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A generalized inverse eigenvalue problem and m -functions[☆]

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ABSTRACT

In this manuscript, a generalized inverse eigenvalue problem is considered that involves a linear pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ of $(n + 1)$ by $(n + 1)$ matrices arising in the theory of rational interpolation and biorthogonal rational functions. In addition to the reconstruction of the Hermitian matrix $\mathcal{H}_{[0,n]}$, characterizations of the rational functions that are components of the prescribed eigenvectors are given. A condition concerning the positive-definiteness of $\mathcal{J}_{[0,n]}$ which is often an assumption in the direct problem is also isolated. Further, the reconstruction of $\mathcal{H}_{[0,n]}$ is viewed through the inverse of the pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ which involves the concept of m -functions.

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1. Introduction

A generalized inverse eigenvalue problem (GIEP) concerns the reconstruction of matrices from a given set of spectral data. The spectral data may be completely or only partially specified in terms of eigenvalues and eigenvectors. Precisely, a GIEP for a pair

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$(\mathcal{H}, \mathcal{J})$ of square matrices involves the generalized eigenvalue equation $\mathcal{H}\Phi = z\mathcal{J}\Phi$. With the prescribed spectral data, the solution to the problem consists in the reconstruction of the matrices $\mathcal{H}$ and/or $\mathcal{J}$ [11,15].

In general, it is often necessary both from the point of view of practical applications and of mathematical interest that the matrices involved have a specified structure [7]. This introduces a structural constraint on the solution in addition to the spectral constraint. Thus, one may require that both the matrices $\mathcal{H}$ and $\mathcal{J}$ or one of them to be, for instance, banded or Hermitian or Hamiltonian [12,16] and so on.

The inverse problem under consideration concerns the generalized eigenvalue equation arising from the continued fraction

$$\frac{1}{u_0(z) - \frac{v_0^L(z)v_0^R(z)}{u_1(z) - \frac{v_1^L(z)v_2^R(z)}{u_2(z) - \ddots}}}, \quad (1.1)$$

where $u_j(z)$, $v_j^L(z)$ and $v_j^R(z)$ are non-vanishing polynomials of degree at most one. Such continued fractions appear in connection to multi-point Padé approximation [3, Chapter 7]. This link is also explicitly explained in [5] (see Definition 1.1 and the continued fractions (1.3) and (2.1)).

If we terminate the continued fraction (1.1) at $u_n(z)$, then it is a rational function denoted by $\mathcal{S}_{n+1}(z) = \mathcal{Q}_{n+1}(z)/\mathcal{P}_{n+1}(z)$ where, the polynomials $\mathcal{Q}_n(z)$ of degree $\leq n-1$ and $\mathcal{P}_n(z)$ of degree $\leq n$ satisfy the three term recurrence relation [14]

$$\mathcal{X}_{n+1}(z) = u_n(z)\mathcal{X}_n(z) - v_{n-1}^L(z)v_{n-1}^R(z)\mathcal{X}_{n-1}(z), \quad n = 0, 1, 2, \dots, \quad (1.2)$$

with the initial conditions (setting $v_{-1}^L(z) = v_{-1}^R(z) = 1$)

$$\mathcal{Q}_{-1}(z) = -1, \quad \mathcal{Q}_0(z) = 0, \quad \mathcal{P}_{-1}(z) = 0 \quad \text{and} \quad \mathcal{P}_0(z) = 1. \quad (1.3)$$

The linear pencil that is associated with $\mathcal{S}_{n+1}(z)$ and (1.2) is $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ where the matrices $\mathcal{H}_{[0,n]}$ and $\mathcal{J}_{[0,n]}$ are tridiagonal.

In the present manuscript, we consider the matrices

$$\mathcal{H}_{[0,n]} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 & 0 \\ \bar{b}_0 & a_1 & b_1 & \cdots & 0 & 0 \\ 0 & \bar{b}_1 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & \bar{b}_{n-1} & a_n \end{pmatrix},$$

$$\mathcal{J}_{[0,n]} = \begin{pmatrix} c_0 & d_0 & 0 & \cdots & 0 & 0 \\ d_0 & c_1 & d_1 & \cdots & 0 & 0 \\ 0 & d_1 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-1} & d_{n-1} \\ 0 & 0 & 0 & \cdots & d_{n-1} & c_n \end{pmatrix}, \quad d_j \neq 0, \quad 0 \leq j \leq n-1,$$

and the following generalized inverse eigenvalue problem.

GIEP 1. *Given: the symmetric matrix $\mathcal{J}_{[0,n]}$, the hermitian matrix $\mathcal{H}_{[0,k]}$, real numbers λ and μ , and vectors $p_{[k,n]}^R = (p_k^R, p_{k+1}^R, \dots, p_n^R)^T$ and $s_{[k,n]}^R = (s_k^R, s_{k+1}^R, \dots, s_n^R)^T$, where for fixed k , $1 \leq k \leq n-1$. To find:*

- (i) *hermitian matrix $\mathcal{H}_{[0,n]}$ with eigenvalues λ and μ such that $\mathcal{H}_{[0,k]}$ is the leading principal sub-matrix of $\mathcal{H}_{[0,n]}$,*
- (ii) *vectors $p_{[0,k-1]}^R = (p_0^R, p_1^R, \dots, p_{k-1}^R)^T$ and $s_{[0,k-1]}^R = (s_0^R, s_1^R, \dots, s_{k-1}^R)^T$ such that*

$$p_{[0,n]}^R = \begin{pmatrix} p_{[0,k-1]}^R \\ p_{[k,n]}^R \end{pmatrix} \quad \text{and} \quad s_{[0,n]}^R = \begin{pmatrix} s_{[0,k-1]}^R \\ s_{[k,n]}^R \end{pmatrix},$$

are the right eigenvectors of the matrix pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$, corresponding to the eigenvalues λ and μ respectively.

The semi-infinite form of the pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$, denoted as $(z\mathcal{J}_{[0,\infty]} - \mathcal{H}_{[0,\infty]})$, is a linear pencil of tridiagonal matrices arising in the theory of biorthogonal rational functions and rational interpolation [5,17]. A particular case is

$$\mathcal{H}_{[0,\infty]} - z\mathcal{J}_{[0,\infty]} = \begin{pmatrix} a_0 - z & (i-z)b_0 & 0 & \cdots \\ (-i-z)b_0 & a_1 - z & (i-z)b_1 & \cdots \\ 0 & (-i-z)b_2 & a_2 - z & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (1.4)$$

that is, the b 's appearing in $\mathcal{H}_{[0,n]}$ are purely imaginary: $b_j = i\mathfrak{b}_j$, $d_j = \mathfrak{b}_j$ and the c 's appearing in $\mathcal{J}_{[0,n]}$ are unity. The matrix pencil (1.4) has its origins in the continued fraction representation of Nevanlinna functions, which in turn are obtained via the Cayley transformation of the continued fraction representation of a Carathéodory function [14] (see also [8]). As further specific illustrations, the rational functions arising as components of eigenvectors in such cases have been related to a class of hypergeometric polynomials orthogonal on the unit circle [10] as well as pseudo-Jacobi polynomials (or Routh-Romanovski polynomials) [8].

The motivation to consider GIEP 1 comes partly from an attempt to look at the pencils (1.4) as inverse problems in a general set-up. Precisely, suppose $c_j = 1$ and

$b_j = id_j$ for $j = 0, 1, \dots, k$. If $\mathcal{H}_{[0,k]} - \lambda \mathcal{J}_{[0,k]}$, $k < n$, is of the form (1.4), does this imply $\mathcal{H}_{[0,n]} - \lambda \mathcal{J}_{[0,n]}$ is also of the form (1.4)? In the direct problem, Wall's algorithm [14, Chapter XV] (see also [8]) generates the sequences $\{c_j, b_j, d_j\}$ so that the above question is answered in the affirmative (in fact for $\mathcal{H}_{[0,\infty]} - \lambda \mathcal{J}_{[0,\infty]}$). However, we will see that this is not so when we consider the inverse case. In fact, we will illustrate the dependence of b_j/d_j to be even purely imaginary on the given spectral points λ and μ .

As further contrast to the direct problem, the components of the right eigenvector of the linear pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ are rational functions with poles at $z = b_j/d_j$ while that of the left eigenvector have poles at $z = \bar{b}_j/d_j$. In addition to poles, the entries of the matrices $\mathcal{J}_{[0,n]}$ and $\mathcal{H}_{[0,n]}$ also completely specify the numerators of such rational functions appearing as polynomial solutions of a three term recurrence relation. Thus, a way to interpret the reconstruction of the matrix $\mathcal{H}_{[0,n]}$ is to determine its entries in terms of rational functions with arbitrary poles. These rational functions enter into the problem as components of a prescribed eigenvector, while the structural constraint of the pencil being tridiagonal characterizes these poles. Moreover, while in the direct approach, the poles are at $\pm i$, it will be shown that for GIEP 1 to be solvable, the poles that are prescribed cannot be real numbers.

It is also known that the zeros of these numerator polynomials are the eigenvalues of the linear pencil under consideration [10, Theorem 1.1]. These numerator polynomials are precisely normalized $\mathcal{P}_j(z)$, the denominator of the convergents $\mathcal{S}_j(z)$ of the continued fraction (1.1). Further, it can be verified with the recurrence relation (1.2) and the initial conditions (1.3) that the following expressions

$$\mathcal{P}_{n+1}(z) = \det(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}), \quad \mathcal{Q}_{n+1}(z) = \det(z\mathcal{J}_{[1,n]} - \mathcal{H}_{[1,n]}),$$

hold which lead to the formula

$$\mathcal{S}_{n+1}(z) = \frac{\mathcal{Q}_{n+1}(z)}{\mathcal{P}_{n+1}(z)} = \langle (z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})^{-1} e_0, e_0 \rangle,$$

with the standard inner product

$$\langle x, y \rangle = \sum_{j=0}^{\infty} x_j \bar{y}_j, \quad x = (x_0, x_1, \dots) \in \ell^2, \quad y = (y_0, y_1, \dots) \in \ell^2,$$

on the space ℓ^2 of complex square summable sequences.

A fundamental object related to a pair $(\mathcal{H}, \mathcal{J})$ of self-adjoint matrices is the function

$$\mathfrak{m}(z) = \langle (z\mathcal{J} - \mathcal{H})^{-1} e_0, e_0 \rangle, \quad z \in \rho(\mathcal{H}, \mathcal{J})$$

called the m -function or the Weyl function of the linear pencil $(z\mathcal{J} - \mathcal{H})$ [5] (see also [2,4]). Here $\sigma(\mathcal{H}, \mathcal{J})$ and $\rho(\mathcal{H}, \mathcal{J}) := \mathbb{C} \setminus \sigma(\mathcal{H}, \mathcal{J})$ are, respectively, the spectrum and the resolvent set of the pencil $(z\mathcal{J} - \mathcal{H})$. We can denote similarly the m -function

$$\mathbf{m}(z, j+1) = \frac{\mathcal{Q}_{j+1}(z)}{\mathcal{P}_{j+1}(z)} = \langle (z\mathcal{J}_{[0,j]} - \mathcal{H}_{[0,j]})^{-1}e_0, e_0 \rangle, \quad z \in \rho(\mathcal{H}_{[0,j]}, \mathcal{J}_{[0,j]}), \quad (1.5)$$

of the finite pencil $(z\mathcal{J}_{[0,j]} - \mathcal{H}_{[0,j]})$.

For the general theory of m -functions arising in the context of orthogonal polynomials, we refer to [13] and for that in the context of linear pencil, we refer to [5]. In particular, the m -functions have been used in inverse spectral analysis (reconstruction) of Jacobi matrices of the form $\mathcal{J}_{[0,n]}$, see [9] and references therein. The scheme in such reconstruction is to define a recursive algorithm with an underlying continued fraction structure to generate a class of m -functions. The asymptotic expansion of such m -functions at infinity is then used to obtain the entries of $\mathcal{J}_{[0,\infty]}$. Motivated by the above reconstruction of Jacobi matrices, we look at a use of m -functions in reconstruction of the Hermitian matrix $\mathcal{H}_{[0,n]}$, again keeping in mind the form (1.4). We will further elaborate on this in Section 4.

Our primary goal in this manuscript is to find a representation of the entries b'_j s of the matrix $\mathcal{H}_{[0,n]}$ in terms of given spectral points and corresponding eigenvectors. We find characterizations of both the given poles and the entries b'_j s which appear in special matrix pencil (1.4). A condition concerning the positive-definiteness of $\mathcal{J}_{[0,n]}$, which is often an assumption in the direct problem, is also isolated. Further, we have a view at the entries b'_j s through the $\mathbf{m}$ -functions (1.5) which, as is obvious, involve a point in the resolvent set and not in the spectrum of the pair $(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]})$.

The manuscript is organized as follows. Section 2 includes preliminary results that illustrate the key role played by the entry b_k in the inverse approach to the linear pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$. The matrix $\mathcal{H}_{[0,n]}$ is reconstructed in Section 3 thereby solving GIEP 1. In Section 4 we have a further look at the problem through m -functions that involves computing the inverse of the matrix $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$.

2. Preliminary results

In this section, we derive some results that will help in solving the GIEP. Though the entries are yet to be determined, we use them as symbols in the computation, with the final result depending only on b_k and given components of the eigenvector. In a way, these results exhibit the role played by the specific entry b_k in the solution.

First of all, it can be seen that if $\mathcal{H}_{[0,n]}$ were completely specified, the leading minors $\mathcal{P}_m(z)$ of $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ satisfy the three term recurrence relation

$$\mathcal{X}_{m+1}(z) = (zc_m - a_m)\mathcal{X}_m(z) - (zd_{m-1} - b_{m-1})(zd_{m-1} - \bar{b}_{m-1})\mathcal{X}_{m-1}(z), \quad (2.1)$$

for $0 \leq m \leq n$, where we define $\mathcal{P}_{-1}(z) := 0$ and $\mathcal{P}_0(z) := 1$. (Since (2.1) is a particular form of (1.2), we will set $d_{-1} = 0$ and $b_{-1} = -1$.) If κ_m is the leading coefficient of $\mathcal{P}_m(z)$, then from (2.1), we have $\kappa_{m+1} = c_m\kappa_m - d_{m-1}^2\kappa_{m-1}$, with $\kappa_0 = 1$ and $\kappa_1 = c_0$. Hence, we assume that

$$\frac{\kappa_m}{\kappa_{m-1}} \neq \frac{d_{m-1}^2}{c_m}, \quad m \geq 1, \quad (2.2)$$

so that $\mathcal{P}_{m+1}(z)$ is a polynomial of degree $m+1$. Further, $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})p_{[0,n]}^R = 0$ yields the following relations

$$(zd_{m-1} - \bar{b}_{m-1})p_{m-1}^R(z) + (zc_m - a_m)p_m^R(z) + (zd_m - b_m)p_{m+1}^R(z) = 0, \quad (2.3)$$

for $m = 0, 1, \dots, n$, where $p_{n+1}^R(z) = 0$ and we define $p_{-1}^R(z) := 0$. The former equality occurs if $z \in \sigma(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]})$. Moreover, with $p_0^R(z)$ a non-vanishing function to be specified, the components of the eigenvector $p_{[0,n]}^R$ can be obtained from (2.3), for instance by induction, in the form of the rational functions

$$p_m^R(z) = \frac{\mathcal{P}_m(z)}{\prod_{j=0}^{m-1} (b_j - zd_j)} p_0^R(z), \quad m = 1, 2, \dots, n-1, \quad (2.4)$$

and $p_n^R(z)$ obtained from $(zc_n - a_n)p_n^R(z) = (\bar{b}_{n-1} - zd_{n-1})p_{n-1}^R(z)$. However, because of the prescribed data, we will assume that the components of the vector $p_{[k,n]}^R$ are given in the form

$$p_m^R(z) = \frac{\mathcal{T}_m(z)}{\prod_{j=0}^{m-1} (\alpha_j - z)}, \quad m = k, k+1, \dots, n, \quad (2.5)$$

where $\mathcal{T}_m(z)$ is a polynomial with leading coefficient ς_m . But, we note that once $\mathcal{H}_{[0,n]}$ is determined completely, because of (2.3), $\mathcal{T}_m(z)$, $m = k, k+1, \dots, n$, will satisfy (2.1) where we put $\mathcal{T}_{k-1}(z) := \mathcal{P}_{k-1}(z)$. We also assume, as in (2.2),

$$\varsigma_k \neq \frac{d_{k-1}^2 \kappa_{k-1}}{c_k} \quad \text{and} \quad \frac{\varsigma_m}{\varsigma_{m-1}} \neq \frac{d_{m-1}^2}{c_m}, \quad m = k+1, \dots, n$$

so that $\mathcal{T}_m(z)$ is a polynomial of degree m . Further, a component of the right eigenvector of $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ must be of the form (2.4). In particular, if we look at $p_k^R(z)$, this would imply that the set $\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$ is necessarily a permutation of the set $\{b_0/d_0, b_1/d_1, \dots, b_{k-1}/d_{k-1}\}$ which is known. The inverse problem thus consists of α_j , $j = k, \dots, n-1$, being arbitrary, on which the determination of b_j , $j = k, \dots, n-1$, depends.

Lemma 2.1. *Given $(\lambda, p_{[0,n]}^R)$ an eigen-pair for $(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]})$, let $\lambda \notin \sigma(\mathcal{H}_{[0,k]}, \mathcal{J}_{[0,k]})$, then the components of $p_{[0,k-1]}^R(z)$ are given by*

$$p_m^R(z) = \frac{(b_k - zd_k) \prod_{j=m}^{k-1} (b_j - zd_j) \mathcal{P}_m(z)}{\mathcal{P}_{k+1}(z)} p_{k+1}^R(z), \quad m = 1, 2, \dots, k-1,$$

at $z = \lambda$ and $p_0^R(\lambda)$ assumed to be a non-vanishing function of λ .

Proof. Since $\lambda \notin \sigma(\mathcal{H}_{[0,k]}, \mathcal{J}_{[0,k]})$, $\mathcal{P}_{k+1}(\lambda) = \det(\lambda \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}) \neq 0$. If we use the form of $p_{k+1}^R(\lambda)$ as suggested by (2.4) with b_k unknown at the moment, we will have that $p_{k+1}^R(\lambda) \neq 0$ and

$$p_0^R(\lambda) = \frac{\prod_{j=0}^k (b_j - \lambda d_j) p_{k+1}^R(\lambda)}{\mathcal{P}_{k+1}(\lambda)} = \frac{(b_k - \lambda d_k) p_{k+1}^R(\lambda)}{\mathcal{P}_{k+1}(\lambda)} \prod_{j=0}^{k-1} (b_j - \lambda d_j) \mathcal{P}_0(\lambda).$$

Substituting the above expression of $p_0^R(\lambda)$ in (2.4) will lead to the required forms of the components $p_m^R(\lambda)$, $m = 1, 2, \dots, k-1$. $\square$

Lemma 2.1 shows that once b_k is computed, the vector $p_{[0,k-1]}^R$ can be uniquely obtained in terms of $p_{k+1}^R(\lambda)$, which is now known in the form given by (2.5). This would determine $p_{[0,k-1]}^R$ at $z = \lambda$ completely.

To proceed further, we will make use of rational functions of the form

$$p_m^L(z) = \frac{\mathcal{P}_m(z)}{\prod_{j=0}^{m-1} (\bar{b}_j - z d_j)} p_0^L(z), \quad m = 1, 2, \dots, n-1, \quad (2.6)$$

and $p_n^L(z)$ obtained from the equation $(z c_n - a_n) p_n^L(z) = (b_{n-1} - z d_{n-1}) p_{n-1}^L(z)$. These arise as components of the left eigenvector $p_{[0,n]}^L$ of $(z \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ corresponding to the eigenvalue $z = \lambda$ and owing to the underlying hermitian character of the problem, satisfy $p_j^L(\lambda) = \bar{p}_j^R(\lambda)$, $0 \leq j \leq n$. Again, we reiterate that $p_j^L(z)$ is specified in the form (2.6) only for $j = 0, 1, \dots, k-1$, while $p_m^L(z)$ is obtained in the form suggested by (2.5) for $m = k, k+1, \dots, n$.

Similarly, we define the rational functions $s_m^L(\mu)$ and $s_m^R(\mu)$ corresponding to the eigenvalue μ . Lemma 2.1 with λ replaced by μ and the assumption $\mu \notin \sigma(\mathcal{H}_{[0,k]}, \mathcal{J}_{[0,k]})$ gives the corresponding expressions for $s_j^R(z)$ at $z = \mu$ for $j = 0, 1, \dots, k-1$.

For ease of notations, we use $p_j^R := p_j^R(\lambda)$ and similarly for others. Let us also denote by $\mathcal{J}_{[k+1,n]}$ and $\mathcal{H}_{[k+1,n]}$, as is clear from the notations, the trailing matrices obtained by removing the first $k+1$ rows and columns from $\mathcal{J}_{[0,n]}$ and $\mathcal{H}_{[0,n]}$ respectively.

Lemma 2.2. Suppose $\lambda, \mu \notin \sigma(\mathcal{H}_{[0,i]}, \mathcal{J}_{[0,i]})$ for $i = k-1, k$. Then, the following identities

$$(\lambda - \mu) p_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} s_{[k+1]}^R = (b_k - \lambda d_k) p_k^L s_{k+1}^R - (\bar{b}_k - \mu d_k) p_{k+1}^L s_k^R, \quad (2.7)$$

$$(\lambda - \mu) s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R = (\bar{b}_k - \mu d_k) s_{k+1}^L p_k^R - (b_k - \lambda d_k) s_k^L p_{k+1}^R, \quad (2.8)$$

hold.

Proof. The relations (2.3) for $m = k + 1 \cdots, n$ can be written as

$$\begin{aligned}
 z \begin{pmatrix} c_{k+1} & d_{k+1} & 0 & \cdots & 0 \\ d_{k+1} & c_{k+2} & d_{k+2} & \cdots & 0 \\ 0 & d_{k+2} & c_{k+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix} \begin{pmatrix} p_{k+1}^R \\ p_{k+2}^R \\ p_{k+3}^R \\ \vdots \\ p_n^R \end{pmatrix} \\
 = \begin{pmatrix} a_{k+1} & b_{k+1} & 0 & \cdots & 0 \\ \bar{b}_{k+1} & a_{k+2} & d_{k+2} & \cdots & 0 \\ 0 & \bar{b}_{k+2} & a_{k+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \begin{pmatrix} p_{k+1}^R \\ p_{k+2}^R \\ p_{k+3}^R \\ \vdots \\ p_n^R \end{pmatrix} + (\bar{b}_k - zd_k) \begin{pmatrix} p_k^R \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
 \end{aligned}$$

or in the compact form

$$\mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R = z \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R - (\bar{b}_k - zd_k) p_k^R \vec{e}_1^T, \quad (2.9)$$

where $\vec{e}_1 = (1, 0, \cdots, 0)^T \in \mathbb{R}^{n-k}$. Post-multiplying (2.9) at $z = \lambda$ by $s_{[k+1,n]}^L$, we obtain

$$\begin{aligned}
 \mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R s_{[k+1,n]}^L \\
 = \lambda \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R s_{[k+1,n]}^L - (\bar{b}_k - \lambda d_k) p_k^R \vec{e}_1^T s_{[k+1,n]}^L,
 \end{aligned} \quad (2.10)$$

while pre-multiplying the conjugate transpose of (2.9) at $z = \mu$ by $p_{[k+1,n]}^R$, we obtain

$$\begin{aligned}
 p_{[k+1,n]}^R s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} \\
 = \mu p_{[k+1,n]}^R s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} - (b_k - \mu d_k) s_k^L p_{[k+1,n]}^R \vec{e}_1^T.
 \end{aligned} \quad (2.11)$$

We proceed with the well-known technique of subtracting traces of the respective sides of (2.10) and (2.11). The left hand side upon subtraction is zero owing to the fact that $\text{Tr}(A.B) = \text{Tr}(B.A)$ for any well-defined matrix product. Consequently, we have

$$\begin{aligned}
 (\lambda - \mu) \text{Tr}[p_{[k+1,n]}^R s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]}] \\
 = (\bar{b}_k - \lambda d_k) p_k^R \text{Tr}[\vec{e}_1^T s_{[k+1,n]}^L] - (b_k - \mu d_k) s_k^L \text{Tr}[p_{[k+1,n]}^R \vec{e}_1^T].
 \end{aligned}$$

The left hand side above is equal to the matrix product $(\lambda - \mu) s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R$, while the right hand side can be simplified to $(\bar{b}_k - \lambda d_k) s_{k+1}^L p_k^R - (b_k - \mu d_k) s_k^L p_{k+1}^R$ which gives (2.7). A similar computation starting from

$$\mathcal{H}_{[0,k]} p_{[0,k]}^R = z \mathcal{J}_{[0,k]} p_{[0,k]}^R - (b_k - zd_k) p_{k+1}^R \vec{e}_{k+1}^T, \quad \vec{e}_{k+1} = (0, 0, \cdots, 1)^T \in \mathbb{R}^{k+1},$$

leads to (2.8). $\square$

The assumptions in Lemma 2.2 are necessary for the right hand sides of (2.7) and (2.8) to be non-vanishing. Later, we will use (2.8) to make an observation regarding the positive-definiteness of $\mathcal{J}_{[0,k]}$. Before that we solve the stated inverse problem GIEP 1.

3. Solution to the GIEP

The given data in GIEP 1 suggest that we write the equation $(\lambda\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})p_{[0,n]}^R = 0$ in the form

$$\begin{pmatrix} \lambda\mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]} & \mathcal{O}_\lambda \\ \mathcal{O}_\lambda^* & \lambda\mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]} \end{pmatrix} \begin{pmatrix} p_{[0,k]}^R \\ p_{[k+1,n]}^R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3.1)$$

where

$$\mathcal{O}_\lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda d_k - b_k & 0 & \cdots & 0 \end{pmatrix}. \quad (3.2)$$

Pre-multiplying (3.1) by $s_{[0,n]}^L$ gives the relation

$$\begin{aligned} s_{[0,k]}^L \mathcal{H}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R - s_{[0,k]}^L \mathcal{O}_\lambda p_{[k+1,n]}^R - s_{[k+1,n]}^L \mathcal{O}_\lambda^* p_{[0,k]}^R \\ = \lambda (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R). \end{aligned} \quad (3.3)$$

Similarly, from $s_{[0,n]}^L (\mu\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = 0$, we obtain

$$\begin{aligned} s_{[0,k]}^L \mathcal{H}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{H}_{[k+1,n]} p_{[k+1,n]}^R - s_{[0,k]}^L \mathcal{O}_\mu p_{[k+1,n]}^R - s_{[k+1,n]}^L \mathcal{O}_\mu^* p_{[0,k]}^R \\ = \mu (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R), \end{aligned}$$

which used with (3.3) to eliminate $\mathcal{H}_{[0,n]}$ and $\mathcal{H}_{[k+1,n]}$ gives

$$\begin{aligned} s_{[0,k]}^L [\mathcal{O}_\mu - \mathcal{O}_\lambda] p_{[k+1,n]}^R + s_{[k+1,n]}^L [\mathcal{O}_\mu^* - \mathcal{O}_\lambda^*] p_{[0,k]}^R \\ = (\lambda - \mu) (s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R). \end{aligned}$$

The left hand side can be further simplified to finally obtain

$$s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R + s_{[k+1,n]}^L \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R + d_k (s_{k+1}^L p_k^R + s_k^L p_{k+1}^R) = 0,$$

which, as $\mu \rightarrow \lambda$, implies

$$d_k \begin{vmatrix} p_k^L & -p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix} = - \left[(p_{[0,k]}^L)^* \mathcal{J}_{[0,k]} p_{[0,k]}^R + (p_{[k+1,n]}^L)^* \mathcal{J}_{[k+1,n]} p_{[k+1,n]}^R \right]. \quad (3.4)$$

The next lemma provides a crucial characterization of the pole α_k , that is, it should not be a real number if the entry b_k is to be determined uniquely.

Lemma 3.1. *Suppose $\alpha_k \notin \mathbb{R}$. Then*

$$\Delta_k := p_{k+1}^L p_k^R \begin{vmatrix} s_k^L & s_k^R \\ s_{k+1}^L & s_{k+1}^R \end{vmatrix} - s_{k+1}^L s_k^R \begin{vmatrix} p_k^L & p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix} \neq 0, \quad (3.5)$$

if $\lambda, \mu \notin \sigma(\mathcal{H}_{[0,i]}, \mathcal{J}_{[0,i]})$ for $i = k-1, k$.

Proof. Using the forms as suggested by (2.5), we first note that

$$\begin{vmatrix} p_k^L & p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix} = -\frac{\mathcal{T}_k(\lambda)\mathcal{T}_{k+1}(\lambda)}{\prod_{j=0}^k |\alpha_j - \lambda|^2} 2i\text{Im}\alpha_k \neq 0,$$

by given assumptions and where $\text{Im}\alpha_k$ is the imaginary part of α_k . Then,

$$\Delta_k = \frac{\mathcal{T}_k(\lambda)\mathcal{T}_k(\mu)\mathcal{T}_{k+1}(\lambda)\mathcal{T}_k(\mu)}{\prod_{j=0}^{k-1} |\alpha_j - \lambda|^2 |\alpha_j - \mu|^2} \left[\frac{1}{|\alpha_k - \lambda|^2 (\overline{\alpha_k} - \mu)} - \frac{1}{|\alpha_k - \mu|^2 (\overline{\alpha_k} - \lambda)} \right] 2i\text{Im}\alpha_k,$$

which simplifies further to give

$$\Delta_k = \frac{\mathcal{T}_k(\lambda)\mathcal{T}_k(\mu)\mathcal{T}_{k+1}(\lambda)\mathcal{T}_k(\mu)}{\prod_{j=0}^k |\alpha_j - \lambda|^2 |\alpha_j - \mu|^2} 2i(\lambda - \mu)\text{Im}\alpha_k.$$

Since $\lambda, \mu \notin \sigma(\mathcal{H}_{[0,i]}, \mathcal{J}_{[0,i]})$, $i = k-1, k$ and $\alpha_k \notin \mathbb{R}$, we have that Δ_k is not zero. $\square$

Now, with Δ_k a non-zero purely imaginary number or equivalently, $i\Delta_k$ a non-vanishing real number, we proceed to show that b_k can be determined uniquely.

Theorem 3.2. *Suppose that the given spectral points $\lambda, \mu \notin \sigma(\mathcal{H}_{[0,i]}, \mathcal{J}_{[0,i]})$ for $i = k-1, k$. If $\alpha_k \notin \mathbb{R}$, then*

$$b_k = (\lambda + \mu)d_k + \frac{d_k}{\Delta_k} \left[\mu s_{k+1}^L s_k^R \begin{vmatrix} p_k^L & p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix} - \lambda p_{k+1}^L p_k^R \begin{vmatrix} s_k^L & s_k^R \\ s_{k+1}^L & s_{k+1}^R \end{vmatrix} \right], \quad (3.6)$$

where Δ_k is given by (3.5).

Proof. For $m \geq k+1$, we have from (2.3)

$$\begin{aligned} (\lambda d_{m-1} - \bar{b}_{m-1})p_{m-1}^R + (\lambda c_m - a_m)p_m^R + (\lambda d_m - b_m)p_{m+1}^R &= 0, \\ (\lambda d_{n-1} - \bar{b}_{n-1})p_{n-1}^R + (\lambda c_n - a_n)p_n^R &= 0, \end{aligned} \quad (3.7)$$

and the corresponding equations for the components of $p_{[0,k]}^L$

$$\begin{aligned} (\lambda d_{m-1} - b_{m-1})p_{m-1}^L + (\lambda c_m - a_m)p_m^L + (\lambda d_m - \bar{b}_m)p_{m+1}^L &= 0, \\ (\lambda d_{n-1} - b_{n-1})p_{n-1}^L + (\lambda c_n - a_n)p_n^L &= 0. \end{aligned} \quad (3.8)$$

Eliminating p_m^R and p_m^L between the first equations of (3.7) and (3.8) we obtain

$$\begin{aligned} (p_{m-1}^L p_m^R b_{m-1} - p_m^L p_{m+1}^R b_m) + (p_{m+1}^L p_m^R \bar{b}_m - p_m^L p_{m-1}^R \bar{b}_{m-1}) \\ = \lambda d_{m-1} (p_{m-1}^L p_m^R - p_m^L p_{m-1}^R) + \lambda d_m (p_{m+1}^L p_m^R - p_m^L p_{m+1}^R), \end{aligned}$$

which on summing respective sides from $m = k + 1$ to $m = n - 1$, gives

$$\begin{aligned} (p_k^L p_{k+1}^R b_k - p_{n-1}^L p_n^R b_{n-1}) + (p_n^L p_{n-1}^R \bar{b}_{n-1} - p_{k+1}^L p_k^R \bar{b}_k) \\ = \lambda d_k (p_k^L p_{k+1}^R - p_{k+1}^L p_k^R) + \lambda d_{n-1} (p_n^L p_{n-1}^R - p_{n-1}^L p_n^R). \end{aligned} \quad (3.9)$$

From the last two relations of (3.7) and (3.8), we get

$$p_{n-1}^L p_n^R b_{n-1} - p_n^L p_{n-1}^R \bar{b}_{n-1} = \lambda d_{n-1} (p_{n-1}^L p_n^R - p_n^L p_{n-1}^R), \quad (3.10)$$

which when added to (3.9) yields

$$p_k^L p_{k+1}^R b_k - p_{k+1}^L p_k^R \bar{b}_k = \lambda d_k (p_k^L p_{k+1}^R - p_{k+1}^L p_k^R). \quad (3.11)$$

A computation similar to the relations (3.7), (3.8) and (3.9) for the eigenpair $(\mu, s_{[0,n]}^R)$ gives

$$s_k^L s_{k+1}^R b_k - s_{k+1}^L s_k^R \bar{b}_k = \mu d_k (s_k^L s_{k+1}^R - s_{k+1}^L s_k^R). \quad (3.12)$$

We solve the system of equations (3.11) and (3.12) for b_k and $\bar{b}_k$. First, the determinant of the system is

$$\begin{aligned} \Delta_k &= p_{k+1}^L p_k^R s_k^L s_{k+1}^R - p_k^L p_{k+1}^R s_{k+1}^L s_k^R = \begin{vmatrix} s_k^L p_{k+1}^L & s_k^R p_{k+1}^R \\ p_k^L s_{k+1}^L & p_k^R s_{k+1}^R \end{vmatrix} \\ &= p_{k+1}^L p_k^R \begin{vmatrix} s_k^L & s_k^R \\ s_{k+1}^L & s_{k+1}^R \end{vmatrix} - s_{k+1}^L s_k^R \begin{vmatrix} p_k^L & p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix}, \end{aligned}$$

which by Lemma 3.1 is non-zero. It is now a matter of computation to get

$$\begin{aligned} \Delta_k b_k &= \lambda d_k p_{k+1}^L p_k^R s_{k+1}^L s_k^R + \mu d_k s_k^L s_{k+1}^R p_{k+1}^L p_k^R \\ &\quad - \lambda d_k p_k^L p_{k+1}^R s_{k+1}^L s_k^R - \mu d_k s_{k+1}^L s_k^R p_{k+1}^L p_k^R, \end{aligned}$$

which can be further simplified to obtain

$$\Delta_k b_k = (\lambda + \mu) d_k \Delta_k + d_k [\lambda p_{k+1}^L p_k^R (s_{k+1}^L s_k^R - s_k^L s_{k+1}^R) - \mu s_{k+1}^L s_k^R (p_{k+1}^L p_k^R - p_k^L p_{k+1}^R)],$$

leading to (3.6) and specifying the entry b_k uniquely. $\square$

A similar computation as in the above proof for $\bar{b}_k$ gives

$$\bar{b}_k = (\lambda + \mu) d_k + \frac{d_k}{\Delta_k} \left[\mu s_k^L s_{k+1}^R \begin{vmatrix} p_k^L & p_k^R \\ p_{k+1}^L & p_{k+1}^R \end{vmatrix} - \lambda p_k^L p_{k+1}^R \begin{vmatrix} s_k^L & s_k^R \\ s_{k+1}^L & s_{k+1}^R \end{vmatrix} \right].$$

Theorem 3.2 finds the unique expression for b_k . Summing (3.9) from $m = j$ to $m = n - 1$ for each $j = k + 1, \dots, n - 1$ yields an expression similar to (3.6) for each b_j , $j = k + 1, \dots, n - 2$. The entry b_{n-1} is found from the system of equations consisting of (3.10) and the equivalent equation in μ . The assumptions are $\alpha_j \notin \mathbb{R}$ and $\lambda, \mu \notin \sigma(\mathcal{H}_{[0,j]}, \mathcal{J}_{[0,j]})$, $j = k + 1, \dots, n - 1$. Thus, with b_j , $j = k, k + 1, \dots, n - 1$ determined, the a'_j s are found using (2.3) as

$$a_i = \begin{cases} \lambda c_i + \frac{(\lambda d_{i-1} - \bar{b}_{i-1}) p_{i-1}^R(\lambda) + (\lambda d_i - b_i) p_{i+1}^R(\lambda)}{p_i^R(\lambda)}, & i = k + 1, k + 2 \dots n - 1; \\ \lambda c_n + \frac{(\lambda d_{n-1} - \bar{b}_{n-1}) p_{n-1}^R(\lambda)}{p_n^R(\lambda)}, & i = n. \end{cases} \quad (3.13)$$

This completes the reconstruction of the matrix $\mathcal{H}_{[0,n]}$.

Remark 3.3. Since λ and μ are zeros of $\mathcal{P}_{n+1}(z)$, the assumptions for the determination of the matrix $\mathcal{H}_{[0,n]}$ require that $\mathcal{P}_j(z)$, $j = k - 1, k, \dots, n$, do not vanish at λ and μ . However, we emphasize that the determination of each entry b_j requires that $\mathcal{P}_{j-1}(z)$ and $\mathcal{P}_j(z)$ do not share a common zero at λ and μ . This condition is often implicit, both in the direct and inverse problems, in the form of the requirement that the zeros of $\mathcal{P}_{j-1}(z)$ and $\mathcal{P}_j(z)$ or equivalently, the eigenvalues of the corresponding pencil matrices satisfy a separation property known as interlacing [7].

Corollary 3.4. For $j = k, k + 1, \dots, n - 1$, b_j is purely imaginary and equals ih_j , where $h_j \in \mathbb{R} \setminus \{0\}$, if and only if

$$\begin{aligned} \left| \begin{vmatrix} p_j^L & p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \right|^2 &= i \frac{2\mu h_j \Delta_j}{\lambda(\lambda - \mu) d_j} \frac{\begin{vmatrix} p_j^L & -p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix}}{\begin{vmatrix} s_j^L & -s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix}}, \\ \left| \begin{vmatrix} s_j^L & s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} \right|^2 &= i \frac{2\lambda h_j \Delta_j}{\mu(\lambda - \mu) d_j} \frac{\begin{vmatrix} s_j^L & -s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix}}{\begin{vmatrix} p_j^L & -p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix}}. \end{aligned} \quad (3.14)$$

Proof. First, let us find the real and imaginary parts of b_j . Since $\alpha_j \notin \mathbb{R}$, we write $b_j = x_j + iy_j$ to obtain from (3.11) and (3.12) the system of equations

$$x_j + i \frac{p_j^L p_{j+1}^R + p_{j+1}^L p_j^R}{p_j^L p_{j+1}^R - p_{j+1}^L p_j^R} y_j = \lambda d_j; \quad x_j + i \frac{s_j^L s_{j+1}^R + s_{j+1}^L s_j^R}{s_j^L s_{j+1}^R - s_{j+1}^L s_j^R} y_j = \mu d_j,$$

which yields

$$\begin{aligned} x_j &= \frac{d_j}{2\Delta_j} \left[\mu \begin{vmatrix} p_j^L & -p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \begin{vmatrix} s_j^L & s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} - \lambda \begin{vmatrix} s_j^L & -s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} \begin{vmatrix} p_j^L & p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \right], \\ y_j &= \frac{(\lambda - \mu)d_j}{2i\Delta_j} \begin{vmatrix} p_j^L & p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \begin{vmatrix} s_j^L & s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix}. \end{aligned} \quad (3.15)$$

Assuming $b_j = ih_j$, the above (which can be noted to be in the form $x_j = A_j X_j - B_j Y_j$ and $y_j = C_j Y_j X_j$) solved further for $x_j = 0$ and $y_j = h_j$, gives the required relations (3.14). Conversely, if the relations (3.14) hold, then (3.11) and (3.12) lead to

$$\left[\frac{\lambda}{\mu} \begin{vmatrix} p_j^L & p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \begin{vmatrix} s_j^L & -s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} - \frac{\mu}{\lambda} \begin{vmatrix} s_j^L & s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} \begin{vmatrix} p_j^L & -p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \right] y_j = 2\Delta_j h_j. \quad (3.16)$$

Eliminating h_j from (3.14) we obtain

$$\frac{\lambda}{\mu} = \frac{\begin{vmatrix} p_j^L & -p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix} \begin{vmatrix} s_j^L & s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix}}{\begin{vmatrix} s_j^L & -s_j^R \\ s_{j+1}^L & s_{j+1}^R \end{vmatrix} \begin{vmatrix} p_j^L & p_j^R \\ p_{j+1}^L & p_{j+1}^R \end{vmatrix}}, \quad j = k, k+1, \dots, n-1, \quad (3.17)$$

which immediately imply $x_j = 0$ in (3.15), and when used in (3.16) along with the expression for Δ_j from (3.5) imply $y_j = h_j$. $\square$

If $\alpha_j \notin \mathbb{R}$ ($\text{Im}\alpha_j \neq 0$), we conclude from the proof of Lemma 3.1 that y_j as given in (3.15) is not equal to zero and so $b_j \notin \mathbb{R}$, $j = k, \dots, n-1$. Hence, if we assume that only the condition (3.17) holds, then from (3.15) we obtain $x_j = 0$ and $b_j = iy_j = i\epsilon_j d_j$, $\epsilon_j \neq 0$, that is b_j/d_j turns out to be purely imaginary for $j = k, \dots, n-1$.

Remark 3.5. The emphasis on b_j/d_j being purely imaginary arises from the particular form (1.4) of the pencil $(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$, where in fact $b_j = id_j$ and $c_j = 1$, $j = 0, 1, \dots, n$. As mentioned in Section 1, such pencils appear in analytic function theory and a case has been made to call such pencils as Wall pencils [8].

Further, as an inverse approach to these pencils, the relation (3.17) shows that the expression on the right hand side of (3.17) must be a constant, (that is, independent of the index j) and equal to the ratio of the given spectral points for b_j to be at least purely imaginary. Hence, if we begin with $b_j = id_j$, $j = 0, \dots, k-1$ and $c_j = 1$, $j = 0, \dots, k$, appropriate conditions can be investigated so that b_j is equal to id_j , $j = k, \dots, n-1$, $n \rightarrow \infty$, and we obtain linear pencil similar in structure to a Wall pencil.

The matrix $\mathcal{J}_{[0,n]}$ in Wall pencils is positive-definite, while no such assumption has been made in the present manuscript. However, since the matrix $\mathcal{H}_{[0,n]}$ has been reconstructed, let us have a look in this direction. Suppose the assumptions of Theorem 3.2 hold. We put $\lambda - \mu = h$ in (2.8) to get

$$s_{[0,k]}^L \mathcal{J}_{[0,k]} p_{[0,k]}^R = \frac{(\bar{b}_k - \mu d_k) s_{k+1}^L p_k^R - (b_k - \lambda d_k) s_k^L p_{k+1}^R}{h}, \quad (3.18)$$

so that as $\lambda \rightarrow \mu$ or $h \rightarrow 0$, we have the left hand side as $(s_{[0,k]}^R)^* \mathcal{J}_{[0,k]} s_{[0,k]}^R$. If $\mu \neq \alpha_j$, $j = 0, 1, \dots, k$, s_{k+1}^R is finite at μ so that by Lemma 2.1, $s_{[0,k]}^R$ is a vector with finite component. By l'Hopital's rule, (3.18) yields

$$\begin{aligned} & (s_{[0,k]}^R)^* \mathcal{J}_{[0,k]} s_{[0,k]}^R \\ &= (\bar{b}_k - \mu d_k) s_{k+1}^R (s_k^R)' - (b_k - \mu d_k) s_k^L (s_{k+1}^R)' + d_k s_k^L s_{k+1}^R, \end{aligned} \quad (3.19)$$

so that if $\mathcal{J}_{[0,k]}$ is a positive-definite matrix, the right hand side above is a positive quantity.

The essence of this observation is the following. For $\Delta_k \neq 0$ to hold, it is necessary that $\alpha_k \notin \mathbb{R}$. Since $\mu \in \mathbb{R}$, $\mu \neq \alpha_k$ follows trivially. Recalling the set $\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$ is just a permutation of $\{b_0/d_0, b_1/d_1, \dots, b_{k-1}/d_{k-1}\}$, it follows that $\mu \neq b_j/d_j$, $j = 0, \dots, k-1$. Hence, for (3.19) to hold as an identity with finite quantities on both sides, no zero of $\mathcal{P}_{n+1}(z) = \det(z\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ should coincide with b_j/d_j , $j = 0, 1, \dots, k-1$.

4. A view with m -functions

In this section we consider the reconstruction of the matrix $\mathcal{H}_{[0,n]}$ using the concept of m -functions. We will need that only b_k has been computed by Theorem 3.2, so that Lemma 2.1 determines the eigenvector $p_{[0,n]}^R$. Of course we might assume that GIEP 1 has been solved completely, but nevertheless we have a look at the entries $b_j^L s$ in terms of $d_j^L s$ and m -functions. Hence, we view this not as a separate generalized inverse eigenvalue problem, but rather as an extension of GIEP 1 when the m -functions $m(z, j+1)$ defined as in (1.5) are known for $j = 0, 1, \dots, n$.

We note that because of the problem under consideration, we will use the representation (1.5) for a point outside the spectrum of the pencil. Further, in addition to $p_m^R(z)$ as defined in (2.4), we will use the rational functions

$$q_0^R(\omega) = 0, \quad q_m^R(\omega) = \frac{\mathcal{Q}_m(\omega)}{\prod_{j=0}^{m-1}(\omega d_j - b_j)}, \quad m = 1, \dots, n,$$

where $\mathcal{Q}_m(\omega)$ satisfy (2.1) with initial conditions (1.3). However, we will see that the rational functions $q_m^R(\omega)$ are intermediary and their only use will be to obtain m -functions using (1.5). Hence at present we may consider the b_j 's appearing in the denominator of $q_m^R(\omega)$ as symbols whose explicit forms are yet to be determined.

A key role will be played by the following relation

$$\mathcal{P}_{m+1}(\omega)\mathcal{Q}_m(\omega) - \mathcal{P}_m(\omega)\mathcal{Q}_{m+1}(\omega) = \prod_{j=0}^{m-1}(\omega d_j - b_j)(\omega d_j - \bar{b}_j), \quad (4.1)$$

called the Liouville-Ostrogradsky formula and which follows by induction from the recurrence relation (2.1) along with the initial conditions (1.3). Using (4.1), the matrix representation of the bounded operator $(\omega\mathcal{J} - \mathcal{H})^{-1}$, $\omega \in \rho(\mathcal{H}, \mathcal{J})$ has been found in terms of $p_m^{R(L)}(\omega)$ and $q_m^{R(L)}(\omega)$ [5, Theorem 2.3]. The inverse of operator pencils has been studied, for instance, in [1], but we follow [5] to obtain the inverse of the pencil $\omega\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}$.

Lemma 4.1. Define $m_j := \mathbf{m}(\omega, j) - \mathbf{m}(\omega, n+1)$. Then $\mathcal{R}_{[0,n]}(\omega) = (\omega\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})^{-1}$, $\omega \in \rho(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]})$ is given by

$$\mathcal{R}_{[0,n]}(\omega) = \begin{pmatrix} p_0^L m_0 p_0^R & p_1^L m_0 p_0^R & p_2^L m_0 p_0^R & \cdots & p_{n-1}^L m_0 p_0^R & p_n^L m_0 p_0^R \\ p_0^L m_0 p_1^R & p_1^L m_1 p_1^R & p_2^L m_1 p_1^R & \cdots & p_{n-1}^L m_1 p_1^R & p_n^L m_1 p_1^R \\ p_0^L m_0 p_2^R & p_1^L m_1 p_2^R & p_2^L m_2 p_2^R & \cdots & p_{n-1}^L m_2 p_2^R & p_n^L m_2 p_2^R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_0^L m_0 p_{n-1}^R & p_1^L m_1 p_{n-1}^R & p_2^L m_2 p_{n-1}^R & \cdots & p_{n-1}^L m_{n-1} p_{n-1}^R & p_n^L m_{n-1} p_{n-1}^R \\ p_0^L m_0 p_n^R & p_1^L m_1 p_n^R & p_2^L m_2 p_n^R & \cdots & p_{n-1}^L m_{n-1} p_n^R & p_n^L m_n p_n^R \end{pmatrix}.$$

Proof. Consider the $1 \times (n+1)$ vector $p_{[0,j]}^L := (p_0^L \ p_1^L \ \cdots \ p_j^L \ 0 \ \cdots \ 0)$ and similarly the vector $q_{[0,j]}^L$. Using (2.3) for the left eigenvector we obtain

$$\begin{aligned} p_{[0,j]}^L(\omega\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) &= -(\omega d_j - \bar{b}_j)p_{j+1}^L \bar{e}_j^T + (\omega d_j - b_j)p_j^L \bar{e}_{j+1}^T, \\ q_{[0,j]}^L(\omega\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) &= \bar{e}_0^T - (\omega d_j - \bar{b}_j)q_{j+1}^L \bar{e}_j^T + (\omega d_j - b_j)q_j^L \bar{e}_{j+1}^T, \end{aligned}$$

which in view of (4.1) leads to

$$[q_j^R p_{[0,j]}^L - p_j^R q_{[0,j]}^L](\omega\mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = \bar{e}_j^T - p_j^R \bar{e}_0^T. \quad (4.2)$$

The vectors $p_{[0,n]}^L$ and $q_{[0,n]}^L$ with the above computation yield

$$[q_{n+1}^R p_{[0,n]}^L - p_{n+1}^R q_{[0,n]}^L](\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = -p_{n+1}^R \bar{e}_0^T,$$

which in terms of m -functions can also be written as

$$[q_{[0,n]}^L - \mathbf{m}(\omega, n+1)p_{[0,n]}^L](\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = \bar{e}_0^T. \quad (4.3)$$

Eliminating $\bar{e}_0^T$ between (4.2) and (4.3), we obtain

$$p_j^R [q_{[0,n]}^L - \mathbf{m}(\omega, n+1)p_{[0,n]}^L - q_{[0,j]}^L + \mathbf{m}(\omega, j)p_{[0,j]}^L](\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}) = \bar{e}_j^T,$$

which upon further simplification gives the matrix $R_{[0,n]}(\omega)$. $\square$

In compact form, the $(i, j)^{th}$ entry of $\mathcal{R}_{[0,n]}(\omega)$ is given by $p_j^L m_{\min(i,j)} p_i^R$. Next, for $\omega \in \rho(\mathcal{H}_{[0,n]}, \mathcal{J}_{[0,n]})$, let us factorize

$$\mathcal{R}_{[0,n]}(\omega) = (\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})^{-1} = \mathcal{L}_{[0,n]}(\omega) \mathcal{D}_{[0,n]}(\omega) \mathcal{U}_{[0,n]}(\omega). \quad (4.4)$$

Then, owing to the Hermitian nature of the eigenvalue equation involved, we choose

$$\mathcal{U}_{[0,n]}(\omega) = \mathcal{L}_{[0,n]}^*(\omega), \quad \text{where} \quad \mathcal{L}_{[0,n]}(\omega) = \begin{pmatrix} p_0^R & 0 & 0 & \cdots & 0 & 0 \\ p_1^R & p_1^R & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_{n-1}^R & p_{n-1}^R & p_{n-1}^R & \cdots & p_{n-1}^R & 0 \\ p_n^R & p_n^R & p_n^R & \cdots & p_n^R & p_n^R \end{pmatrix},$$

and $\mathcal{D}_{[0,n]}(\omega)$ is the diagonal matrix $\text{diag}(d_0(\omega), d_1(\omega), \dots, d_n(\omega))$ where

$$d_0(\omega) = m_0, \quad d_j(\omega) = (m_j - m_{j-1}) = \mathbf{m}(\omega, j) - \mathbf{m}(\omega, j-1) \quad j = 1, 2, \dots, n.$$

As a matter of verification, with $m_{-1} := 0$, we have in the right hand side of (4.4)

$$i^{th} \text{ row} \times j^{th} \text{ column} = p_i^R \sum_{k=0}^{\min(i,j)} (m_k - m_{k-1}) p_j^L = p_j^L m_{\min(i,j)} p_i^R.$$

With this decomposition we can easily invert $\mathcal{R}_{[0,n]}(\omega)$ again, so that $[\mathcal{R}_{[0,n]}(\omega)]^{-1} = (\omega \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]})$ will be a matrix in which the entries are given in terms of the m -functions.

For the remaining results, it will be useful to look at the following block matrix representation of $[w \mathcal{J}_{[0,n]} - \mathcal{H}_{[0,n]}]^{-1}$

$$\begin{pmatrix} w \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]} & \mathcal{O}_\omega \\ \mathcal{O}_\omega^* & w \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]} \end{pmatrix}^{-1} = \begin{pmatrix} \Psi_{[0,k]}(\omega) & * \\ * & \Psi_{[k+1,n]}(\omega) \end{pmatrix}, \quad (4.5)$$

where $\mathcal{O}_\omega$ is given by (3.2).

Lemma 4.2. Suppose $\mathbf{m}(\omega, k+1) \neq \mathbf{m}(\omega, n+1)$ and $\mathbf{m}(\omega, i) \neq \mathbf{m}(\omega, i-1)$, $i = k+2, \dots, n$. Let $[\Psi_{[k+1,n]}(\omega)]^{-1}$ denote the inverse of the block $\Psi_{[k+1,n]}(\omega)$. The entries of $[\Psi_{[k+1,n]}(\omega)]^{-1}$ are given by

$$[[\Psi_{[k+1,n]}(\omega)]^{-1}]_{i,j} = \begin{cases} \frac{1}{p_i^L[\mathbf{m}(\omega, i) - \mathbf{m}(\omega, i-1)]p_i^R} + \frac{1}{p_j^L[\mathbf{m}(\omega, j+1) - \mathbf{m}(\omega, j)]p_j^R}, & i=j; \\ \frac{-1}{p_i^L[\mathbf{m}(\omega, j) - \mathbf{m}(\omega, i)]p_j^R}, & i < j, \\ 0, & |i-j| > 1, \end{cases} \quad (4.6)$$

for $i, j = k+2, k+3, \dots, n-1$, while

$$[[\Psi_{[k+1,n]}(\omega)]^{-1}]_{i,j} = \begin{cases} \frac{1}{p_i^L[\mathbf{m}(\omega, i) - \mathbf{m}(\omega, n+1)]p_i^R} + \frac{1}{p_j^L[\mathbf{m}(\omega, j+1) - \mathbf{m}(\omega, j)]p_j^R}, & i=j=k+1; \\ \frac{1}{p_i^L[\mathbf{m}(\omega, i) - \mathbf{m}(\omega, i-1)]p_i^R}, & i=j=n. \end{cases} \quad (4.7)$$

Proof. We note from (4.5) that the block $\Psi_{[k+1,n]}(\omega)$ is obtained by removing the first $k+1$ rows and columns of $\mathcal{R}_{[0,n]}(\omega)$. Hence, by Lemma 4.1, we have

$$\Psi_{[k+1,n]}(\omega) = \begin{pmatrix} p_{k+1}^L m_{k+1} p_{k+1}^R & p_{k+2}^L m_{k+1} p_{k+1}^R & \cdots & p_n^L m_{k+1} p_{k+1}^R \\ p_{k+1}^L m_{k+1} p_{k+2}^R & p_{k+2}^L m_{k+2} p_{k+2}^R & \cdots & p_n^L m_{k+2} p_{k+1}^R \\ \ddots & \ddots & \ddots & \ddots \\ p_{k+1}^L m_{k+1} p_n^R & p_{k+2}^L m_{k+2} p_n^R & \cdots & p_n^L m_n p_n^R \end{pmatrix}.$$

The decomposition $\Psi_{[k+1,n]}(\omega)(\omega) = \mathcal{L}_{[k+1,n]}(\omega)\mathcal{D}_{[k+1,n]}(\omega)\mathcal{U}_{[k+1,n]}(\omega)$ yields the choice

$$\mathcal{U}_{[k+1,n]}(\omega) = \mathcal{L}_{[k+1,n]}^*(\omega) \quad \text{with}$$

$$\mathcal{L}_{[k+1,n]}(\omega) = \begin{pmatrix} p_{k+1}^R & 0 & 0 & \cdots & 0 & 0 \\ p_{k+2}^R & p_{k+2}^R & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_{n-1}^R & p_{n-1}^R & p_{n-1}^R & \cdots & p_{n-1}^R & 0 \\ p_n^R & p_n^R & p_n^R & \cdots & p_n^R & p_n^R \end{pmatrix},$$

and $\mathcal{D}_{[k+1,n]} = \text{diag}\{d_{k+1}, d_{k+2}, \dots, d_n\}$ given by

$$d_{k+1}(\omega) = m_{k+1} = \mathbf{m}(\omega, k+1) - \mathbf{m}(\omega, n+1),$$

$$d_j(\omega) = (m_j - m_{j-1}) = \mathbf{m}(\omega, j) - \mathbf{m}(\omega, j-1), \quad j = k+2, k+3, \dots, n.$$

It can be easily verified that

$$[\mathcal{L}_{[k+1,n]}(\omega)]^{-1} = \begin{pmatrix} 1/p_{k+1}^R & 0 & 0 & \cdots & 0 & 0 \\ -1/p_{k+1}^R & 1/p_{k+2}^R & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/p_{n-1}^R & 0 \\ 0 & 0 & 0 & \cdots & -1/p_{n-1}^R & 1/p_n^R \end{pmatrix},$$

and $[\mathcal{U}_{[k+1,n]}(\omega)]^{-1} = [\mathcal{L}_{[k+1,n]}^*(\omega)]^{-1}$. Then,

$$[\Psi_{[k+1,n]}(\omega)]^{-1} = [\mathcal{U}_{[k+1,n]}(\omega)]^{-1} [\mathcal{D}_{[k+1,n]}(\omega)]^{-1} [\mathcal{L}_{[k+1,n]}(\omega)]^{-1},$$

gives the required entries (4.6) and (4.7). $\square$

We are now ready to view the entries of $\mathcal{H}_{[0,n]}$ in terms of m -functions. Let us denote

$$\mathfrak{m}_i^j(\omega) = \frac{1}{p_i^L[\mathfrak{m}(\omega, j) - \mathfrak{m}(\omega, i)]p_i^R} \quad \text{and} \quad \widehat{\mathfrak{m}}_i^j(\omega) = \frac{1}{p_i^L[\mathfrak{m}(\omega, j) - \mathfrak{m}(\omega, i)]p_j^R}.$$

Theorem 4.3. Suppose the m -functions $\mathfrak{m}(\omega, j)$ of the pencil $(\omega \mathcal{J}_{[0,j]} - \mathcal{H}_{[0,j]})$ are known and satisfy the assumptions of Lemma 4.2 for $j = k+1, k+2, \dots, n, n+1$. If $\omega \in \rho(\mathcal{H}_{0,k}, \mathcal{J}_{[0,k]})$, then the matrix $\mathcal{H}_{[0,n]}$ can be reconstructed with the entries given by

$$b_j = \omega d_j + \widehat{\mathfrak{m}}_{j+1}^j(\omega), \quad j = k+1, k+2, \dots, n-1, \quad (4.8)$$

and

$$a_j = \begin{cases} \omega c_j - \mathfrak{m}_j^{j+1}(\omega) + \mathfrak{m}_j^{n+1}(\omega) + \frac{|\omega d_k - b_k|^2}{\mathfrak{m}_{j-1}^2(\omega)} & j = k+1; \\ \omega c_j + \mathfrak{m}_j^{j-1}(\omega) - \mathfrak{m}_j^{j+1}(\omega), & j = k+2, \dots, n-1, \\ \omega c_j + \mathfrak{m}_j^{j-1}(\omega), & j = n. \end{cases} \quad (4.9)$$

Proof. We use the following representation of the inverse of a 2×2 block matrix obtained through the concept of Schur's complement [6]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & [D - CA^{-1}B]^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}, \quad (4.10)$$

where I is the identity matrix of appropriate order. Since $\omega \in \rho(\mathcal{H}_{[0,k]}, \mathcal{J}_{[0,k]})$, we can substitute $A = \omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}$, $B = \mathcal{O}_\omega$ given by (3.2) and $D = \omega \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}$. Then comparing the respective blocks of (4.10) and (4.5), we get

$$(\omega \mathcal{J}_{[k+1,n]} - \mathcal{H}_{[k+1,n]}) = [\Psi_{[k+1,n]}(\omega)]^{-1} + \mathcal{O}_\omega^*(\omega \mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]})^{-1} \mathcal{O}_\omega. \quad (4.11)$$

We use Lemma 4.1 to obtain the inverse $\mathcal{R}_{[0,k]}$ of $\omega\mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}$ so that

$$\mathcal{O}_\omega^*[\omega\mathcal{J}_{[0,k]} - \mathcal{H}_{[0,k]}]^{-1}\mathcal{O}_\omega = p_k^L[\mathbf{m}(\omega, k) - \mathbf{m}(\omega, k+1)]p_k^R|\omega d_k - b_k|^2\vec{e}_0\vec{e}_0^T,$$

where $\vec{e}_0 \in \mathbb{R}^{k+1}$. The entries of $[\Psi_{[k+1,n]}]^{-1}$ obtained from (4.6) and (4.7) and used in (4.11) yields the required expressions (4.8) and (4.9). $\square$

As mentioned earlier, the rational functions $q_j^{R(L)}(\omega)$ are only intermediary since the final expressions for the entries depend on the m -functions defined by (1.5). As opposed to (3.13), the a'_j s in (4.9) except for a_{k+1} , are determined independent of b'_j s while for a_{k+1} , in addition to b_k , we need prior information on $\mathbf{m}(\omega, n+1)$. Finally, the entries (4.8) show that obtaining b_j/d_j as purely imaginary is highly non-trivial in the finite dimensional case (that is $j = 0, 1, \dots, n$) if the b'_j s are expressed in terms of m -functions. Hence it would be interesting to see if b_j/d_j does approach a purely imaginary number in the limiting case, that is as j tends to infinity.

Declaration of competing interest

There is no competing interest.

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