Supplementary information for "Cumulative emissions accounting of greenhouse gases due to path independence for a sufficiently rapid emissions cycle"

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1 Repeated integrals of emissions graph

As noted in the paper, we consider during the increasing phase of emissions $0 \leq x < \frac{1}{2}$, the stylized form of emissions $\hat{m}(x) = \beta x^{\gamma}$, and for the decreasing phase $\frac{1}{2} \leq x \leq 1$, $\hat{m}(x) = \beta (1-x)^{\gamma}$, which is a mirror-image. Repeated integrals for $0 \leq x < \frac{1}{2}$ can be directly obtained as

$$\hat{m_1}(x) = \int_0^x \hat{m}(h)\, dh = \frac{\beta}{\gamma+1} x^{\gamma+1} \tag{1}$$

and so on, for

$$\hat{m_i}(x) = \int_0^x \hat{m}_{i-1}(h)\, dh = \frac{\beta}{(\gamma+1)\ldots(\gamma+i)} x^{\gamma+i} \tag{2}$$

As for the domain $\frac{1}{2} \leq x \leq 1$,

$$\hat{m_1}(x) = \int_0^{\frac{1}{2}} \beta h^{\gamma} dh + \int_{\frac{1}{2}}^x \beta (1-h)^{\gamma} dh \tag{3}$$

simplifying to

$$\hat{m_1}(x) = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1} - \frac{\beta}{\gamma+1}(1-x)^{\gamma+1} \tag{4}$$

and the 2$^{\text{nd}}$ repeated-integral

$$\hat{m_2}(x) = \int_0^{\frac{1}{2}} \frac{\beta}{\gamma+1} h^{\gamma+1} dh + \int_{\frac{1}{2}}^x \left\{ \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1} - \frac{\beta}{\gamma+1}(1-h)^{\gamma+1} \right\} dh \tag{5}$$

simplifying to

$$\hat{m_2}(x) = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\left(x - \frac{1}{2}\right) + \frac{\beta}{(\gamma+1)(\gamma+2)}(1-x)^{\gamma+2} \tag{6}$$

for the 3$^{\text{rd}}$ integral

$$\hat{m_3}(x) = \int_0^{\frac{1}{2}} \frac{\beta}{(\gamma+1)(\gamma+2)} h^{\gamma+2} dh + \int_{\frac{1}{2}}^x \left\{ \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\left(h - \frac{1}{2}\right) + \frac{\beta}{(\gamma+1)(\gamma+2)}(1-h)^{\gamma+2} \right\} dh \tag{7}$$

which becomes

$$\hat{m_3}(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^2}{2!} - \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)}(1-x)^{\gamma+3} \quad (8)$$

whereas for the 4^{th} integral

$$\hat{m_4}(x) = \int_0^{\frac{1}{2}} \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)} h^{\gamma+3} dh + \int_{\frac{1}{2}}^{x}\left\{\frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(h-\frac{1}{2}\right)^2}{2!} - \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)}(1-h)^{\gamma+3}\right\} dh \quad (9)$$

becoming

$$\hat{m_4}(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\left(x-\frac{1}{2}\right) + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^3}{3!} + \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}(1-x)^{\gamma+4} \quad (10)$$

and for the 5^{th} integral

$$\hat{m_5}(x) = \int_0^{\frac{1}{2}} \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} h^{\gamma+4} dh + \int_{\frac{1}{2}}^{x} \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\left(h-\frac{1}{2}\right) + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(h-\frac{1}{2}\right)^3}{3!} dh + \int_{\frac{1}{2}}^{x} \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)}(1-h)^{\gamma+4} dh \quad (11)$$

which becomes

$$\hat{m_5}(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+5}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\frac{\left(x-\frac{1}{2}\right)^2}{2!} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^4}{4!} - \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)}(1-x)^{\gamma+5} \quad (12)$$

and for the 6^{th} integral

$$\hat{m_6}(x) = \int_0^{\frac{1}{2}} \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} h^{\gamma+5} dh + \int_{\frac{1}{2}}^{x} \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+5}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} dh + \int_{\frac{1}{2}}^{x} \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\frac{\left(h-\frac{1}{2}\right)^2}{2!} dh + \int_{\frac{1}{2}}^{x} \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(h-\frac{1}{2}\right)^4}{4!} dh - \int_{\frac{1}{2}}^{x} \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)}(1-h)^{\gamma+5} dh \quad (13)$$

becoming

$$\hat{m}_6(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+5}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)}\left(x-\frac{1}{2}\right) + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\frac{\left(x-\frac{1}{2}\right)^3}{3!} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^5}{5!} + \frac{\beta}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)(\gamma+6)}(1-x)^{\gamma+6} \tag{14}$$

There is a pattern here. For odd i, we obtain, for the domain $\frac{1}{2} \leq x \leq 1$

$$\hat{m}_i(x) = \left\{\frac{2\beta\left(\frac{1}{2}\right)^{\gamma+i}}{(\gamma+1)\dots(\gamma+i)} + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+i-2}}{(\gamma+1)\dots(\gamma+i-2)}\frac{\left(x-\frac{1}{2}\right)^2}{2!} + \dots + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^{i-1}}{(i-1)!}\right\} - \frac{\beta}{(\gamma+1)\dots(\gamma+i)}(1-x)^{\gamma+i} \tag{15}$$

whereas for even i

$$\hat{m}_i(x) = \left\{\frac{2\beta\left(\frac{1}{2}\right)^{\gamma+i-1}}{(\gamma+1)\dots(\gamma+i-1)}\left(x-\frac{1}{2}\right) + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+i-3}}{(\gamma+1)\dots(\gamma+i-3)}\frac{\left(x-\frac{1}{2}\right)^3}{3!} + \dots + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(x-\frac{1}{2}\right)^{i-1}}{(i-1)!}\right\} + \frac{\beta}{(\gamma+1)\dots(\gamma+i)}(1-x)^{\gamma+i} \tag{16}$$

Recall that $x = 1$ corresponds to the end of the emissions cycle in our idealized model, where $\hat{m}(x)$ has returned to zero. In this limit,

$$\hat{m}_1(x) = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1} \tag{17}$$

$$\hat{m}_2(x) = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{1}{2} = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+2} \tag{18}$$

$$\hat{m}_3(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(\frac{1}{2}\right)^2}{2!} = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+3}\left\{\frac{1}{(\gamma+2)(\gamma+3)} + \frac{1}{2!}\right\} \tag{19}$$

$$\hat{m}_4(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\left(\frac{1}{2}\right) + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(\frac{1}{2}\right)^3}{3!} = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+4}\left\{\frac{1}{(\gamma+2)(\gamma+3)} + \frac{1}{3!}\right\} \tag{20}$$

$$\hat{m}_5(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+5}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(\frac{1}{2}\right)^4}{4!} = \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+5}\left\{\frac{1}{(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{1}{2!(\gamma+2)(\gamma+3)} + \frac{1}{4!}\right\}$$

and

$$\hat{m}_6(x) = \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+5}}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)}\left(\frac{1}{2}\right) + \frac{2\beta\left(\frac{1}{2}\right)^{\gamma+3}}{(\gamma+1)(\gamma+2)(\gamma+3)}\frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1}\frac{\left(\frac{1}{2}\right)^5}{5!}$$
$$= \frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+6}\left\{\frac{1}{(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{1}{3!(\gamma+2)(\gamma+3)} + \frac{1}{5!}\right\} \tag{21}$$

Clearly the repeated integrals are decreasing rapidly, with

$$\frac{\hat{m}_{i+1}(x=1)}{\hat{m}_i(x=1)} \leq \frac{1}{2} \tag{22}$$

for example,

$$\frac{\hat{m}_6(x=1)}{\hat{m}_5(x=1)} = \frac{1}{2}\frac{\left\{\frac{1}{(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{1}{3!(\gamma+2)(\gamma+3)} + \frac{1}{5!}\right\}}{\left\{\frac{1}{(\gamma+2)(\gamma+3)(\gamma+4)(\gamma+5)} + \frac{1}{2!(\gamma+2)(\gamma+3)} + \frac{1}{4!}\right\}} < \frac{1}{2} \tag{23}$$

whereas

$$\frac{\hat{m}_2(x=1)}{\hat{m}_1(x=1)} = \frac{1}{2} \tag{24}$$

For $0 \leq x < \frac{1}{2}$, the ratio is

$$\frac{\hat{m}_{i+1}(x)}{\hat{m}_i(x)} = \frac{1}{\gamma+i+1}x < \frac{1}{2} \tag{25}$$

so the decrease in the corresponding series is also rapid.

2 Condition for path independence during increasing phase of emissions

As noted in the paper, the condition for path independence $0 \leq \left|\frac{\tau_M}{\tau_r}\right| \leq \theta \ll 1$ is

$$\left|\left\{1 - \alpha\frac{\hat{m}_1(x)}{\hat{m}(x)} + \alpha^2\frac{\hat{m}_2(x)}{\hat{m}(x)} - \ldots\right\} - \left\{1 - \alpha\frac{\hat{m}_2(x)}{\hat{m}_1(x)} + \alpha^2\frac{\hat{m}_3(x)}{\hat{m}_1(x)} - \ldots\right\}\right| \leq \theta\left|\left\{1 - \alpha\frac{\hat{m}_2(x)}{\hat{m}_1(x)} + \alpha^2\frac{\hat{m}_3(x)}{\hat{m}_1(x)} - \ldots\right\}\right| \tag{26}$$

which we truncate to third order in α. This becomes

$$-\left[\alpha\left\{\frac{\hat{m}_2(x)}{\hat{m}_1(x)} - \frac{\hat{m}_1(x)}{\hat{m}(x)}\right\} - \alpha^2\left\{\frac{\hat{m}_3(x)}{\hat{m}_1(x)} - \frac{\hat{m}_2(x)}{\hat{m}(x)}\right\} + \alpha^3\left\{\frac{\hat{m}_4(x)}{\hat{m}_1(x)} - \frac{\hat{m}_3(x)}{\hat{m}(x)}\right\}\right] \leq \theta\left\{1 - \alpha\frac{\hat{m}_2(x)}{\hat{m}_1(x)} + \alpha^2\frac{\hat{m}_3(x)}{\hat{m}_1(x)} - \alpha^3\frac{\hat{m}_4(x)}{\hat{m}_1(x)} + \ldots\right\} \tag{27}$$

Using, for the increasing phase with $0 \leq x < \frac{1}{2}$,

$$\frac{\hat{m}_1(x)}{\hat{m}(x)} = \frac{x}{\gamma+1} \tag{28}$$

$$\frac{\hat{m}_2(x)}{\hat{m}(x)} = \frac{x^2}{(\gamma+1)(\gamma+2)} \tag{29}$$

$$\frac{\hat{m}_3(x)}{\hat{m}(x)} = \frac{x^3}{(\gamma+1)(\gamma+2)(\gamma+3)} \quad (30)$$

and

$$\frac{\hat{m}_2(x)}{\hat{m}_1(x)} = \frac{x}{\gamma+2} \quad (31)$$

$$\frac{\hat{m}_3(x)}{\hat{m}_1(x)} = \frac{x^2}{(\gamma+2)(\gamma+3)} \quad (32)$$

$$\frac{\hat{m}_4(x)}{\hat{m}_1(x)} = \frac{x^3}{(\gamma+2)(\gamma+3)(\gamma+4)} \quad (33)$$

we find

$$\frac{\hat{m}_2(x)}{\hat{m}_1(x)} - \frac{\hat{m}_1(x)}{\hat{m}(x)} = -\frac{x}{(\gamma+1)(\gamma+2)} \quad (34)$$

$$\frac{\hat{m}_3(x)}{\hat{m}_1(x)} - \frac{\hat{m}_2(x)}{\hat{m}(x)} = -\frac{2x^2}{(\gamma+1)(\gamma+2)(\gamma+3)} \quad (35)$$

$$\frac{\hat{m}_4(x)}{\hat{m}_1(x)} - \frac{\hat{m}_3(x)}{\hat{m}(x)} = -\frac{3x^3}{(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} \quad (36)$$

Substituting these expressions into the main expression in Eq. (27) of this SI, and using $y = \alpha x$ as defined in the paper, we obtain the cubic polynomial in y

$$\frac{\frac{3}{\gamma+1}+\theta}{(\gamma+2)(\gamma+3)(\gamma+4)}y^3 - \frac{\frac{2}{\gamma+1}+\theta}{(\gamma+2)(\gamma+3)}y^2 + \frac{\frac{1}{\gamma+1}+\theta}{(\gamma+2)} - \theta \leq 0 \quad (37)$$

which we write as

$$g(y) = \{3+\theta(\gamma+1)\}y^3 - \{2+\theta(\gamma+1)\}(\gamma+4)y^2 + \{1+\theta(\gamma+1)\}(\gamma+3)(\gamma+4)y - \theta(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4) \leq 0 \quad (38)$$

In the limit of small θ, so that $\theta(\gamma+1) \ll 1$, we obtain the approximate cubic polynomial

$$\widetilde{g}(y) = 3y^3 - 2(\gamma+4)y^2 + (\gamma+3)(\gamma+4)y - \theta(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4) \leq 0 \quad (39)$$

3 The cubic polynomial has a single root

Zero slope of the cubic polynomial in Eq. (38), which we write as $g(y) = a_3y^3 + a_2y^2 + a_1y + a_0$, corresponds to vanishing of its 1st-derivative

$$g'(y) = 3a_3y^2 + 2a_2y + a_1 \quad (40)$$

at real roots of the quadratic Eq. (40), requiring the latter's discriminant

$$4a_2^2 - 12a_3a_1 = -4(\gamma+4)(2\theta^2\gamma^3 + 9\theta^2\gamma^2 + 12\theta^2\gamma + 5\theta^2 + 8\theta\gamma^2 + 28\theta\gamma + 20\theta + 5\gamma + 11) \quad (41)$$

to be positive. This is an impossibility since θ and γ are both positive, and hence $g'(y) > 0$ implying that $g(y)$ is an increasing function. It has a single real-valued root.

4 Cumulative emissions timescale during the decreasing phase of emissions

During the decreasing phase of our idealized emissions cycle, for $\frac{1}{2} \leq x \leq 1$,the cumulative emissions timescale is, from Eq. (25) of the paper, and applying Eq. (4) of this SI

$$\tau_M(x) = T\frac{\frac{2\beta}{\gamma+1}\left(\frac{1}{2}\right)^{\gamma+1} - \frac{\beta}{\gamma+1}(1-x)^{\gamma+1}}{\beta(1-x)^{\gamma}} \quad (42)$$

which simplifies to

$$\tau_M(x) = \frac{T}{\gamma+1}\left\{\left(\frac{1}{2(1-x)}\right)^{\gamma} - (1-x)\right\} \tag{43}$$

which is increasing in x. Hence the cumulative emissions timescale grows as the emissions cycle proceeds. Furthermore, $\tau_M(x)$ is shorter for short T or large γ, as during the increasing phase of emissions.