TABLE II
TYPICAL HUFFMAN CODES

	Prefix Value	Huffman Code
High compr. image = EWeek		
	0	000100
	1	000101
	2	00011
	3	0011
	4	0010
	5	0000
	6	010
	7	011
	8	1
Medium compr. image = IBMad	Prefix Value	Huffman Code
	0	0110
	1	111
	2	110
	3	010
	4	000
	5	001
	6	101
	7	0111
	8	100
Low compr. image = Aerial	Prefix Value	Huffman Code
	0	011
	1	111
	2	001
	3	10
	4	000
	5	110
	6	0101
	7	01001
	8	01000

TABLE III
PERFORMANCE WITH FIXED HUFFMAN CODES

	PKARC	Indiv	EWeek	Aerial	IBMad
IBMad	13	27	16	24	27
Derin	33	51	50	33	45
EWeek	41	55	55	33	47
Andy	35	52	52	33	46
Marilyn	30	40	36	30	39
Karanne	26	47	46	32	43
Girl	24	29	18	25	28
Couple	27	36	31	28	35
Moon	13	23	6	21	21
City	7	15	−7	15	12
Aerial	7	16	−6	16	14
Hat	7	24	10	22	23

TABLE IV
COMPARISON WITH ENTROPY OF THE IMAGE AND DIFFERENCE IMAGE

		Entropy	Differential Entropy	Deficiency
IBMad	5.84	6.68	5.44	0.4
Derin	3.92	7.47	3.71	0.21
EWeek	3.60	7.39	3.46	0.14
Andy	3.84	7.13	3.91	0.07
Marilyn	4.80	6.27	4.63	0.17
Karanne	4.24	7.15	4.16	0.08
Girl	5.68	6.44	5.04	0.64
Couple	5.12	6.22	4.83	0.29
Moon	6.17	6.71	5.51	0.66
City	6.80	7.31	6.15	0.65
Aerial	6.72	7.31	5.97	0.75
Hat	6.08	7.37	5.27	0.31

TABLE V
MINIMUM AND MAXIMUM CODEWORD LENGTHS FOR HUFFMAN CODES

	Image		Difference Image	
	Largest Codeword Length	Smallest Codeword Length	Largest Codeword Length	Smallest Codeword Length
IBMad	25	5	27	3
Derin	23	4	27	2
EWeek	23	4	27	1
Andy	20	4	27	2
Marilyn	25	4	27	2
Karanne	16	4	27	2
Girl	24	5	25	3
Couple	25	4	26	2
Moon	20	6	25	4
City	21	6	26	4
Aerial	21	7	25	5
Hat	20	4	25	4

REFERENCES

[1] D. A. Huffman, "A method for the construction of minimum-redundancy codes," *Proc. IRE*, vol. 40, pp. 1098–1101, 1952.

[2] J. Risannen and G. G. Langdon, "Universal modeling and coding," *IEEE Trans. Inform. Theory*, vol. 27, no. 1, pp. 12–23, 1981.

[3] D. E. Knuth, "Dynamic Huffman coding," *J. Algorithms*, vol. 6, pp. 163–180, 1985.

[4] J. Ziv and A. Lempel, "Coding theorems for individual sequences via variable-rate coding," *IEEE Trans. Inform. Theory*, vol. 24, no. 5, pp. 530–536, 1978.

[5] T. A. Welch, "A technique for high-performance data compression," *IEEE Comput. Mag.*, vol. 17, no. 6, pp. 8–19, 1984.

[6] D. L. Neuhoff and N. Moayeri, "Tree searched vector quantization with interblock noiseless coding," in *Proc. Conf. Inform. Sci. Syst.* (Princeton, NJ), 1988, pp. 781–783.

[7] R. W. Hamming, *Coding and Information Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1986.

On Two-Dimensional Maximum Entropy Spectral Estimation

N. Srinivasa, K. R. Ramakrishnan, and K. Rajgopal

***Abstract*—A novel method of two-dimensional (2-D) spectral estimation, which we introduced recently using the Radon transform and a**

Manuscript received March 31, 1989; revised November 26, 1990.

N. Srinivasa was with the Department of Electrical Engineering, Indian Institute of Science, Bangalore 560012, India.

K. R. Ramakrishnan and K. Rajgopal are with the Department of Electrical Engineering, Indian Institute of Science, Bangalore 560012, India.

IEEE Log Number 9104023.

1053-587X/92$03.00 © 1992 IEEE

one dimensional (1-D) autoregressive model, led us to investigate the maximization of entropy subject to the correlation matching constraints in the Radon space. Instead of solving the 2-D maximum entropy spectral estimation problem, we convert it into a problem which is easier to solve. It is shown that a radial slice of the 2-D ME spectrum can be obtained by 1-D AR modeling of the projections (Radon transform) of a stationary random field (SRF). The advantages and limitations of using this new duality relation to estimate the complete 2-D ME spectra on a polar raster are discussed.

I. Introduction

Maximum entropy spectral estimation (MESE) has received considerable interest ever since Burg introduced this new approach of estimating a power spectral density (PSD) estimate from finite data set [1]–[3]. The rationale behind general ME methods is very elegantly discussed by Jaynes [4], [5], and in the spectral estimation context by Ables [6]. In the one-dimensional (1-D) case, the duality between the ME spectrum and the spectrum obtained by assuming an autoregressive (AR) model for the underlying time series data has been very well established [3], [7]. The ME spectrum can be calculated by solving a Toeplitz system of linear equations, when autocorrelations are available [7]. These equations can be efficiently solved using the Levinson recursion [8]. Alternatively, Burg's method [2] can be used to estimate the ME spectrum directly from the basic time series data without the need for estimating and extrapolating the autocorrelation function. This procedure follows truly the ME principle, as it does not make any assumptions about the unmeasured data.

Assuming a Gaussian probability density function, the MESE problem in the two-dimensional (2-D) case can be stated as follows:

P1: Maximize

$$H(S) = \iint_B \ln \left(S(\omega_1, \omega_2)\right) d\omega_1 \, d\omega_2 \tag{1}$$

subject to the correlation matching constraints,

$$r(m, n) = \iint_B S(\omega_1, \omega_2) \exp^{i(n\omega_1 + m\omega_2)} d\omega_1 \, d\omega_2, \qquad \text{for } (m, n) \in A \tag{2}$$

where $\boldsymbol{B}$ is the region in the frequency domain (ω_1, ω_2) over which the spectrum is assumed to be nonzero. This region can be limited arbitrarily and is problem dependent, but could be specified through a condition of band limitedness for a time series or a cutoff wave number in spatial wave theory. The coarray $\boldsymbol{A}$ is the set of points (m, n) for which the autocorrelations $r(m, n)$'s are known. In general, this can also be an arbitrary region. However, the available segment of the autocorrelation function is assumed to be a part of some positive definite function, so that it ensures positive values for the 2-D PSD. Under this condition Woods [9] has shown that a ME spectrum exists.

The 1-D AR model based ME spectral estimation procedure does not have a natural extension to the 2-D case. This is due to the differences in 1-D and 2-D discrete systems theory. A key difference is that the mathematics for describing 2-D systems is less complete than for 1-D systems. Polynomials encountered in 1-D rational systems can always be factorized, while 2-D polynomials in general can not be factorized due to the lack of a fundamental theorem of algebra in 2-D. The fact that spectral factorization is not guaranteed in two dimensions has been pointed by Burg (quoted by Woods [10]). Another difference is that 2-D systems have many more degrees of freedom than 1-D systems. For instance, parametric models characterized by recursive difference equations in 1-D can be extended to 2-D with many possible difference equation descriptions not all of which are recursively computable [11]. Unlike in the 1-D case, the 2-D MESE is unwieldy to compute and hence there is a continuing interest in developing efficient algorithms for estimating the 2-D ME spectrum. Some of them utilize an optimization framework using iterative procedures [12], [13], the correspondence between the 2-D ME spectrum and the 2-D noncausal Gauss–Markov random field (GMRF) model [9], [14], [15], or Fourier transform the extrapolated autocorrelation function [16]–[18]. The reader is referred to [19], [20] for a review of various methods of 2-D MESE methods.

Srinivasa [20] has recently studied some applications of LP modeling and filtering in the Radon space. In particular, a novel method of 2-D spectral estimation utilizing 1-D AR model in the Radon space was proposed [21]. This approach, outlined briefly in Section II, led us to investigate the maximization of entropy subject to the correlation matching constraints in the Radon space. In this correspondence, we introduce a modification to the 2-D autocorrelation matching constraint. Instead of solving P1, we modify it to a problem which is much easier to solve. This finds application especially in tomography and radar, and may be of interest in other areas. A new closed form expression for a radial slice of the 2-D ME spectrum is derived in Section III. The advantages and limitations of using this new duality relation for estimating the 2-D ME spectrum are discussed in comparison with other approaches.

II. 2-D Spectral Estimation Utilizing AR Modeling in the Radon Space

The Radon transform is utilized to convert the 2-D spectral estimation problem into a set of independent 1-D spectral estimation problems. Let $p_\theta(t)$ denote the projection of a SRF $f(x, y)$ at an angle θ. Using $\mathcal{R}$ to denote the Radon transform

$$\begin{aligned} p_\theta(t) &= \mathcal{R}\,[f(x, y)] = \int_{AB} f(x, y)\, ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \delta(x \cos\theta + y \sin\theta - t)\, dx\, dy \end{aligned} \tag{3}$$

where ds is the elemental distance on the line AB represented by the equation

$$x \cos\theta + y \sin\theta = t. \tag{4}$$

Let us denote the 1-D PSD of $p_\theta(t)$ by $P_\theta(\omega)$. Let the 2-D PSD of the SRF in polar coordinates be denoted by $S(\omega, \theta)$ while $S_\theta(\omega)$ represent a radial slice (θ being a parameter which indicates the angle at which the slice is taken). Note that

$$S(\omega_1, \omega_2) = S(\omega \cos\theta, \omega \sin\theta). \tag{5}$$

Jain and Ansari [22] have reintroduced the modification of $\mathcal{R}$ for SRF's, which is originally due to Ludwig [23], and have shown that the 1-D PSD of the projection of the SRF is related to the 2-D PSD by the following relation:

$$S(\omega \cos\theta, \omega \sin\theta) = |\omega| P_\theta(\omega). \tag{6}$$

Thus, the 2-D PSD estimate can be built up slice by slice on a polar raster from the 1-D PSD of the projections.

Srinivasa *et al.* [21] have proposed the use of 1-D AR modeling for the projection data. A slice of the 2-D PSD estimated using 1-D

AR model is then given by

$$S_\theta(\omega) = \frac{|\omega| \nu_{\theta,K}}{\left|1 - \sum_{k=1}^{K} a_\theta(k) \exp^{-ik\omega}\right|^2} \quad (7)$$

where $a_\theta(k)$, for $1 \le k \le K$ represents the coefficients of a Kth order AR model and $\nu_{\theta,K}$ is the variance of the residual sequence for the projection angle θ. The details of the procedure for 2-D PSD estimation from the observation of a SRF and the advantages of the Radon transform approach are not described here as they are readily available in [20], [21].

Starting from the mathematical expression for the entropy and using the ME principle we next establish the correspondence between (7) and the radial slice of the 2-D ME spectrum. The key point here is to convert P1 into an equivalent set of 1-D problems by using the Radon transform.

III. A New Duality Relation for 2-D ME Spectrum

In the following, the region $\boldsymbol{B}$ in (1) is assumed to be a circular region with the origin as the center and radius C, while A is taken to be a regular lattice of points on a square grid. As described earlier, in the Radon transform approach of 2-D PSD estimation the spectrum is obtained on a polar raster. The polar coordinates (ω, θ) were thus found necessary. It has been pointed out by Shannon and Weaver [24] that the entropy H of a continuous function S is relative to the coordinate system. A change in coordinate system will change the entropy in general and is related to the Jacobian of the coordinate transformation. Hence, changing from Cartesian to polar coordinates, the integral in (1) is written as

$$H(S) = \int_0^\pi \int_{-C}^{C} \ln (S(\omega, \theta))|\omega| \, d\omega \, d\theta. \quad (8)$$

It is to be noted that θ lies only in the range $[0, \pi)$ when both positive and negative values of ω are considered, and the property

$$S(\omega, \theta + \pi) = S(-\omega, \theta) \quad (9)$$

is used. The set of constraints in problem P1 are now transformed into constraints in the Radon space by defining the Radon transform of $r(m, n)$. Defining $r_\theta(t_k)$ as

$$r_\theta(t_k) = \boldsymbol{R}[r(m, n)] \quad (10)$$

and assuming $r_\theta(t_k)$ is available for K discrete values of k, the equivalent set of constraints to be met by the slice of 2-D PSD at the particular angle θ are

$$r_\theta(t_k) = \int_{-C}^{C} S_\theta(\omega) \exp^{i\omega k} d\omega, \qquad \text{for } |k| \le K. \quad (11)$$

It is important to note that the $\mathcal{R}$ transform is a whitening transform in θ for stationary random fields [22]. A projection at a particular angle θ does not contribute to any other slice of the 2-D PSD other than the considered angle θ. Hence, the entropy maximization problem can now be carried out by maximizing the inner integral in (8) for all θ. Thus, the equivalent set of problems P2 that are to be considered are

P2: Maximize

$$H'(S) = \int_{-C}^{C} \ln (S_\theta(\omega))|\omega| \, d\omega \quad (12)$$

subject to the constraints

$$r_\theta(t_k) = \int_{-C}^{C} S_\theta(\omega) \exp^{ik\omega} d\omega, \qquad \text{for } |k| \le K \quad (13)$$

for all θ.

Now consider P2 for any angle θ. The ME principle requires the entropy to be stationary with respect to the unknown autocorrelations, i.e., $r_\theta(t_k)$ for $|k| > K$. Hence the partial derivatives of $H'(S)$ with respect to $r_\theta(t_k)$ for $|k| > K$ can be set to zero

$$\frac{\partial H'(S)}{\partial r_\theta(t_k)} = \int_{-C}^{C} |\omega|[S_\theta(\omega)]^{-1} \exp^{-ik\omega} d\omega = 0, \qquad \text{for } |k| > K. \quad (14)$$

Equation (14) implies that $|\omega|[S_\theta(\omega)]^{-1}$ can be expressed as a finite Fourier series. Hence,

$$|\omega|[S_\theta(\omega)]^{-1} = \sum_{k=-K}^{K} \Psi_\theta(k) \exp^{-ik\omega}$$

or

$$S_\theta(\omega) = \frac{|\omega|}{\sum_{k=-K}^{K} \Psi_\theta(k) \exp^{-ik\omega}}. \quad (15)$$

Following the argument for the equivalence of the ME spectrum with AR spectrum in the 1-D case [3], the 1-D polynomial in the denominator of (15) can be factorized and thus a radial slice of the 2-D ME spectrum can be written as

$$S_{\mathrm{ME}\theta}(\omega) = \frac{|\omega| \rho_{\theta,K}}{\left|1 - \sum_{k=1}^{K} \alpha_\theta(k) \exp^{-ik\omega}\right|^2}. \quad (16)$$

Comparing (7) and (16) the following theorem can be stated.

Theorem: A radial slice of the 2-D maximum entropy spectrum $S_{\mathrm{ME}\theta}(\omega)$ of a stationary random field is obtained by multiplying the 1-D AR spectrum of the projection of the stationary random field by a $|\omega|$ function.

We next discuss the advantages and limitations in using this new result for estimating the 2-D ME spectrum.

Remarks:

1) The Radon transform essentially reduces the 2-D MESE problem P1 into a set of 1-D MESE problems which is in turn handled by AR modeling procedures. Hence in this method of 2-D spectral estimation the problem of 2-D spectral factorization does not arise. The 2-D ME spectrum obtained here turns out to be a function of the prediction error of a set of 1-D LP filter for the projections of SRF. Smylie *et al.* [25] have expressed the entropy rate as a function of a set of 2-D row and column filters. The problems encountered while using such 2-D corner filters, or with filters of other shapes, (e.g., loss of stability and distortion of spectra [26]–[28]), are not encountered while using the new approach.

2) By using the Burg method, or its modifications, for estimating the AR parameters of the projection data the spectrum can be estimated directly from the observations of a SRF. The Radon transform approach of 2-D spectral estimation is thus a stochastic approach of spectral estimation.

3) From the computational point of view as well, this new ap-

proach is very advantageous as it requires solving a set of a system of linear equations. Further, this method is amenable for parallel processing. Procedures which estimate the 2-D ME spectrum by solving linear equations are applicable only in special cases, e.g., isotropic random field [29], and GMRF when the model order is exactly known [15].

4) The only approximation required in using the Radon transform approach is in computing the line integrals of the 2-D observation set. The evaluation of the Radon transform involves integration from $-\infty$ to ∞ along a line. Given the autocorrelation function on a finite domain, we cannot compute its true Radon transform. On the other hand, if we compute the projections of the given lags and then apply the above method, we would have to change the value of the given lags if we have to match the Radon transform values. Hence in using this approach for ME spectral estimation, we are not solving exactly the problem P1, but solving an easier problem P3. The details of using a Riemann sum approximation for estimating the line integrals of the available finite data is given in [21].

5) Regarding the nature of the 2-D spectrum that is obtained, it is discrete in the angular variable θ, while along each slice a continuous spectrum is indeed obtained. The number of slices can be increased indefinitely by computing the necessary projections at each angle. However, from many computer simulations that have been carried out it has been observed that for a data array of size (32×32) an increment of 1^0 is sufficient to get a good estimate of the 2-D spectrum [20], [21]. The other features of the RT approach of 2-D spectral estimation have been discussed in [20], [21].

IV. Conclusion

This correspondence has established and discussed an interesting theoretical result that is obtained when an LP model is used in the Radon space for the 2-D spectral estimation problem. It is shown that 1-D autoregressive spectrum of the projections of a stationary random field when multiplied by a $|\omega|$ function results in a radial slice of the 2-D ME spectrum. The elegance of this new approach along with the advantages and its limitations is brought out by comparing it with other methods that have been proposed earlier for 2-D MESE. The approximation made in estimating the Radon transform has been mentioned. The effect of enforcing the matching constraints in the Radon space given finite lags, and the statistical properties of the resulting ME spectral estimate have to be investigated further. Lastly, the extension of the new duality relation derived for the 2-D case, to the m-D case is an interesting problem.

Acknowledgment

The authors wish to express their appreciation to Prof. R. L. Kashyap, Prof. V. U. Reddy, and the reviewers for their advice and critical comments in the preparation of the manuscript.

References

[1] J. P. Burg, "Maximum entropy spectral analysis," presented at the 37th Annu. Meeting Soc. Exploration Geophysicists, Oklahoma City, OK, 1967; reprinted in D. G. Childers, *Modern Spectrum Analysis*. New York: IEEE Press, 1978.

[2] J. P. Burg, "A new analysis technique for time series data," presented at the NATO Advanced Study Institute Signal Processing, Enschede, Netherlands, 1968; reprinted in D. G. Childers, *Modern Spectral Analysis*. New York: IEEE Press, 1978.

[3] J. P. Burg, "Maximum entropy spectral analysis," Ph.D. dissertation, Dep. Geophys., Stanford Univ., Stanford, CA, May 1975.

[4] E. T. Jaynes, "Information theory and statistical mechanics," *Phys. Rev.*, vol. 106, pp. 620-630, 1957.

[5] E. T. Jaynes, "On the rationale of maximum-entropy methods," *Proc. IEEE*, vol. 70, pp. 939-952, Sept. 1982.

[6] J. B. Ables, "Maximum entropy spectral analysis," *Astron. Astrophys. Suppl. Ser.*, vol. 15, pp. 383-393, 1974.

[7] A. van den Bos, "Alternative interpretation of maximum entropy spectral analysis," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 493-494, July 1971.

[8] N. Levinson, "The Wiener RMS (root mean square) error criterion in filter design and prediction," *J. Math. Phys.*, vol. 25, no. 4, pp. 261-278, 1947.

[9] J. W. Woods, "Two-dimensional Markov spectral estimation," *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 552-559, Sept. 1976.

[10] J. W. Woods, "Two-dimensional discrete Markov random fields," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 232-240, Mar. 1972.

[11] A. K. Jain, "Advances in mathematical models for image processing," *Proc. IEEE*, vol. 69, pp. 349-389, May 1981.

[12] A. K. Jain and S. Ranganath, "Two-dimensional spectrum estimation," in *Proc. RADC Spectrum Estimation Workshop*, May 1978, pp. 151-157.

[13] S. W. Lang and J. H. McClellan, "Multidimensional MEM spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 880-887, Dec. 1982.

[14] R. Chellappa, Y.-H. Hu, and S.-Y. Kung, "On two-dimensional Markov spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 836-841, Aug. 1983.

[15] G. Sharma and R. Chellappa, "Two-dimensional spectral estimation using noncausal autoregressive models," *IEEE Trans. Inform. Theory*, vol. IE-32, pp. 268-275, Mar. 1986.

[16] S. E. Roucos and D. G. Childers, "A two-dimensional maximum entropy spectral estimator," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 554-560, Sept. 1980.

[17] H. Lev-Ari, "Multidimensional maximum entropy covariance extension," in *Proc. ICASSP '85* (Tampa, FL), Apr. 1985, pp. 816-819.

[18] J. S. Lim and N. A. Malik, "A new algorithm for two-dimensional maximum entropy power spectrum estimation," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-29, pp. 401-413, June 1981.

[19] J. H. McClellan, "Multidimensional spectral estimation," *Proc. IEEE*, vol. 70, pp. 1029-1039, 1982.

[20] N. Srinivasa, "Applications of linear prediction and filtering in the Radon space," Ph.D. dissertation, Dep. Elec. Eng., Indian Institute of Science, Bangalore, India, 1988.

[21] N. Srinivasa, K. R. Ramakrishnan, and K. Rajgopal, "Two-dimensional spectral estimation: A Radon transform approach," *IEEE J. Oceanic Eng.*, vol. OE-12, no. 1, pp. 90-96, Jan. 1987.

[22] A. K. Jain and S. Ansari, "Radon transform theory for random fields and optimum image reconstructions from noisy projections," in *Proc. ICASSP '84* (San Diego, CA), Apr. 1984, 12A.7.

[23] D. Ludwig, "The Radon transform on Euclidean space," *Commun. Pure Appl. Math.*, vol. 19, pp. 49-81, 1966.

[24] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*. Urbana, IL: University of Illinois Press, 1959.

[25] D. E. Smylie, G. K. C. Clarke, and T. J. Ulrych, "Anlaysis of irregularities in the earth's rotation," in *Methods in Computational Physics*, vol. 13. New York: Acadmic, 1973, pp. 391-430.

[26] J. V. Pendrell and D. E. Smylie, "The maximum entropy principle in two-dimensional spectral analysis," *Astron. Astrophys.*, vol. 112, pp. 181-189, 1982.

[27] D. Tjostheim, "Autoregressive modeling and spectral analysis of array data in the plane," *IEEE Trans. Geosci. Remote Sensing*, vol. GE-19, pp. 15-24, Jan. 1981.

[28] T. J. Ulrych and C. J. Walker, "High resolution 2-dimensional power spectral estimation," in *Applied Time Series Analysis II*, D. F. Findley, Ed. New York: Academic, 1981.

[29] A. H. Tewfik, B. C. Levy, and A. S. Willsky, "An efficient maximum entropy technique for 2-D isotropic random fields," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 797-812, May 1988.