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On graphs whose eternal vertex cover number and vertex cover number coincide[☆,☆☆]

Jasine Babu[a,*], L. Sunil Chandran[b], Mathew Francis[c], Veena Prabhakaran[a], Deepak Rajendraprasad[a], Nandini J. Warrier[d]

[a] *Indian Institute of Technology Palakkad, India*
[b] *Indian Institute of Science, Bangalore, India*
[c] *Indian Statistical Institute, Chennai, India*
[d] *National Institute of Technology Calicut, India*

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ABSTRACT

The eternal vertex cover problem is a variant of the classical vertex cover problem defined in terms of an infinite attacker–defender game played on a graph. In each round of the game, the defender reconfigures guards from one vertex cover to another in response to a move by the attacker. The minimum number of guards required in any winning strategy of the defender when this game is played on a graph G is the eternal vertex cover number of G, denoted by $\mathrm{evc}(G)$. It is known that given a graph G and an integer k, checking whether $\mathrm{evc}(G) \le k$ is NP-hard. Further, it is known that for any graph G, $\mathrm{mvc}(G) \le \mathrm{evc}(G) \le 2\,\mathrm{mvc}(G)$, where $\mathrm{mvc}(G)$ is the vertex cover number of G.

Though a characterization is known for graphs for which $\mathrm{evc}(G) = 2\,\mathrm{mvc}(G)$, a characterization of graphs for which $\mathrm{evc}(G) = \mathrm{mvc}(G)$ remained as an open problem, since 2009. We achieve such a characterization for a class of graphs that includes chordal graphs and internally triangulated planar graphs. For biconnected chordal graphs, our characterization leads to a polynomial time algorithm for precisely determining $\mathrm{evc}(G)$ and an algorithm for determining a safe strategy for guard movement in each round of the game using only $\mathrm{evc}(G)$ guards.

Though the eternal vertex cover problem is only known to be in PSPACE in general, it follows from our new characterization that the problem is in NP for locally connected graphs, a graph class which includes all biconnected internally triangulated planar graphs. We also provide reductions establishing NP-completeness of the problem for biconnected internally triangulated planar graphs. As far as we know, this is the first NP-completeness result known for the problem for any graph class.

1. Introduction

A vertex cover of a graph $G(V, E)$ is a set $S \subseteq V$ such that for every edge in E, at least one of its endpoints is in S. A minimum vertex cover of G is a vertex cover of G of minimum cardinality and its cardinality is the vertex cover number

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* Corresponding author.
E-mail addresses: jasine@iitpkd.ac.in (J. Babu), sunil@iisc.ac.in (L.S. Chandran), mathew@isichennai.res.in (M. Francis), veenaprabhakaran7@gmail.com (V. Prabhakaran), deepak@iitpkd.ac.in (D. Rajendraprasad), nandini.wj@gmail.com (N.J. Warrier).

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of G, denoted by $\mathrm{mvc}(G)$. Equivalently, if we imagine that a guard placed on a vertex v can monitor all edges incident at v, then $\mathrm{mvc}(G)$ is the minimum number of guards required to ensure that all edges of G are monitored.

The notion of "eternal vertex cover", first introduced by Klostermeyer and Mynhardt [11], is an extension of the above formulation in the context of a two-player multi-round game, where a "defender" uses mobile guards placed on some vertices of G in order to protect the edges of G from an "attacker". We focus on a well studied variant of the game in which at most one guard is allowed to be present at a vertex at any point of time. The game begins with the defender placing guards on some vertices, with at most one guard per vertex. The total number of guards remains the same throughout the game. In each round of the game, the attacker chooses an edge to attack. In response, the defender moves the guards in such a way that each guard either stays at its current location or moves to an adjacent vertex. If a guard crosses the attacked edge during this move, then the attack is considered to have been successfully defended. The movement of all guards in a round is assumed to happen in parallel. Note that it is possible for the two guards on the two endpoints of an edge to exchange their positions; in fact, only such a move is enough when an edge that has guards on both endpoints is attacked. The game then proceeds to the next round of attack–defense. The defender wins if any sequence of attacks can be defended. If an attack cannot be defended in some round, the attacker wins.

Clearly, if the vertices occupied by the guards do not form a vertex cover at the beginning of some round, there is an attack that cannot be defended, namely an attack on an edge that has no guards on its endpoints. If $\mathcal{C}$ is a family of vertex covers of G of the same cardinality, such that the defender can choose any vertex cover from $\mathcal{C}$ as the starting configuration and successfully keep on defending attacks forever by moving among configurations in $\mathcal{C}$ itself, then $\mathcal{C}$ is an *eternal vertex cover class* of G and each vertex cover in $\mathcal{C}$ is an *eternal vertex cover* of G. If S is an eternal vertex cover belonging to an eternal vertex cover class $\mathcal{C}$, we say that S is a *configuration* in $\mathcal{C}$. The eternal vertex cover number of G, denoted by $\mathrm{evc}(G)$, is the minimum cardinality of an eternal vertex cover of G. An eternal vertex cover of cardinality $\mathrm{evc}(G)$ is said to be a *minimum eternal vertex cover* and an eternal vertex cover class that consists of minimum eternal vertex covers is said to be a *minimum eternal vertex cover class*. It is easy to see that for any graph G, $\mathrm{mvc}(G) \leq \mathrm{evc}(G)$. Also, since a graph with vertex set V has the family $\{V\}$ as an eternal vertex cover class, $\mathrm{evc}(G)$ is finite and therefore well-defined for every graph G.

The *eternal vertex cover problem* is the problem of deciding, given a graph G and an integer k as input, whether $\mathrm{evc}(G) \leq k$. Fomin et al. [8] discuss the computational complexity of the eternal vertex cover problem and derive some algorithmic results for it. They show that given a graph $G(V, E)$ and an integer k, it is NP-hard to decide whether $\mathrm{evc}(G) \leq k$. The paper gives two algorithms for the problem: an exact algorithm with $2^{O(n)}$ time complexity and exponential space complexity and an FPT algorithm with eternal vertex cover number as the parameter. They also describe a simple polynomial time 2-factor approximation algorithm for the eternal vertex cover problem using maximum matchings. It is of interest to note here that although Fomin et al. use a variant of the eternal vertex cover problem in which more than one guard can be placed on a single vertex, their results can be carried forward (with minor modifications in proofs) to the original model which allows at most one guard per vertex. It is not yet known whether the problem is in NP, though Fomin et al. [8] had shown that it is in PSPACE. It is also unknown whether the eternal vertex cover problem for bipartite graphs is NP-hard. Some related graph parameters based on multi-round attacker–defender games and their relationship with the eternal vertex cover number were investigated by Anderson et al. [1] and Klostermeyer and Mynhardt [12].

Klostermeyer and Mynhardt [11] showed that for any graph G, $\mathrm{mvc}(G) \leq \mathrm{evc}(G) \leq 2\,\mathrm{mvc}(G)$. If G is a cycle, then $\mathrm{mvc}(G) = \mathrm{evc}(G)$ and if G is a path on an odd number of vertices, then $\mathrm{evc}(G) = 2\,\mathrm{mvc}(G)$. Thus, even for the class of bipartite graphs, both the lower bound and the upper bound mentioned above are tight. Klostermeyer and Mynhardt [11] gave a characterization of graphs whose eternal vertex cover number is twice their vertex cover number. The characterization follows a nontrivial constructive method starting from any tree T which requires $2\,\mathrm{mvc}(T)$ guards to protect it. They also give a few examples of graphs for which the eternal vertex cover number and the vertex cover number coincide; such as the complete graph on n vertices (K_n), the Petersen graph, $K_m \square K_n$, $C_m \square C_n$ (where $\square$ represents the box product) and the $n \times m$ grid, where either n or m is even. However, they mention that an elegant characterization of graphs for which $\mathrm{evc}(G) = \mathrm{mvc}(G)$ seems to be difficult. Here, we achieve such a characterization that works for all chordal graphs, internally triangulated planar graphs and locally connected graphs.[1]

Without loss of generality, we only consider connected graphs for this characterization. The characterization (see Theorem 1) has the following implications:

- For chordal graphs, deciding whether $\mathrm{evc}(G) = \mathrm{mvc}(G)$ can be done in polynomial time. If the parameters are equal, then a safe strategy of guard movement in each round of the game, with $\mathrm{evc}(G)$ guards, can be determined in polynomial time.
- For biconnected chordal graphs, $\mathrm{evc}(G)$ can be computed in polynomial time. Further, a safe strategy of guard movement in each round of the game, with $\mathrm{evc}(G)$ guards, can be determined in polynomial time.
- For biconnected internally triangulated graphs, there is a PTAS for computing $\mathrm{evc}(G)$. Further, for internally triangulated planar graphs, deciding whether $\mathrm{evc}(G) = \mathrm{mvc}(G)$ is in P^{NP}.
- Deciding whether $\mathrm{evc}(G) \leq k$ is in NP for locally connected graphs, a graph class that includes the class of biconnected internally triangulated planar graphs.

[1] A graph is locally connected, if the open neighborhood of each vertex induces a connected subgraph.

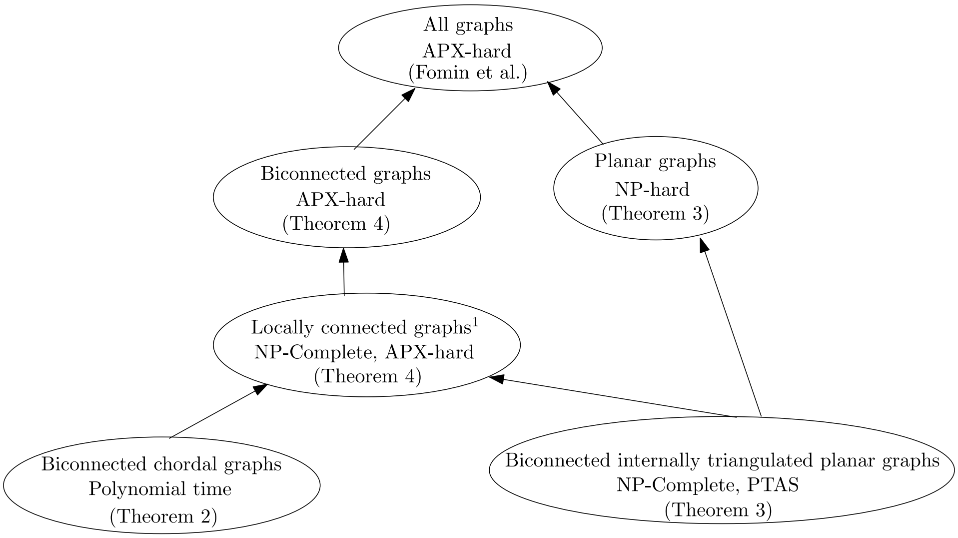

Fig. 1. Complexity of deciding whether $\text{evc}(G) \le k$.

Other results included in this paper are the following:

- Deciding whether $\text{evc}(G) \le k$ is NP-complete for locally connected graphs and biconnected internally triangulated planar graphs. To the best of our knowledge, these are the first NP-completeness results known for the problem for any graph class. Various NP-hardness and approximation hardness results obtained are summarized in Fig. 1.
- Klostermeyer and Mynhardt [11] had posed a question whether it is necessary for every edge e of G to be present in some maximum matching, to satisfy $\text{evc}(G) = \text{mvc}(G)$. We present an example which answers this question in the negative.

2. A basic necessary condition for evc(G) = mvc(G)

In this section, we derive some basic necessary conditions for a graph G to have $\text{evc}(G) = \text{mvc}(G)$.

We first introduce some basic definitions. Given a graph $G(V, E)$ and a set $U \subseteq V$, we denote by $G[U]$ the subgraph induced in G by U, and we denote by $G \setminus U$ the subgraph induced in G by $V \setminus U$.

Definition 1. For any subset U of vertices of a graph G, we define $\text{evc}_U(G)$ as the minimum integer k such that G has an eternal vertex cover class $\mathcal{C}$ in which every configuration is a vertex cover of cardinality k that contains all the vertices in U. We define $\text{mvc}_U(G)$ as the minimum cardinality of a vertex cover of G that contains all the vertices of U.

Note that when $U = \emptyset$, $\text{mvc}_U(G) = \text{mvc}(G)$ and $\text{evc}_U(G) = \text{evc}(G)$. Also, since for a graph $G(V, E)$, the family $\{V\}$ is an eternal vertex cover class, we have that $\text{evc}_U(G)$ is finite for any $U \subseteq V$. Clearly, for any $U \subseteq V$, we have $\text{mvc}(G) \le \text{mvc}_U(G)$, $\text{evc}(G) \le \text{evc}_U(G)$ and $\text{mvc}_U(G) \le \text{evc}_U(G)$.

The following is an easy observation.

Observation 1. *Let $G(V, E)$ be a graph with no isolated vertices. If $\text{evc}(G) = \text{mvc}(G)$, then for every vertex $v \in V$, there is some minimum vertex cover of G containing v.*

Proof. Suppose that $\text{evc}(G) = \text{mvc}(G)$ and that $\mathcal{C}$ is a minimum eternal vertex cover class of G. Let $C \in \mathcal{C}$. If $v \in C$, then C is a minimum vertex cover of G containing v and we are done. So let us assume that $v \notin C$. Let u be a neighbor of v. As C is a vertex cover, $u \in C$. By the definition of an eternal vertex cover class, the defender can choose C to be the starting configuration and there is a configuration $C' \in \mathcal{C}$ to which the guards can be moved if the edge uv is attacked in the first round. Clearly, the guard on u has to move to v in order to successfully defend such an attack. Therefore, $v \in C'$. Then C' is a minimum vertex cover of G containing v. □

It is easy to see that the simple necessary condition stated above is not sufficient for many graphs. For example, for a path P_n on n vertices, where $n > 2$ is an even number, each vertex belongs to some minimum vertex cover; but still $\frac{n}{2} = \text{mvc}(G) < \text{evc}(G) = n - 1$. In fact, among graphs which are not biconnected, it is easy to find several such examples. Therefore, we generalize Observation 1 to get a stronger necessary condition for a graph G with cut vertices.

The following is an easy generalization of Observation 1 and it can be proven by a straightforward generalization of the proof of Observation 1.

Observation 2. *Let $G(V, E)$ be a graph with no isolated vertices and $U \subseteq V$. If $\text{evc}_U(G) = \text{mvc}_U(G)$, then for every vertex $v \in V \setminus U$, $\text{mvc}_{U\cup\{v\}}(G) = \text{mvc}_U(G)$.*

The next lemma shows that if the vertex cover number and the eternal vertex cover number of a graph G coincide, then these parameters also coincide with $\text{evc}_X(G)$, where X is the set of cut vertices of G. In particular, we show that for a graph G for which these two parameters coincide, every minimum eternal vertex cover of G contains all the vertices in X.

Lemma 1. *Let $G(V, E)$ be any graph. Let $X \subseteq V$ be the set of cut vertices of G. If $\text{evc}(G) = \text{mvc}(G)$, then any minimum eternal vertex cover of G contains all the vertices of X. Consequently, if $\text{evc}(G) = \text{mvc}(G)$, then $\text{evc}_X(G) = \text{mvc}_X(G) = \text{evc}(G) = \text{mvc}(G)$.*

Proof. Suppose that $\text{evc}(G) = \text{mvc}(G) = k$. If $X = \emptyset$, the result holds trivially. If $X \neq \emptyset$, we will show that all the vertices of X are contained in any minimum eternal vertex cover of G.

Suppose for the sake of contradiction that there exist a cut vertex x of G and a minimum eternal vertex cover C of G such that $x \notin C$. Let H be a connected component of $G \setminus \{x\}$ that is not a connected component of G, $H_1 = G[V(H) \cup \{x\}]$ and $H_2 = G[V \setminus V(H)]$. Note that H_1 and H_2 are edge-disjoint subgraphs of G with x being their only common vertex and also that x has non-zero degree in both of them. Let $k_1 = \text{mvc}(H_1)$ and $k_2 = \text{mvc}(H_2)$. Clearly, $k \le k_1 + k_2$. It is easy to see that if S is any vertex cover of G, then $S \cap V(H_1)$ and $S \cap V(H_2)$ are vertex covers of H_1 and H_2 respectively. This implies that $|C \cap V(H_1)| \ge k_1$, and $|C \cap V(H_2)| \ge k_2$. As $x \notin C$ and $|C| = k$, we now have that $k = k_1 + k_2$, $|C \cap V(H_1)| = k_1$, and $|C \cap V(H_2)| = k_2$. If both H_1 and H_2 have minimum vertex covers that contain x, then $\text{mvc}(G) = k$ would have been at most $k_1 + k_2 - 1$. Therefore, we can assume without loss of generality that no minimum vertex cover of H_1 contains x. As C is an eternal vertex cover, an attack on an edge ux of H_1 while the guards are placed according to the configuration C can be defended by moving the guards to another configuration, say C'. Noting that during this move the guard on u has to be shifted to x, it can be seen that in C', there are at most k_1 vertices of $V(H_1)$ with guards on them. Then the vertices in $C' \cap V(H_1)$ form a minimum vertex cover of H_1 containing x, which is a contradiction. □

By combining Observation 2 and Lemma 1, we can derive the following basic necessary condition for a graph to have its vertex cover number and eternal vertex cover number coincide.

Lemma 2 (*Necessary Condition*)**.** *Let $G(V, E)$ be any graph with no isolated vertices. Let $X \subseteq V$ be the set of cut vertices of G. If $\text{evc}(G) = \text{mvc}(G)$, then for every vertex $v \in V \setminus X$, there is a minimum vertex cover S_v of G such that $X \cup \{v\} \subseteq S_v$.*

Proof. Suppose $\text{evc}(G) = \text{mvc}(G)$. Then, by Lemma 1, we have $\text{evc}_X(G) = \text{mvc}_X(G)$. Hence, by Observation 2, for every vertex $v \in V \setminus X$, $\text{mvc}_{X\cup\{v\}}(G) = \text{mvc}_X(G)$. Since we also have $\text{mvc}_X(G) = \text{mvc}(G)$ by Lemma 1, this implies that for every vertex $v \in V \setminus X$, $\text{mvc}_{X\cup\{v\}}(G) = \text{mvc}(G)$. □

From Lemma 2, it is evident that if $\text{evc}(G) = \text{mvc}(G)$, then G must have a minimum vertex cover containing all its cut vertices. The following is another interesting corollary of Lemma 2.

Corollary 1. *For any connected graph G with at least three vertices and minimum degree 1, $\text{evc}(G) \neq \text{mvc}(G)$.*

The corollary holds because a degree one vertex and its neighbor (which is a cut vertex, if the graph itself is not just an edge) cannot be simultaneously present in a minimum vertex cover of G.

3. Main characterization

In this section, we prove the main structure theorem of this paper. We show that the necessary condition mentioned in Lemma 2 is also sufficient for a large class of graphs including all chordal graphs, internally triangulated planar graphs and locally connected graphs, to have $\text{evc}(G) = \text{mvc}(G)$.

The following lemma specifies a sufficient condition under which the converse of Observation 2 holds. The proof of this lemma involves repeated applications of Hall's marriage theorem [5] and forms the most nontrivial ingredient in the proof of Theorem 1, our main characterization theorem.

Lemma 3. *Let $G(V, E)$ be a graph with at least two vertices. Let $U \subseteq V$ and suppose that every vertex cover S of G that contains U and has cardinality $\text{mvc}_U(G)$ is connected. If for every vertex $v \in V \setminus U$, $\text{mvc}_{U\cup\{v\}}(G) = \text{mvc}_U(G)$, then $\text{evc}_U(G) = \text{mvc}_U(G)$.*

Proof. Let $k = \text{mvc}_U(G)$. Suppose that every vertex cover S of G with $U \subseteq S$ and $|S| = k$ is connected. Further suppose that for every vertex $v \in V \setminus U$, $\text{mvc}_{U\cup\{v\}}(G) = k$.

We define $\mathcal{C}$ to be the set of all vertex covers S of G with $U \subseteq S$ and $|S| = k$. We shall show that $\mathcal{C}$ is an eternal vertex cover class of G, thereby proving the lemma. To show that $\mathcal{C}$ is an eternal vertex cover class of G, we show that if an attack on some edge occurs when the guards occupy the vertices in $S \in \mathcal{C}$, the attack can be defended by moving the guards such that they occupy the vertices in some $S' \in \mathcal{C}$.

Consider an attack on the edge uv such that $u \in S$ and $v \notin S$. Let $\Gamma = \{Q \in \mathcal{C} : v \in Q\}$. We know that $\Gamma \neq \emptyset$. Let S' be a set in Γ such that there is no set $S'' \in \Gamma$ with $S' \cap S \subset S'' \cap S$. For example, a set in Γ, the size of whose symmetric difference with S is as small as possible, is one possible choice for S'. We show that it is possible to safely defend the attack on uv by moving from S to S'.

Let $T = S \cap S'$ and let A, B be such that $S = T \uplus A$ and $S' = T \uplus B$. As S and S' are vertex covers, $V \setminus S$ and $V \setminus S'$ are independent sets, implying that A and B are independent sets. Hence, $H = G[A \uplus B]$ is a bipartite graph. Further, since $|S| = |S'|$, we have $|A| = |B|$, and as $v \in S' \setminus S$, we also have $|A| = |B| \geq 1$.

Claim 1. *H has a perfect matching.*

Proof of Claim 1. Note that $U \subseteq T$. Consider any $B' \subseteq B$. Since $V \setminus S$ is an independent set, we have $N_G(B') \subseteq S = T \uplus A$. If $|N_H(B')| < |B'|$, then $T \uplus (B \setminus B') \uplus N_H(B')$ is a vertex cover containing U and of size smaller than k, violating the fact that $\text{mvc}_U(G) = k$. Therefore, $\forall B' \subseteq B$, $|N_H(B')| \geq |B'|$ and by Hall's theorem [5], H has a perfect matching. □

Claim 2. *$\forall x \in A$, the bipartite graph $H \setminus \{x, v\}$ has a perfect matching.*

Proof of Claim 2. If $H \setminus \{x, v\}$ has no vertices, then the claim holds trivially. Consider any non-empty subset $B' \subseteq (B \setminus \{v\})$. By Claim 1, $|N_H(B')| \geq |B'|$. If $|N_H(B')| = |B'|$, then $Q = T \uplus (B \setminus B') \uplus N_H(B')$ is a vertex cover of G with $|Q| = k$ and $(U \cup \{v\}) \subseteq Q$. This contradicts the choice of S', since $S' \cap S \subset Q \cap S$. Therefore, $|N_H(B')| \geq |B'| + 1$ and $|N_H(B') \setminus \{x\}| \geq |B'|$. Hence, for all subsets $B' \subseteq (B \setminus \{v\})$, $|N_H(B') \setminus \{x\}| \geq |B'|$ and by Hall's theorem, $H \setminus \{x, v\}$ has a perfect matching. □

With the help of Claims 1 and 2, we can now complete the proof of Lemma 3. We will describe how to move the guards from configuration S to configuration S', in order to defend the attack on the edge uv, in such a way that each guard either stays at its previous location or moves to a neighboring vertex, no two guards are on the same vertex at any given time, and the guard at u moves to v.

- Case 1. $u \in A$:
  By Claim 2, there exists a perfect matching M in $H \setminus \{u, v\}$. In order to defend the attack, move the guard on u to v and also all the guards on $A \setminus \{u\}$ to $B \setminus \{v\}$ along the edges of the matching M.
- Case 2. $u \in T$:
  Recall that $|A| = |B| \geq 1$. By our assumption, the vertex cover $S = T \uplus A$ is connected. Let P be a shortest path from A to u in $G[S]$. By the minimality of P, it has exactly one vertex x from A and there exists a perfect matching M in $H \setminus \{x, v\}$. In order to defend the attack, move the guard on u to v, x to z_1 and z_i to z_{i+1}, $\forall i \in [t-1]$. In addition, move all the guards on $A \setminus \{x\}$ to $B \setminus \{v\}$ along the edges of the matching M.

It is clear that in both cases, the attack can be defended by moving the guards as mentioned and the new configuration is S'. □

A vertex cover S of a graph G is called a *connected vertex cover* if $G[S]$ is connected. The *connected vertex cover number* of G is the size of a minimum cardinality connected vertex cover of G. Klostermeyer and Mynhardt [11] observed that $\text{evc}(G)$ is at most one more than the size of a connected vertex cover of G. Combining this result with Lemma 3, the following observation is immediate.

Observation 3. *Let $G(V, E)$ be a graph with no isolated vertices for which every minimum vertex cover is connected. If for every vertex $v \in V$, there exists a minimum vertex cover S_v of G such that $v \in S_v$, then $\text{evc}(G) = \text{mvc}(G)$. Otherwise, $\text{evc}(G) = \text{mvc}(G) + 1$.*

The following theorem is the main structure theorem in this paper. The theorem gives a necessary and sufficient condition for a graph G to satisfy $\text{evc}(G) = \text{mvc}(G)$, if every minimum vertex cover of G that contains all cut vertices is connected. As mentioned before, Lemma 3 supplies the non-trivial direction of the characterization, while the more obvious direction is supplied by Lemma 2.

Theorem 1 (*Main Characterization Theorem*)**.** *Let $G(V, E)$ be a connected graph with at least two vertices and X be the set of cut vertices of G. If every minimum vertex cover S of G with $X \subseteq S$ is connected, then the following characterization holds: $\text{evc}(G) = \text{mvc}(G)$ if and only if for every vertex $v \in V \setminus X$, there exists a minimum vertex cover S_v of G such that $(X \cup \{v\}) \subseteq S_v$.*

Proof. If for every vertex $v \in V \setminus X$ there exists a minimum vertex cover S_v of G such that $(X \cup \{v\}) \subseteq S_v$, then it is easy to see that $\text{mvc}_X(G) = \text{mvc}(G)$. Hence, by our assumption, it follows that every vertex cover of G of cardinality $\text{mvc}_X(G)$ that contains X is connected. Therefore, by Lemma 3, we have $\mathrm{evc}_X(G) = \mathrm{mvc}_X(G) = \mathrm{mvc}(G)$. Since $\mathrm{mvc}(G) \le \mathrm{evc}(G) \le \mathrm{evc}_X(G)$, it follows that $\mathrm{evc}(G) = \mathrm{mvc}(G)$.

Conversely, if $\mathrm{evc}(G) = \mathrm{mvc}(G)$, by Lemma 2, for every vertex $v \in V \setminus X$, there exists a minimum vertex cover S_v of G such that $(X \cup \{v\}) \subseteq S_v$. □

Note. By going through the proofs presented, it can be verified that Theorem 1 is valid also for the variant of the game where more than one guard is allowed on a vertex simultaneously.

Chordal graphs and internally triangulated planar graphs satisfy the property that if the graph is connected, then every vertex cover of the graph that contains all its cut vertices forms a connected vertex cover of the graph. Hence, our characterization theorem is directly applicable to these graph classes, as stated below.

Observation 4. *Let G be a chordal graph or an internally triangulated planar graph. Suppose G is connected and has at least two vertices. Then $\mathrm{evc}(G) = \mathrm{mvc}(G)$ if and only if for every non-cut vertex v of G, there exists a minimum vertex cover S_v of G such that v and all cut vertices of G are contained in S_v.*

In Section 6, we show that a similar observation (Observation 5) holds for a larger class of graphs. In the next three sections, we present some of the algorithmic and complexity theoretic results derived using Theorem 1.

4. Eternal vertex cover number of chordal graphs

A graph is *chordal* if it contains no induced cycle of length four or more. In this section, we discuss the algorithmic consequences of Theorem 1, in the context of chordal graphs. To the best of our knowledge, prior to this work, polynomial time algorithms for computing eternal vertex cover number were only known for simple graph classes like trees, cycles and grids. We derive a polynomial time algorithm for computing eternal vertex cover number of biconnected chordal graphs. Further, we show that deciding whether the eternal vertex cover number of a chordal graph coincides with its vertex cover number or not, can be done in polynomial time. At the end of this section, we also provide a generalization of these results to a superclass of chordal graphs.

It is well-known that the computation of the vertex cover number of a chordal graph can be done in polynomial time [9]. This fact, together with Theorem 1, gives us the following result for chordal graphs.

Theorem 2. *For any chordal graph G,*

1. *deciding whether* $\mathrm{evc}(G) = \mathrm{mvc}(G)$ *can be done in polynomial time,*
2. *if* $\mathrm{evc}(G) = \mathrm{mvc}(G)$, *then there is a polynomial time (per-round) strategy for guard movements using* $\mathrm{evc}(G)$ *guards, and*
3. *if G is a biconnected, then* $\mathrm{evc}(G)$ *can be computed in polynomial time and there is a polynomial time (per-round) strategy for guard movements using* $\mathrm{evc}(G)$ *guards.*

Proof. Without loss of generality, we may assume that G is connected and has at least two vertices. We can compute the vertex cover number of any induced subgraph of G in polynomial time [9]. This implies that for any $Y \subseteq V$, we can decide in polynomial time whether G has a minimum vertex cover containing Y, since this is equivalent to checking whether $\mathrm{mvc}(G) = \mathrm{mvc}(G') + |Y|$ where $G' = G \setminus Y$. There are well-known polynomial time algorithms, like the one using depth first search, for computing the set of cut vertices of a graph. It is also well known that, for any connected chordal graph G, every vertex cover of G that contains all its cut vertices forms a connected vertex cover of G. Let X be the set of cut vertices of G.

1. By Theorem 1, to decide whether $\mathrm{mvc}(G) = \mathrm{evc}(G)$, it is enough to check for every vertex $v \in V \setminus X$ whether G has a minimum vertex cover containing $X \cup \{v\}$. As observed above, this can be done in polynomial time.
2. Suppose $\mathrm{evc}(G) = \mathrm{mvc}(G) = k$. Then, by Lemma 1, $\mathrm{evc}_X(G) = \mathrm{mvc}_X(G) = k$. Therefore, by Observation 2, for every vertex $v \in V \setminus X$, $\mathrm{mvc}_{X \cup \{v\}}(G) = k$. We construct a polynomial time algorithm that, given any minimum vertex cover S of G with $X \subseteq S$ and an edge uv, computes another minimum vertex cover S' of G with $X \subseteq S'$. The set S' is such that an attack on the edge uv, while the guards are in configuration S, can be defended by moving them to configuration S'. We construct the algorithm by extending the basic ideas used in the proof of Lemma 3. Assume without loss of generality that $u \in S$ and $v \notin S$ (if both $u, v \in S$, then the current configuration of guards does not need to be changed). Starting with setting $Y = X \cup \{v\}$, we keep extending Y by adding vertices from $S \setminus Y$ to it, as long as there is a minimum vertex cover of G containing Y. We stop when adding any vertex from $S \setminus Y$ to Y will result in a set that is not contained in any minimum vertex cover of G. Compute some minimum vertex cover S' that contains this Y. Note that the set S' now has the property that there is no set S'' that contains $X \cup \{v\}$ such that $S' \cap S \subset S'' \cap S$. The construction of Y can be done in polynomial time as any vertex in $S \setminus Y$ that cannot be added to Y at one point will never become suitable for addition to Y later. We now move the guards from configuration S to configuration S' as explained in the proof of Lemma 3. The basic computational steps involved in this process are computing minimum vertex covers of some induced subgraphs of G, and finding a maximum matching in a bipartite graph. All these computations can be performed in polynomial time [10].

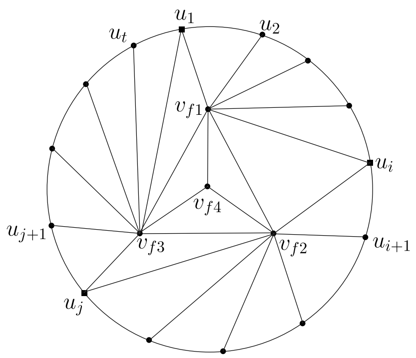

Fig. 2. Triangulating an internal face $f = u_1, u_2, \ldots, u_t, u_1$ of G with $t > 3$, by adding new vertices and edges.

3. Let G be a biconnected chordal graph. Since every vertex cover of a biconnected chordal graph forms a connected vertex cover of the graph, by Observation 3, $\text{evc}(G) \in \{\text{mvc}(G), \text{mvc}(G)+1\}$. Therefore, by using Part 1 of this theorem, evc(G) can be decided exactly, in polynomial time. If $\text{evc}(G) = \text{mvc}(G)$, using Part 2 of this theorem, we can complete the proof. If $\text{evc}(G) = \text{mvc}(G) + 1$, we will make use of the fact that every minimum vertex cover of G is connected. We will fix a minimum vertex cover S and initially place guards on all vertices of S and also on one additional vertex. Using the method given by Klostermeyer and Mynhardt [11] to show that evc(G) is at most one more than the connected vertex cover number of G, we will be able to keep defending attacks while maintaining guards on all vertices of S after the end of each round of the game. □

A class of graphs $\mathcal{H}$ is called hereditary, if every graph $G \in \mathcal{H}$ has the property that the deletion of any set of vertices from G yields another graph in $\mathcal{H}$. Chordal graphs form a well-known hereditary graph class. The proposition below gives a generalization of Theorem 2 to a class of hereditary graphs. The proof of this proposition can be obtained by an easy generalization of the proof of Theorem 2.

Proposition 1. *Let $\mathcal{H}$ be a hereditary graph class such that there is a polynomial time algorithm for computing vertex cover number of graphs in $\mathcal{H}$. Further, suppose that every connected graph $G \in \mathcal{H}$ satisfies the property that every minimum vertex cover S of G that contains all the cut vertices of G forms a connected vertex cover of G. Then, given a graph G that belongs to $\mathcal{H}$,*

1. *deciding whether* $\text{evc}(G) = \text{mvc}(G)$ *can be done in polynomial time,*
2. *if* $\text{evc}(G) = \text{mvc}(G)$*, then there is a polynomial time (per-round) strategy for guard movements using* evc(G) *guards, and*
3. *if G is biconnected, then* evc(G) *can be computed in polynomial time and there is a polynomial time (per-round) strategy for guard movements using* evc(G) *guards.*

5. A PTAS for computing eternal vertex cover number of biconnected internally triangulated planar graphs

In general, the problem of computing eternal vertex cover number is NP-hard and is only known to be in PSPACE [8]. To the best of our knowledge, no NP-completeness results for the eternal vertex cover problem were known for any graph classes, prior to this work. A graph is an *internally triangulated planar graph* if it has a planar embedding in which all internal faces are triangles. In this section, using Theorem 1, we show that the eternal vertex cover problem is NP-complete for biconnected internally triangulated planar graphs. We also show that there is a PTAS for computing the eternal vertex cover number of biconnected internally triangulated planar graphs. The section concludes by stating that the problem is in NP for a superclass of biconnected internally triangulated planar graphs.

The NP-hardness of the eternal vertex cover problem for biconnected internally triangulated planar graphs is shown using a sequence of simple reductions. First, we will show that the classical vertex cover problem is NP-hard for biconnected internally triangulated planar graphs. Then, we will show that an additive one approximation to vertex cover is also NP-hard for the same class and use it to derive the required conclusion.

Proposition 2. *Given a biconnected internally triangulated planar graph G and an integer k as input, it is NP-complete to decide if* $\text{mvc}(G) \le k$.

Proof. The vertex cover problem on biconnected planar graphs is known to be NP-hard [13]. We show a reduction from the vertex cover problem on biconnected planar graphs to the vertex cover problem on biconnected internally triangulated planar graphs. Suppose we are given a biconnected planar graph G and an integer k. We construct G' such that G is an induced subgraph of G'. First, compute a planar embedding of G in polynomial time [14]. We know that in any planar

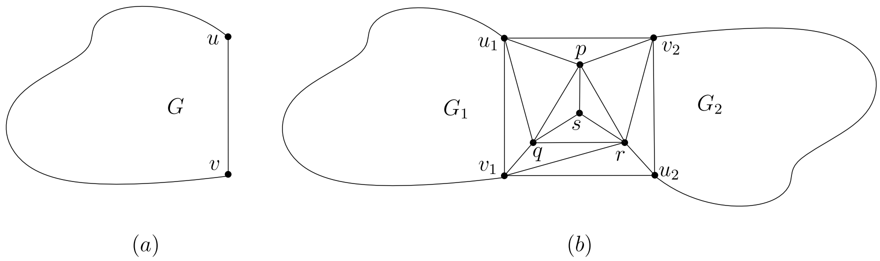

Fig. 3. NP-hardness reduction for additive one approximation of vertex cover number of biconnected internally triangulated planar graphs.

embedding, each face of a biconnected planar graph is bounded by a cycle [5]. To construct G', each internal face of G with more than three vertices on its boundary is triangulated by adding four new vertices and some edges (see Fig. 2). Let $f = u_1, u_2, \ldots, u_t, u_1$ be a cycle bounding an internal face of G, with $t > 3$. Let i and j be two distinct indices from $[2, t]$. Add three vertices v_{f1}, v_{f2} and v_{f3} inside f. Now, add edges $v_{f1}u_1, v_{f1}u_2, \ldots, v_{f1}u_i, v_{f2}u_i, v_{f2}u_{i+1}, \ldots, v_{f2}u_j, v_{f3}u_j, v_{f3}u_{j+1}, \ldots, v_{f3}u_t$ and $v_{f3}u_1$. Furthermore, add the edges $v_{f1}v_{f2}$, $v_{f2}v_{f3}$ and $v_{f3}v_{f1}$. Note that the graph remains planar after the addition of these vertices and edges. Add a new vertex v_{f4} inside the triangle formed by v_{f1}, v_{f2} and v_{f3} and make it adjacent to v_{f1}, v_{f2} and v_{f3}. Repeat this construction procedure for all internal faces of G bounded by more than 3 vertices. As per the construction, it is clear that the resultant graph G' is biconnected, internally triangulated and planar. It can be seen easily that the biconnected planar graph G has a vertex cover of size at most k if and only if the biconnected internally triangulated planar graph G' has a vertex cover of size at most $k' = k + 3f'$ where f' is the number of internal faces of G bounded by more than 3 vertices. □

To show the NP-hardness of the eternal vertex cover problem of biconnected internally triangulated planar graphs, we will show that if the vertex cover number of biconnected internally triangulated planar graphs can be approximated to within an additive one error, then it can be used to precisely compute the vertex cover number of graphs of the same class.

Proposition 3. *Getting an additive* 1*-approximation for computing the vertex cover number of biconnected internally triangulated planar graphs is NP-hard.*

Proof. Let $G(V, E)$ be the given biconnected internally triangulated planar graph. Consider a fixed planar internally triangulated embedding of G. The reduction algorithm constructs a new graph G' as follows. Make two copies of G namely, G_1 and G_2. For each vertex $v \in V$, let v_1 and v_2 denote its corresponding vertices in G_1 and G_2 respectively. Choose any arbitrary edge $e = uv$ on the outer face of G. Add new edges u_1v_2 and v_1u_2 maintaining the planarity. Now, the new graph is biconnected and planar; but the face with boundary $u_1v_1u_2v_2$ needs to be triangulated. For this, we follow the same procedure we used in the proof of Proposition 2 which adds four new vertices p, q, r, s and some new edges inside this face (see Fig. 3). The resultant graph G' is biconnected, internally triangulated and planar.

Consider a minimum vertex cover S of G such that $\mathrm{mvc}(G) = |S| = k$. It is clear that either u or v is in S. It is easy to see that $S' = \{x_1 : x \in S\} \cup \{x_2 : x \in S\} \cup \{p, q, r\}$ is a vertex cover of G' with size $2k + 3$. Similarly, at least k vertices from G_1 and G_2 and at least 3 vertices among $\{p, q, r, s\}$ has to be chosen for a minimum vertex cover of G'. This shows that $\mathrm{mvc}(G') = 2k + 3$.

Suppose there exists a polynomial time additive 1-approximation algorithm for computing the vertex cover number of biconnected internally triangulated planar graphs. Let k' be the approximate value of the vertex cover number of G' that is computed by this algorithm. Then, $k' \in \{\mathrm{mvc}(G'), \mathrm{mvc}(G')+1\}$. This implies that $\mathrm{mvc}(G) = \left\lfloor \frac{(k'-3)}{2} \right\rfloor$, giving a polynomial time algorithm to compute $\mathrm{mvc}(G)$. Hence, by Proposition 2, getting an additive 1-approximation for computing the vertex cover number of biconnected internally triangulated planar graphs is NP-hard. □

Now, we are ready to prove the NP-completeness of the eternal vertex cover problem for biconnected internally triangulated planar graphs.

Theorem 3. *Given a biconnected internally triangulated planar graph G and an integer k as input, it is NP-complete to decide if* $\mathrm{evc}(G) \leq k$*. Further, there exists a polynomial time approximation scheme for computing the eternal vertex cover number of biconnected internally triangulated planar graphs.*

Proof. First, we show that the problem of deciding whether $\mathrm{evc}(G) \leq k$, given a biconnected internally triangulated planar graph G and an integer k as input, is in NP. For this, it is sufficient to show that this problem has a polynomial time verifiable certificate. Since every minimum vertex cover of a biconnected internally triangulated planar graph is connected, by Observation 3, we can easily derive that $\mathrm{evc}(G) = \min\{k : \forall v \in V(G), G$ has a vertex cover of size k

containing v}. Using this, it is easy to obtain a polynomial time verifiable certificate to check if $\mathrm{evc}(G) \leq k$, as follows. The certificate can consist of one subset of $V(G)$ corresponding to each vertex $v \in V(G)$ such that for a yes instance of the problem, the subset corresponding to each $v \in V(G)$ is a vertex cover of G that contains v and is of size at most k.

Now, we will prove the NP-hardness of the problem. Since every vertex cover of a biconnected internally triangulated planar graph G is connected, by Observation 3, $\mathrm{mvc}(G) \leq \mathrm{evc}(G) \leq \mathrm{mvc}(G) + 1$. Therefore, a polynomial time algorithm to compute $\mathrm{evc}(G)$ would give a polynomial time additive 1-approximation for $\mathrm{mvc}(G)$. Hence, by Proposition 3, we can conclude that given a biconnected internally triangulated planar graph G and an integer k, it is NP-hard to decide if $\mathrm{evc}(G) \leq k$. This completes the proof of NP-completeness stated in the theorem.

As mentioned earlier, for a biconnected internally triangulated planar graph, $\mathrm{evc}(G) = \min\{k : \forall v \in V(G), G$ has a vertex cover of size k containing $v\}$. We also know that there is a PTAS designed by Baker et al. [3] for computing the vertex cover number of planar graphs. Combining these two facts, it is easy to derive a polynomial time approximation scheme for computing the eternal vertex cover number of biconnected internally triangulated planar graphs. □

The following proposition shows that the eternal vertex cover problem is in NP not only for biconnected internally triangulated planar graphs, but also for a superclass of it. The proof of this result is similar to the first part of the proof of Theorem 3.

Proposition 4. *Given a graph G for which every minimum vertex cover is connected and an integer k as input, deciding whether* $\mathrm{evc}(G) \leq k$ *is in NP.*

6. Complexity results of the eternal vertex cover problem for locally connected graphs

A graph G is *locally connected* if for every vertex v of G having degree at least 2, its open neighborhood $N_G(v)$ induces a connected subgraph in G. In this section, we discuss the implications of Theorem 1 to the complexity of the problem of computing eternal vertex cover problem for locally connected graphs. We show that given a locally connected graph G and an integer k as input, deciding whether $\mathrm{evc}(G) \leq k$ is NP-complete and the corresponding optimization problem is APX-hard. We also derive some related results in the context of locally connected graphs, obtained using Theorem 1.

A connected locally connected graph is always biconnected unless it is an edge or a single vertex. Biconnected chordal graphs and biconnected internally triangulated graphs are well known examples of locally connected graphs. Erdös, Palmer and Robinson [7] showed that when the probability $p(n)$ of adding an edge is $\sqrt{\frac{3\log n}{2n}}$ or higher, almost all graphs in $\mathcal{G}(n, p)$ are locally connected. Some other sufficient conditions for a graph to be locally connected were given by Chartrand and Pippert [4] and Vanderjagt [15].

A *block* in a connected graph G is either a maximal biconnected subgraph or a bridge of G. If each block of a graph is locally connected, we call it a *blockwise locally connected* graph. The following observation enables us to apply Theorem 1 to connected graphs which are blockwise locally connected.

Proposition 5. *Let* $G(V, E)$ *be a connected graph. If G is blockwise locally connected, then every vertex cover of G that contains all cut vertices of G is a connected vertex cover of G.*

Proof. For contradiction, suppose that S is a vertex cover of G containing all the cut vertices of G such that $G[S]$ is not connected. Then, since $V \setminus S$ is an independent set and G is connected, there exist a vertex $v \in V \setminus S$ and two distinct components C_1 and C_2 of $G[S]$ such that v is adjacent to vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$. As $V \setminus S$ is an independent set, we have $N_G(v) \subseteq S$. Note that in a blockwise locally connected graph, the neighborhood of every vertex that is not a cut vertex induces a connected subgraph. Since v is not a cut vertex, we then have that $N_G(v)$ induces a connected subgraph in G and therefore, v_1 and v_2 must belong to the same component of $G[S]$, which is a contradiction. □

From Proposition 5, it follows that Theorem 1 is applicable to all connected graphs which are blockwise locally connected. Combining this with Observation 3, we get the following result, which is a generalization of Observation 4.

Observation 5. *For a connected graph G that is blockwise locally connected and has at least two vertices,* $\mathrm{evc}(G) = \mathrm{mvc}(G)$ *if and only if for every vertex v of G, there is a minimum vertex cover of G that contains v and all the cut-vertices of G. Further, for any connected graph G that is locally connected,* $\mathrm{evc}(G) \in \{\mathrm{mvc}(G), \mathrm{mvc}(G) + 1\}$.

The next theorem proves the NP-completeness and the hardness of approximation of the eternal vertex cover number problem of locally connected graphs.

Theorem 4. *Given a locally connected graph G and an integer k as input, it is NP-complete to decide if* $\mathrm{evc}(G) \leq k$. *Moreover, the eternal vertex cover number of locally connected graphs cannot be approximated to within any factor smaller than* $10\sqrt{5} - 21$ *in polynomial time, unless P=NP.*

Proof. Let G be a locally connected graph. Without loss of generality, we may assume that G is biconnected. By Proposition 5, every vertex cover of G is connected and hence, by Proposition 4, we can see that deciding whether $\text{evc}(G) \leq k$ is in NP. Since biconnected internally triangulated graphs form a subclass of locally connected graphs for which the problem is NP-hard (Theorem 3), the NP-completeness of the problem for locally connected graphs follows easily.

By Observation 5, it follows that $\text{evc}(G) \in \{\text{mvc}(G), \text{mvc}(G) + 1\}$. Hence, for proving approximation hardness of the eternal vertex cover problem, it is enough to prove the approximation hardness of computing vertex cover number of locally connected graphs. We prove this below, using the approximation hardness of the vertex cover problem.

A famous result by Dinur and Safra [6] states that it is NP-hard to approximate the vertex cover number of connected graphs within any factor smaller than $10\sqrt{5} - 21$. For a given connected graph G and integer k, we can construct a locally connected graph G' by adding a new vertex to G and making it adjacent to all the existing vertices of G. It can be seen easily that $\text{mvc}(G') = \text{mvc}(G) + 1$. Therefore, even for locally connected graphs, the vertex cover number is NP-hard to approximate to within any factor smaller than $10\sqrt{5} - 21$.

As noted above, this implies that the eternal vertex cover number of locally connected graphs cannot be approximated to within any factor smaller than $10\sqrt{5} - 21$ in polynomial time, unless P=NP. □

The following is another implication of Theorem 1.

Theorem 5. *Given a blockwise locally connected graph G as input, deciding whether* $\text{evc}(G) = \text{mvc}(G)$ *is in* P^{NP}. *In particular, given an internally triangulated planar graph G as input, deciding whether* $\text{evc}(G) = \text{mvc}(G)$ *is in* P^{NP}.

Proof. Since internally triangulated graphs are blockwise locally connected, we need to only prove the first part of the theorem. Without loss of generality, we may assume that G is a connected graph that is blockwise locally connected. Using polynomially many queries to an NP oracle that answers whether G has a vertex cover of size at most k or not, we can compute $\text{mvc}(G)$. Let $t = \text{mvc}(G)$ and X be the set of cut vertices of G. Computing X can be done in polynomial time. By Theorem 1, it suffices to check whether for every vertex $v \in V \setminus X$, there exists a vertex cover S_v of G of size t such that $(X \cup \{v\}) \subseteq S_v$, which is equivalent to checking whether the graph $G \setminus (X \cup \{v\})$ has a vertex cover of size $t - |X| - 1$. This decision problem is also in NP. Thus, the entire procedure of deciding whether $\text{evc}(G) = \text{mvc}(G)$ requires only polynomially many queries to an NP oracle. □

7. Is the necessary condition sufficient?

It is interesting to ask if the necessary condition stated in Lemma 2 is sufficient for every graph to have its eternal vertex cover number and vertex cover number coincide. Here, we give a biconnected bipartite planar graph of maximum degree 4 which answers this question in negative. Consider the bipartite graph $G(X \cup Y, E)$ with $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ shown in Fig. 4. This graph consists of two copies of $K_{2,3}$ on vertex sets $\{x_1, x_2, x_3, y_4, y_5\}$ and $\{y_1, y_2, y_3, x_4, x_5\}$ connected by two edges x_1y_1 and x_5y_5. From the figure, it can be easily seen that $\text{mvc}(G) = 5$ and that for each vertex, there is a minimum vertex cover containing it. Note that, there is only one minimum vertex cover $S_{x_2} = \{x_1, x_2, x_3, x_4, x_5\}$ that contains x_2. Therefore, the attacker can force the guards into configuration S_{x_2} from any other configuration by attacking an edge incident on the vertex x_2. When the guards are in configuration S_{x_2}, if there is an attack on the edge x_5y_5, the guards have to move to a configuration containing y_5. The only minimum vertex covers of G containing y_5 are $S_1 = \{y_1, y_2, y_3, y_4, y_5\}$, $S_2 = \{x_4, x_5, y_4, y_5, x_1\}$ and $S_3 = \{x_4, x_5, y_4, y_5, y_1\}$. Since the edge x_5y_5 does not belong to any maximum matching of G, a transition from S_{x_2} to S_1 is not legal. Configurations S_2 and S_3 both contain x_5. Following the attack on x_5y_5 in configuration S_{x_2}, when the guard on x_5 moves to y_5, no other guard can move to x_5, because no neighbor of x_5 is occupied in S_{x_2}. Thus, transitions to S_2 and S_3 are also not legal. Hence, the attack on x_5y_5 cannot be handled and therefore $\text{evc}(G) \neq \text{mvc}(G)$.

This example shows that the necessary condition is not sufficient for planar graphs or bipartite graphs, even when they are biconnected.

8. A graph G with an edge not contained in any maximum matching but $\text{mvc}(G) = \text{evc}(G)$

Klostermeyer and Mynhardt [11] proved that if a graph G has two disjoint minimum vertex covers and each edge is contained in a maximum matching then $\text{mvc}(G) = \text{evc}(G)$. They had asked whether in a graph G having $\text{mvc}(G) = \text{evc}(G)$, it is necessary for every edge to belong to some maximum matching. Here, we give a graph G for which the answer is negative. The graph G shown in Fig. 5 has $\text{mvc}(G) = 5$. Therefore, $\text{evc}(G) \geq 5$. It can also be easily seen that G has a maximum matching of size 4 and that the edge $(8, 4)$ is not contained in any maximum matching. It can be verified that G has an evc class with two configurations, namely $S_1 = \{1, 8, 3, 5, 7\}$ and $S_2 = \{6, 8, 2, 4, 7\}$, and hence $\text{evc}(G) = 5$. Note that the graph we constructed is a biconnected internally triangulated planar graph.

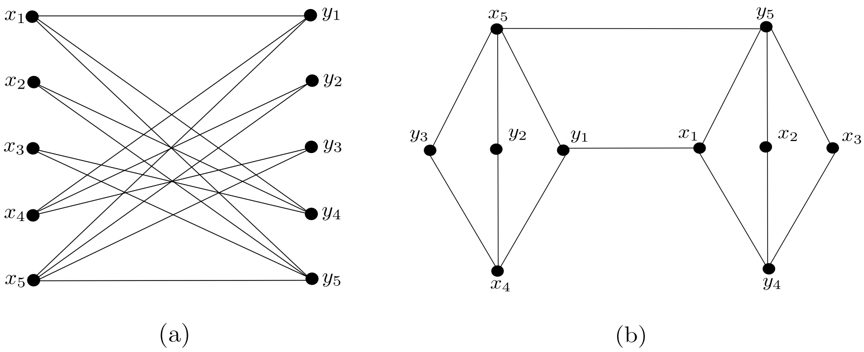

Fig. 4. (a) A biconnected bipartite planar graph with all vertices in some minimum vertex cover and $evc(G) \neq mvc(G)$. (b) A planar drawing of the same graph.

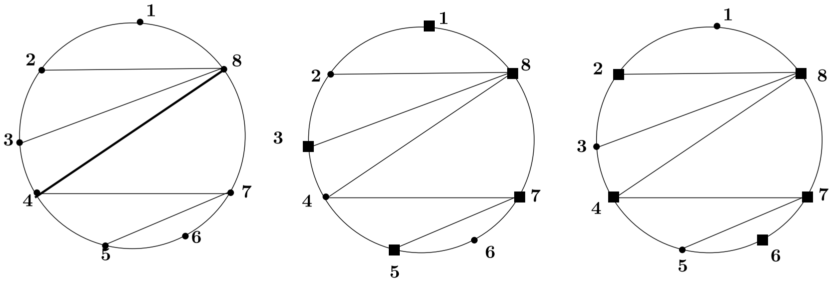

Fig. 5. $\text{mvc}(G) = \text{evc}(G) = 5$ and size of maximum matching is 4. Edge $(8, 4)$ is not contained in any maximum matching.

9. Conclusion and open problems

This paper presents an attempt to derive a characterization of graphs for which the eternal vertex cover number coincides with the vertex cover number. A characterization that works for a graph class that includes chordal graphs and internally triangulated planar graphs is obtained. The characterization is derived from a simple to state necessary condition; and has several implications, including a polynomial time algorithm for deciding whether a chordal graph G has $\text{evc}(G) = \text{mvc}(G)$ and a polynomial time algorithm for computing the eternal vertex cover number of biconnected chordal graphs. A characterization of bipartite graphs for which the eternal vertex cover number and vertex cover number coincide remains open.

The characterization also leads to NP-completeness results for the eternal vertex cover problem on some graph classes like locally connected graphs and biconnected internally triangulated planar graphs. Even though it was known that the general problem is NP-hard, to the best of our knowledge, results obtained here are the first NP-completeness results known for the eternal vertex cover problem.

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