Supplementary information for Characterizing decoherence rates of a superconducting qubit by direct microwave scattering

Yong Lu,[1,*] Andreas Bengtsson,[1] Jonathan J. Burnett,[1,2] Emely Wiegand,[1] Baladitya Suri,[1,3] Philip Krantz,[1] Anita Fadavi Roudsari,[1] Anton Frisk Kockum,[1] Simone Gasparinetti,[1] Göran Johansson,[1] and Per Delsing[1,†]

[1] *Department of Microtechnology and Nanoscience (MC2), Chalmers University of Technology, SE-412 96 Göteborg, Sweden*

[2] *National Physical Laboratory, Hampton Road, Teddington, Middlesex, TW11 0LW, United Kingdom*

[3] *Indian Institute of Science, Department of Instrumentation and Applied Physics, Bangalore 560012, India*

A. Power Spectrum

Here, we follow the method in Ref. [1] to calculate our circuit model. Our qubit Hamiltonian is ($\hbar = 1$)

$$H = -\frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x, \tag{1}$$

where $\Delta = \omega_{\mathrm{p}} - \omega_{01}$; ω_{p} and ω_{01} are the pump frequency and the qubit $|0\rangle \leftrightarrow |1\rangle$ transition frequency, respectively.

The Lindblad master equation, describing the qubit dynamics with decoherence included, is given by

$$\frac{d}{dt}\rho = \mathcal{L}\rho = -i[H,\rho] + \mathcal{L}_\gamma \rho, \tag{2}$$

where the Liouvilian $\mathcal{L}_\gamma$ is

$$\mathcal{L}_\gamma \rho = \Gamma_1 D[\sigma_-]\rho + \frac{\Gamma_\phi}{2} D[\sigma_z]\rho, \tag{3}$$

in which $D[c]\rho = c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)$.

In the frame rotating with ω_{p}, the corresponding equations of motion for $s_1(t) \equiv \rho_{10}(t) = \langle \sigma_-(t)\rangle\, e^{i\omega_{\mathrm{p}} t}$ and $s_2(t) \equiv \rho_{11}(t) = \langle \sigma_+(t)\sigma_-(t)\rangle$ are obtained from Eq. (2)

$$\frac{d}{dt}\begin{pmatrix} s_1 \\ s_1^* \\ s_2 \end{pmatrix} = M \begin{pmatrix} s_1 \\ s_1^* \\ s_2 \end{pmatrix} + B, \tag{4}$$

where

$$M = \begin{pmatrix} i\Delta - \Gamma_2 & 0 & i\Omega \\ 0 & -i\Delta - \Gamma_2 & -i\Omega \\ i\Omega/2 & -i\Omega/2 & -\Gamma_1 \end{pmatrix}, B = \begin{pmatrix} -i\Omega/2 \\ i\Omega/2 \\ 0 \end{pmatrix}. \tag{5}$$

Here, Γ_1, Γ_ϕ, and $\Gamma_2 = \frac{1}{2}\Gamma_1 + \Gamma_\phi$ are the total relaxation rate of the qubit, the pure dephasing rate, and the decoherence rate, respectively.

The qubit reaches its stationary state for $t \gg \Gamma_{1,2}^{-1}$. The stationary values, $\overline{s}_1 = s_1(\infty)$ and $\overline{s}_2 = s_2(\infty)$, are

$$\overline{s_1} = \frac{\Omega\Gamma_1(\Delta - i\Gamma_2)}{2(\Omega^2\Gamma_2 + \Gamma_1(\Delta^2 + \Gamma_2^2))}, \tag{6}$$

$$\overline{s_2} = \frac{\Omega^2\Gamma_2}{2(\Omega^2\Gamma_2 + \Gamma_1(\Delta^2 + \Gamma_2^2))}. \tag{7}$$

* e-mail:yongl@chalmers.se

† e-mail:per.delsing@chalmers.se

To determine the two-time correlation function of the atom, three quantities are defined:

$$s_3(\tau) = \langle \sigma_+(t)\sigma_-(t+\tau)\rangle\, e^{i\omega_{\mathrm{p}}\tau}, \tag{8}$$

$$s_4(\tau) = \langle \sigma_+(t)\sigma_+(t+\tau)\rangle\, e^{-i\omega_{\mathrm{p}}(2t+\tau)}, \tag{9}$$

$$s_5(\tau) = \langle \sigma_+\sigma_+(t+\tau)\sigma_-(t+\tau)\rangle\, e^{-i\omega_{\mathrm{p}}\tau}, \tag{10}$$

all of which are time-independent when stationary. From Eq. (4), we have equations of motion for these quantities as

$$\frac{d}{dt}\begin{pmatrix} s_3 \\ s_4 \\ s_5 \end{pmatrix} = M \begin{pmatrix} s_3 \\ s_4 \\ s_5 \end{pmatrix} + B, \tag{11}$$

with initial values $s_3(0) = \overline{s}_2$ and $s_4(0) = s_5(0) = 0$. In the $\tau \to \infty$ limit, the stationary values are $\overline{s}_3 = |s_1|^2$, $\overline{s}_4 = (\overline{s}_1^*)^2$, and $\overline{s}_5 = \overline{s}_1^*\overline{s}_2$. Using new variables, $\delta s_j(\tau) = s_j(\tau) - \overline{s}_j (j = 3,4,5)$, the above equations are rewritten as

$$\frac{d}{dt}\begin{pmatrix} \delta s_3 \\ \delta s_4 \\ \delta s_5 \end{pmatrix} = M \begin{pmatrix} \delta s_3 \\ \delta s_4 \\ \delta s_5 \end{pmatrix} = M * \delta S. \tag{12}$$

Here, $\delta s_3(\infty) = \delta s_4(\infty) = s_5(\infty) = 0$. Taking the Fourier transforms of $\delta s_j(\tau)$ by $I_j(\omega) = \int_0^\infty d\tau e^{i(\omega-\omega_{\mathrm{p}})\tau}\delta s_j(\tau)$ with partial integration, we have

$$I(\omega) = [M + i(\omega - \omega_{\mathrm{p}})\mathbb{1}]^{-1}[\lim_{\tau\to\infty} \delta S(\tau) e^{i(\omega-\omega_{\mathrm{p}})\tau} - \delta S(0)]. \tag{13}$$

Because $\lim_{\tau\to\infty} \delta S(\tau) e^{i(\omega-\omega_{\mathrm{p}})\tau} = 0$,

$$I(\omega) = -[M + i(\omega - \omega_{\mathrm{p}})\mathbb{1}]^{-1}\delta S(0). \tag{14}$$

Specifically, $I_3(\omega)$ is given by

$$I_3(\omega) = \frac{|\overline{s}_1|^2 - \overline{s}_2}{\mu_1} + \frac{\Omega^2(\overline{s}_1^*)^2 - \Omega^2(|\overline{s}_1|^2 - \overline{s}_2)\mu_2/\mu_1 - 2i\Omega\overline{s}_1^*\overline{s}_2\mu_2}{2\mu_1\mu_2\mu_3 + \Omega^2(\mu_1 + \mu_2)}, \tag{15}$$

where $\mu_1 = -\Gamma_2 + i\delta\omega_{01}$, $\mu_2 = -\Gamma_2 + i(\omega + \omega_{01} - 2\omega_{\mathrm{p}})$, and $\mu_3 = -\Gamma_1 + i\delta\omega_{01}$ with $\delta\omega_{01} = \omega - \omega_{01}$. Combining the above results, the incoherent part of the spectrum is obtained as

$$S_{\mathrm{i}}(\omega) = \frac{\Gamma_{\mathrm{r}}}{\pi} Re[I_3(\omega)], \tag{16}$$

which is the same as Ref [1].

When the pump is on resonance with the qubit, if the pump power is strong ($\Omega \gg \Gamma_{1,2}$), $s_2 \approx \frac{1}{2}$ and $s_1 \approx \frac{-i\Gamma_1}{2\Omega}$. Then, $I_3(\omega)$ can be simplified to

$$\begin{aligned} I_3(\omega) &\approx -\frac{1}{4\mu_1} - \frac{1}{4\mu_1}\frac{\mu_3+\Gamma_1}{\mu_1\mu_3+\Omega^2} \\ &\approx -\frac{1}{4\mu_1} - \frac{1}{4}\left\{\frac{1}{\Gamma_s+i(\delta\omega_{01}+\Omega)} + \frac{1}{\Gamma_s+i(\delta\omega_{01}-\Omega)}\right\}, \end{aligned} \tag{17}$$

where $\Gamma_\mathrm{r} = \Gamma_1 - \Gamma_\mathrm{n}$ and $\Gamma_\mathrm{s} = \frac{\Gamma_1+\Gamma_2}{2}$. Therefore, Eq. (16) becomes

$$\begin{aligned} S_\mathrm{i}(\omega) \approx \frac{1}{\pi}\frac{\hbar\omega_{01}\Gamma_\mathrm{r}}{4}\left\{\frac{\Gamma_\mathrm{s}}{(\delta\omega_{01}+\Omega)^2+\Gamma_\mathrm{s}^2}\right. \\ \left. + \frac{2\Gamma_2}{(\delta\omega_{01})^2+\Gamma_2^2} + \frac{\Gamma_\mathrm{s}}{(\delta\omega_{01}-\Omega)^2+\Gamma_\mathrm{s}^2}\right\}. \end{aligned} \tag{18}$$

B. Asymmetric Mollow triplet

In this section, we explain how dephasing leads to asymmetry in the off-resonant Mollow triplet. For the driven qubit, the states in the dressed-state basis can be written as

$$|n,+\rangle = \sin\theta|g,n+1\rangle + \cos\theta|e,n\rangle, \tag{19}$$

$$|n,-\rangle = \cos\theta|g,n+1\rangle - \sin\theta|e,n\rangle, \tag{20}$$

where $|g\rangle$ ($|e\rangle$) is the ground (excited) state of the qubit, n is the number of drive photons, and θ is defined by

$$\tan 2\theta = -\frac{\sqrt{\Delta^2+\Omega^2}}{\Delta}. \tag{21}$$

A sketch of the dressed states is shown in Fig. 4(a). To find the transition rates between the dressed states caused by relaxation, i.e., coupling of an environment to σ_x, we calculate the matrix elements

$$\langle n,+|\sigma_x|n+1,+\rangle = \sin\theta\cos\theta, \tag{22}$$

$$\langle n,-|\sigma_x|n+1,+\rangle = \cos^2\theta, \tag{23}$$

$$\langle n,+|\sigma_x|n+1,-\rangle = -\sin^2\theta, \tag{24}$$

$$\langle n,-|\sigma_x|n+1,-\rangle = -\sin\theta\cos\theta. \tag{25}$$

Thus, Fermi's golden rule gives that the transition rates are

$$\Gamma_{++} \propto \sin^2\theta\cos^2\theta, \tag{26}$$

$$\Gamma_{+-} \propto \cos^4\theta, \tag{27}$$

$$\Gamma_{-+} \propto \sin^4\theta, \tag{28}$$

$$\Gamma_{--} \propto \sin^2\theta\cos^2\theta. \tag{29}$$

In the case of resonant drive, $\Delta = 0$, we have $\theta = \pi/4$ and all the transition matrix elements are equal.

As illustrated in Fig. 4(b), the transitions caused by relaxation are either between or within the $+$ and $-$ subspaces. Due to energy conservation, the product of the transition rate from the $+$ subspace to the $-$ subspace, Γ_{+-}, and the occupation probability of state this subspace, P_+, equals the product of the transition rate from the $-$ subspace to the $+$ subspace, Γ_{-+}, and the occupation probability of this subspace, P_-:

$$\Gamma_{+-}P_+ = \Gamma_{-+}P_-. \tag{30}$$

If the drive is off-resonant, the transition rates are not the same. For $\delta < 0$, we have $\Gamma_{+-} > \Gamma_{-+}$, and for $\delta > 0$, we have $\Gamma_{-+} > \Gamma_{+-}$, i.e., the sideband that is closest to the qubit frequency has the highest transition rate. However, the emission spectrum is still symmetric, since the number of emitted photons in each sideband is given by the product the corresponding occupation probability and transition rate.

The presence of pure dephasing adds an additional term $H_\phi \propto \sigma_z(a+a^\dagger)$, where a and $a^\dagger$ are annihilation and creation operators for a bath, to the interaction Hamiltonian. The effect that this has on the dressed states can be understood by calculating the transition-matrix elements of σ_z between the dressed states. We find

$$\langle n,+|\sigma_z|n,-\rangle = -2\sin\theta\cos\theta. \tag{31}$$

All matrix elements of σ_z for transitions between states with different number of drive photons are zero. The pure dephasing thus causes transitions as sketched in Fig. 4(a) and (b). Both upward and downward transitions are almost equally likely, since the corresponding transition energies are small compared to k_BT.

The pure dephasing thus modifies the condition for equilibrium from Eq. (30) to

$$(\Gamma_{+-}+\Gamma_\phi)P_+ = (\Gamma_{-+}+\Gamma_\phi)P_-. \tag{32}$$

This means that a non-zero Γ_ϕ pushes the state of the system closer to $P_- = P_+$ than it otherwise would have been. However, since the transition rates corresponding to relaxation remain the same as before, the result is that more photons are emitted at the frequency of the transition with the larger transition rate. This leads to an asymmetric power spectrum, where more photons are emitted in the sideband closest to the qubit frequency than in the sideband furthest away from the qubit frequency.

C. Reflection coefficient

From the input-output relation, the output coherent field α_out is the sum of the incoming coherent field α_in and the field emitted by the atom:

$$\alpha_\mathrm{out} = \alpha_\mathrm{in} - i\sqrt{\Gamma_\mathrm{r}}\,\langle\sigma_-(t)\rangle\,, \tag{33}$$

where $\alpha_\mathrm{in} = \frac{\Omega}{2\sqrt{\Gamma_\mathrm{r}}}$. Combining this with Eq. (7), the reflection coefficient, $r = \frac{\alpha_\mathrm{out}}{\alpha_\mathrm{in}}$, becomes:

$$r = 1 - \frac{i\Gamma_\mathrm{r}\Gamma_1(\Delta - i\Gamma_2)}{\Omega^2\Gamma_2 + \Gamma_1(\Delta^2+\Gamma_2^2)}. \tag{34}$$

In the case of a weak probe ($\Omega \ll \Gamma_2$), Eq. (34) becomes

$$r = 1 - \frac{i\Gamma_{\rm r}}{\Delta + i\Gamma_2}. \tag{35}$$

For a resonant probe ($\Delta = 0$), Eq. (35) is simplified to

$$r = 1 - \frac{1}{\frac{\Omega^2}{\Gamma_1\Gamma_{\rm r}} + \frac{\Gamma_2}{\Gamma_{\rm r}}}. \tag{36}$$

D. Power Dissipation

At resonance ($\Delta = 0$), the input power is given by (setting $\hbar\omega_{01} = 1$)

$$P_{\rm in} = |\alpha_{\rm in}|^2 = \Omega^2/(4\Gamma_{\rm r}). \tag{37}$$

The output power is a sum of coherent and incoherent contributions:

$$P_{\rm out} = P_{\rm coh} + P_{\rm incoh}, \tag{38}$$

with

$$P_{\rm coh} = P_{\rm in}|r|^2 = \frac{\Omega^2}{4\Gamma_{\rm r}}(1 - \frac{\Gamma_1\Gamma_{\rm r}}{\Omega^2 + \Gamma_2\Gamma_1})^2, \tag{39}$$

$$\begin{aligned} P_{\rm incoh} &= \Gamma_{\rm r}(\langle\sigma_+\sigma_-\rangle - \langle\sigma_+\rangle\langle\sigma_-\rangle) \\ &= \frac{\Gamma_{\rm r}}{2}\frac{\Omega^2(\Gamma_1\Gamma_\phi + \Omega^2)}{(\Gamma_1\Gamma_2 + \Omega^2)^2}. \end{aligned} \tag{40}$$

In particular, when $\Gamma_\phi \ll \Gamma_1$, we have $P_{\rm incoh} \simeq 2\Gamma_{\rm r}\rho_{11}^2$. The net power loss is $P_{\rm loss} = P_{\rm in} - P_{\rm coh} - P_{\rm incoh}$.

$$P_{\rm loss} = \Gamma_{\rm n}\frac{\Omega^2}{2(\Gamma_1\Gamma_2 + \Omega^2)} = \Gamma_{\rm n}\rho_{11}. \tag{41}$$

When $\Omega^2 \gg \Gamma_1\Gamma_2$, the qubit is saturated. Then, we have $P_{\rm in} \approx P_{\rm coh} \approx \frac{\Omega^2}{4}$, $P_{\rm incoh} \approx \frac{\Gamma_{\rm r}}{2}$, and $P_{\rm loss} \approx \frac{\Gamma_{\rm n}}{2}$.

[1] Koshino, K. & Nakamura, Y. Control of the radiative level shift and linewidth of a superconducting artificial atom through a variable boundary condition. *New Journal of Physics* **14**, 043005 (2012).