the interval $(0,t]$ or, equivalently, the triple (A,B,C) is controllable if, for each $\phi \in C$, the rank of $Q(A,B,C)$ equals n where

$$Q(A,B,C) = [Q_1^1,\cdots,Q_1^n,Q_2^2,\cdots,Q_2^n,\cdots,Q_n^n]$$

and

$$Q_1^1 = C \quad Q_j^{k+1} = AQ_j^k + BQ_{j-1}^k \qquad Q_j^k = 0, \qquad \text{for } j = 0 \text{ or } j > k$$

(i.e., controllable in the sense of Kirillova and Churakova [6]).

The algebraic criterion for complete controllability given in Definition 1 is equivalent to [7]

$$\text{rank} \int_0^t X(t-s)CC'X'(t-s)\,ds = n$$

for $0 < t < \infty$.

Definition 2: System (1) is pointwise complete on the interval $(0,t]$ or, equivalently, the pair (A,B) is pointwise complete if the rank of $R(A,B)$ equals n where

$$R(A,B) = [B, AB,\cdots, A^{n-1}B]$$

(i.e., pointwise complete in the sense of Ono, Yamasaki, and Sawaragi [8]).

The algebraic criterion for pointwise completeness given in Definition 2 is equivalent to [8]

$$\text{rank} \int_{-\tau}^0 X(t-s-\tau)BB'X'(t-s-\tau)\,ds = n$$

for $0 < t < \infty$.

In the following theorem, * denotes the Penrose–Moore pseudoinverse.

Theorem: If (A,C) is controllable and (A,B) is pointwise complete, and the conditions a) $Q \geq 0$, b) $B'Q^*Q$, and c) $Q - 4WB'Q^*BW \geq 0$ are satisfied where

$$Q = \exp(-AT)CC'\exp(-A'T) + CC'$$

and

$$W = \int_0^T \exp(-As)CC'\exp(-A's)\,ds, \qquad T > 0,$$

then (1) is stabilized by the control law

$$u(x(t)) = -C'W^{-1}x(t)$$

where

$$V(x_t) = 2x'(t)W^{-1}x(t) + \int_{t-\tau}^t x'(s)W^{-1}QW^{-1}x(s)\,ds$$

is the Lyapunov functional associated with the stabilized system.

(See [9] or [10] for details regarding Lyapunov functionals and their derivatives.)

Proof: Since (A,C) is controllable, $W > 0$. Let $\hat{A} \triangleq A - CC'W^{-1}$.

A straightforward calculation shows that verifying that $V(x_t)$ is a Lyapunov functional for the closed-loop system

$$\dot{x}(t) = \hat{A}x(t) + Bx(t-\tau) \tag{2}$$

is equivalent to verifying that

$$V_1(x_t) = 2x'(t)Wx(t) + \int_{t-\tau}^t x'(s)Qx(s)\,ds$$

is a Lyapunov functional for

$$\dot{x}(t) = \hat{A}'x(t) + B'x(t-\tau). \tag{3}$$

Differentiating $V_1(x_t)$a long the solutions of (3) yields the quadratic form

$$\dot{V}_1(x_t) = -\begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}' \begin{bmatrix} Q & -2WB' \\ -2BW & Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}. \tag{4}$$

The center matrix in (4) will be positive semidefinite if and only if conditions a), b), and c) are satisfied [11].

In addition, (3) will be asymptotically stable if $\dot{V}_1(x_t) \not\equiv 0$ along any nontrivial solution of (3) [9].

To show asymptotic stability, note that the solution of (3) possesses the representation

$$x(t,0,\phi) = X(t)\phi(0) + \int_{-\tau}^0 X(t-s-\tau)B'\phi(s)\,ds. \tag{5}$$

Substitution of (5) into (4) clearly implies $\dot{V}_1(x_t) \not\equiv 0$ along a nontrivial solution provided

$$X'(t-s)C \not\equiv 0 \tag{6}$$

$$X'(t-s-\tau)B \not\equiv 0. \tag{7}$$

$X'(t-s)$ is the fundamental solution associated with (2).

Assume that (6) is not satisfied. Then the controllability of the triple $(\hat{A},B,C)$ is contradicted since (A,C) controllable $\Rightarrow$ $(\hat{A},C)$ controllable $\Rightarrow$ rank $Q(\hat{A},B,C) = n$.

Assume that (7) is not satisfied. Then the pointwise completeness of $(\hat{A},B)$ is contradicted since (A,B) pointwise complete $\Rightarrow$ rank $R(\hat{A},B) = n$.

Hence the asymptotic stability of (3) is established which, in turn, establishes the asymptotic stability of (2).

This completes the proof.

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Improved Conditions for the L_2-Stability of Nonstationary Feedback Systems

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***Abstract*—Concerning the L_2-stability of feedback systems containing a linear time-varying operator, some of the stringent restrictions imposed on the multiplier as well as the linear part of the system, in the criteria presented earlier, are relaxed.**

The authors' paper [1] presents new criteria for the L_2-stability of linear and nonlinear systems containing a linear time-varying operator G in L_{2e} in an otherwise time-invariant negative feedback loop.

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These criteria permit a new class of linear causal operators M in L_{2e} with nonstationary kernels to be used as multipliers. However, the criteria in [1] require G, M, and M^{-1} to belong to a class ϑ_{EA} of linear causal operators in L_{2e} defined by the following condition: if $H \in \vartheta_{EA}$, then $Hx(t) = \int_0^\infty h(t,\tau)x(\tau)\,d\tau \;\forall\; x(\cdot) \in L_{2e}$, with the kernel satisfying the twin conditions $h(t,\tau) = 0 \;\forall\; \tau > t$ and $\int_0^\infty \int_0^\infty |h(t,\tau)|^2\,dt\,d\tau < \infty$. It has of late been noticed that the latter integral condition renders the class ϑ_{EA} to be of limited use in practical situations, since it excludes even the simple convolutions. The purpose of this correspondence is to remove this stringent requirement from the criteria of [1].

A careful study of the stability criteria of [1] reveals that the condition G, M, and $M^{-1} \in \vartheta_{EA}$ is imposed only to ensure that these are operators in L_2 with finite gain. Let us now define certain new classes ϑ_{EB} and ϑ_{EC} of linear causal operators H in L_{2e} as follows.

1) if $H \in \vartheta_{EB}$,

then

$$Hx(t) = \int_0^\infty h(t,\tau)x(\tau)\,d\tau \;\forall\; x(\cdot) \in L_{2e}, \tag{1}$$

$$h(t,\tau) = 0 \;\forall\; \tau > t, \tag{2}$$

and $|h(t,\tau)| \leqslant K_1 \exp(-K_2(t-\tau))$ for some positive constants K_1 and K_2, (3)

2) if $H \in \vartheta_{EC}$, then H is defined by (1) and (2) as earlier, with the additional condition

$$\sup_t \int_0^\infty |h(t,\tau)|\,d\tau = N_1 < \infty,\; \sup_\tau \int_0^\infty |h(t,\tau)|\,dt = N_2 < \infty. \tag{4}$$

It may be noted that $H \in \vartheta_{EB}$ implies $\|Hx(t)\| \leqslant K_1\|\exp(-K_2 t) \circledast x(t)\|$ (where $\|\cdot\|$ denotes the L_2-norm and $*$ denotes convolution), and hence H is an operator in L_2 with finite gain. Further, $H \in \vartheta_{EC}$ implies (see Dunford and Schwartz [2]) $\|Hx(\cdot)\| \leqslant (N_1N_2)^{\frac{1}{2}} \|x(\cdot)\|$, and hence H is an operator in L_2 with finite gain. With this, it is now simple to realize that the stability criteria [1, theorems 1 and 2] hold in toto with the operators G, M, and M^{-1} permitted to be members of the class $\vartheta_{EA} \cup \vartheta_{EB} \cup \vartheta_{EC}$.

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A Square-Root Data Array Solution of the Continuous-Discrete Filtering Problem

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***Abstract*—The Dyer–McReynolds [1] discrete square-root filtering algorithm is extended to accommodate continuous dynamics. Differential equations are given to represent the time evolution of the filter data array. These equations are nonlinear, but it is shown that the nonlinearities act to enhance the stability of the solution.**

Introduction

In this correspondence we consider the linear dynamic model

$$\frac{d}{dt}x = Fx + w \tag{1}$$

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where $F = F(t)$, and $w = w(t)$ is a white-noise process with

$$E[w(t)] = 0, \quad E[w(t)w^T(\tau)] = Q\delta(t-\tau), \qquad Q = Q(t).$$

The model is observed at discrete times t_j, $t_0 < t_1 < \cdots < t_N \leq T$ by

$$z_j = A_jx_j + \nu_j \tag{2}$$

where z_j is an m_j vector, $x_j = x(t_j)$, and $\{\nu_j\}$ is a zero-mean white-noise sequence that is statistically independent of $w(t)$ and $x(t_0)$.

The problem of interest in this correspondence is that of obtaining filtered estimates of x. By introducing the state transition matrix corresponding to $F(t)$, we can replace (1) by

$$x_{j+1} = \Phi(t_{j+1},t_j)x_j + \omega_{j+1}$$

$$\omega_{j+1} = \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},\sigma)w(\sigma)\,d\sigma \tag{$\hat{1}$}$$

where $\{\omega_j\}$ is a discrete white-noise sequence with covariance $\{\Lambda_\omega(j)\}$,

$$\Lambda_\omega(j+1) = \int_{t_j}^{t_{j+1}} \Phi(t_{j+1},\sigma)Q(\sigma)\Phi^T(t_{j+1},\sigma)\,d\sigma. \tag{3}$$

Evaluation of the Φ and Λ_ω matrices generally adds computational complexity to the filtering problem.

Motivation for conversion from the continuous formulation (1) to the discrete formulation ($\hat{1}$) is that the latter problem may be solved using square-root filtering techniques (cf. [3]). These techniques are generally more accurate than is the discrete Kalman filter. In this correspondence it is shown that the Dyer–McReynolds (D–M) square-root filter can be applied directly to the continuous-discrete system (1) and (2). By writing differential equations for the D–M data array matrix, one circumvents the problem of computing Φ and $\Lambda_\omega^{\frac{1}{2}}$ which are used by the discrete square-root filter.

The Continuous-Discrete Dyer–McReynolds Filter

The D–M filter [1] consists of the data array (R,d) where $R^TR = (\text{cov}\,(x))^{-1}$ and $d = Rx^*$, x^* being the minimum variance estimate of x (time arguments are suppressed for notational simplicity). This filter is most useful for problems that involve state estimates at relatively few times, for example, after a quantity of measurements have been processed. Except at these times, it is possible to deal only with the data array (4) and (5). Each time an estimate is required, it is necessary to solve a matrix equation $Rx = d$, and each time the covariance of the estimate is required, one must compute $P = R^{-1}R^{-T}$.[1] When estimates and/or covariances are required at numerous points, the computational burden becomes excessive. The algorithm follows.

Propagation $t < t \leq t_{j+1}$:

$$\frac{d}{dt}[R \mid d] = [-RF + \Gamma R \mid \Gamma d] \tag{4}$$

where $\Gamma = -\frac{1}{2}RQR^T$.

The differential equation is initialized at times t_j by $[R \mid d]_{t=t_j} = [\hat{R}_j \mid \hat{d}_j]$.

Updating (at time t_{j+1}): Choose an orthonormal transformation T_{j+1} such that

$$T_{j+1}\begin{bmatrix} R(t_{j+1}) & d(t_{j+1}) \\ R_\nu(j+1)A_{j+1} & R_\nu(j+1)z_{j+1} \end{bmatrix} = \begin{bmatrix} \hat{R}_{j+1} & \hat{d}_{j+1} \\ 0 & e_{j+1} \end{bmatrix} \tag{5}$$

where cov $\nu_{j+1} = R_\nu^{-1}(j+1)R_\nu^{-T}(j+1)$ and the matrix on the right is upper triangular. Computation of the triangularization (5) is not discussed here. References [1], [3]–[5] explain in detail how the computation is performed.

[1] $R^{-T} \triangleq (R^{-1})^T$.