly shows that the F-norm of the sum of the cross-correlation matrices is proportional to the correlation coefficient.

Now, consider the F-norm of the sum of the cross-correlation matrices that result with redundancy averaging. Let $(\Phi_{ds} + \Phi_{sd})_T$ denote this sum. From (5) and (6), it follows that the third term in (7) represents each element on the lth diagonal of $(\Phi_{ds} + \Phi_{sd})_T$. We can then show that

$$\|(\Phi_{ds} + \Phi_{sd})_T\|_F^2 = |\alpha|^2\left(\frac{1}{M}\left|\frac{\sin M\omega_0(T_s - T_d)}{\sin \omega_0(T_s - T_d)}\right|^2 + 2\sum_{l=1}^{M-2}\frac{1}{(M-l)}\left|\frac{\sin (M-l)\omega_0(T_s - T_d)}{\sin \omega_0(T_s - T_d)}\right|^2 + 2\right). \tag{16}$$

We note the following from (16):

i) When the angular separation of the sources is very small compared to the beamwidth of the array, the F-norm is approximately equal to

$$\|(\Phi_{ds} + \Phi_{sd})_T\|_F^2 \approx 4\sigma_s^2\sigma_d^2 (\mathrm{Re}\,(\rho))^2 M^2. \tag{17}$$