ELSEVIER

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Interval hulls of N -matrices and almost P -matrices

Projesh Nath Choudhury^{a,*}, M. Rajesh Kannan^b

^a Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

^b Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

ARTICLE INFO

Article history:

Received 9 October 2020

Accepted 13 January 2021

Available online 19 January 2021

Submitted by P. Semrl

MSC:

primary 15B48

secondary 15A24, 65G30

Keywords:

Sign non-reversal property

Interval hull of matrices

N -matrices

Almost P -matrices

Semipositive matrices

ABSTRACT

We establish a characterization of almost P -matrices via a sign non-reversal property. In this we are inspired by the analogous results for N -matrices. Next, the interval hull of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, denoted by $\mathbb{I}(A, B)$, is the collection of all matrices $C \in \mathbb{R}^{n \times n}$ such that each c_{ij} is a convex combination of a_{ij} and b_{ij} . Using the sign non-reversal property, we identify a finite subset of $\mathbb{I}(A, B)$ that determines if all matrices in $\mathbb{I}(A, B)$ are N -matrices/almost P -matrices. This provides a test for an entire class of matrices simultaneously to be N -matrices/almost P -matrices. We also establish analogous results for semipositive and minimally semipositive matrices. These characterizations may be considered similar in spirit to that of P -matrices by Bialas-Garloff [1] and Rohn-Rex [16], and of positive definite matrices by Rohn [15].

© 2021 Elsevier Inc. All rights reserved.

* Corresponding author.

E-mail addresses: projeshc@iisc.ac.in, projeshnc@alumni.iitm.ac.in (P.N. Choudhury), rajeshkannan@maths.iitkgp.ac.in, rajeshkannan1.m@gmail.com (M.R. Kannan).

1. Introduction

Let $\mathbb{R}^{m \times n}$ denote the space of all real $m \times n$ matrices. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$, the interval hull of the matrices A and B , denoted by $\mathbb{I}(A, B)$, is defined as follows:

$$\mathbb{I}(A, B) = \{C \in \mathbb{R}^{m \times n} : c_{ij} = t_{ij}a_{ij} + (1 - t_{ij})b_{ij}, t_{ij} \in [0, 1]\}.$$

If $A \neq B$, the interval hull contains uncountably many matrices. One of the interesting questions, related to interval hulls of matrices, considered in the literature is the following: Suppose a finite subset of matrices in $\mathbb{I}(A, B)$ has some property, say $\mathcal{S}$. Does the entire interval hull $\mathbb{I}(A, B)$ have the property $\mathcal{S}$? For example, it was shown in [11] that if the matrices $A, B \in \mathbb{R}^{n \times n}$ and $A \leq B$ (entrywise), then all the matrices in the interval hull $\mathbb{I}(A, B)$ are invertible and they each have an entrywise nonnegative inverse if and only if $(A, B$ are invertible and) A^{-1}, B^{-1} are entrywise nonnegative. In [15], the author considered the positive definiteness and stability of the interval hulls of matrices. For a collection of several matrix classes having interval hull characterizations, see the recent survey [6].

In this article, given $A, B \in \mathbb{R}^{m \times n}$, we provide necessary and sufficient conditions for the entire interval hull of matrices $\mathbb{I}(A, B)$ to be contained in one of the following classes: N -matrices, almost P -matrices, (minimally) semipositive matrices, by reducing it in each case to a finite set of test matrices.

Definition 1.1. Let $m, n \geq 1$ be integers.

1. A matrix $A \in \mathbb{R}^{n \times n}$ is an N -matrix if all its principal minors are negative.
2. A matrix $A \in \mathbb{R}^{n \times n}$ ($n \geq 2$) is an *almost P -matrix* if all its proper principal minors are positive, and the determinant of A is negative.
3. An $m \times n$ real matrix A is a *semipositive matrix*, if there exists a vector $x \geq 0$ such that $Ax > 0$. An $m \times n$ real matrix A is a *minimally semipositive matrix*, if it is semipositive and no column-deleted submatrix of A is semipositive.
4. Given matrices A and B , we say that the interval hull $\mathbb{I}(A, B)$ is an N -matrix if all the matrices in $\mathbb{I}(A, B)$ are N -matrices. Similarly for the other classes of matrices discussed in this article.

N -matrices were introduced by Inada in 1971 [7]. These matrices have rich applications in univalence theory (injectivity of differential maps in $\mathbb{R}^n$) and the Linear Complementarity Problem [14]. Recently in [3], the first author in joint work with Tsatsomeros has established an algorithm to detect whether a given matrix is an N -matrix or not; as well as an algorithm to construct every N -matrix recursively. The sign non-reversal property for N -matrices was established by Mohan–Sridhar [13], and Parthasarathy–Ravindran [14]. Coming to the other classes of matrices studied in this work: (a) The concept of almost P -matrices was introduced by Ky Fan in 1966 [4]. A characterization of almost P -matrices (with nonpositive off diagonal entries) in terms of the Linear

Complementarity Problem was discussed by Miao [12]. **(b)** The notion of a semipositive matrix was considered by Stiemke [17] in connection with the problem of existence of positive solutions of linear systems. In 1994, Johnson, Kerr, and Stanford [8] introduced the notion of minimally semipositive matrices. These classes of matrices play a vital role in the study of M -matrices, in convergence theory for sets of matrices and in linear programming [18]. In this paper, we establish a characterization of almost P -matrices using the sign non-reversal property (Theorems 4.2, 4.4). We further obtain the interval hull characterization for all of these classes. Our results are summarized in Table 1.

Table 1

Summary of results. Here, J denotes a nonempty proper subset of $\langle n \rangle = \{1, \dots, n\}$.

Matrix Class	Sign non-reversal property	$\mathbb{I}(A, B)$	Testing set for $\mathbb{I}(A, B)$
N -matrices of the first category with respect to J	[13, Theorem 4.3]	Theorem 3.7	$I_z, z \in \{\pm 1\}^n \setminus \{\pm e^J\}$
N -matrices of the second category	[14, Theorem 2]	Theorem 3.6	$I_z, z \in \{\pm 1\}^n \setminus \{\pm e\}$
Almost P -matrices of the first category with respect to J	Theorem 4.4	Theorem 4.5	$I_z, z \in \{\pm 1\}^n, I_{P_J}$
Almost P -matrices of the second category	Theorem 4.2	Theorem 4.3	$I_z, z \in \{\pm 1\}^n, I_u$
Semipositive	N/A	Theorem 5.1	I_l
Minimally semipositive	N/A	Theorem 5.1	I_l, I_u

We conclude by explaining the notation used above.

Definition 1.2. Fix integers $m, n \geq 1$ and matrices $A, B \in \mathbb{R}^{m \times n}$, with interval hull $\mathbb{I}(A, B)$.

1. Define the matrices

$$(I_u)_{ij} := \max\{a_{ij}, b_{ij}\}, \quad (I_l)_{ij} := \min\{a_{ij}, b_{ij}\}, \quad I_c := \frac{B + A}{2}, \quad \Delta := \frac{I_u - I_l}{2}.$$

2. If $m = n$, given $z = (z_1, \dots, z_n) \in \{\pm 1\}^n$, define the matrices

$$D_z := \text{diag}(z_1, \dots, z_n), \quad I_z := I_c - D_z \Delta D_z.$$

This article is organized as follows: In section 2, we collect the needed definitions and known results. Section 3 contains results for the interval hull of N -matrices. In Section 4, we establish a characterization of almost P -matrices via a sign non-reversal property and as an application, we study the interval hull of such matrices. Section 5 contains similar results for semipositive and minimally semipositive matrices.

2. Preliminaries

We begin with notation, which will be used throughout the paper without further reference. For a matrix $A \in \mathbb{R}^{m \times n}$, $A \geq 0$ ($A > 0$) signifies that all the components of

the matrix A are nonnegative (positive), and let $|A| := (|a_{ij}|)$. For any positive integer n , define $\langle n \rangle := \{1, \dots, n\}$. Let $\mathbb{R}_{\pm}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \pm x_i \geq 0 \text{ for all } i \in \langle n \rangle\}$. For a subset X of $\mathbb{R}^n$, let $\text{int}(X)$ denote the interior of X in $\mathbb{R}^n$ with respect to the Euclidean metric. Let e^i denote the vector whose i -th entry is 1, and other entries are zero. For $J \subseteq \langle n \rangle$, define $e^J \in \mathbb{R}^n$ such that $e_i^J = 1$ for all $i \in J$ and $e_i^J = -1$ for all $i \notin J$. Also define $e := e^{\langle n \rangle}$.

A square matrix A is a P -matrix if all its principal minors are positive. In [5], Gale and Nikaidô established the sign non-reversal property for P -matrices.

Theorem 2.1 (*Sign non-reversal property*). *A matrix $A \in \mathbb{R}^{n \times n}$ is a P -matrix if and only if $x \in \mathbb{R}^n$ and $x_i(Ax)_i \leq 0$ for all i imply $x = 0$.*

Using the sign non-reversal property of P -matrices, in [16], Rohn and Rex showed that the interval hull of matrices $\mathbb{I}(A, B)$, where $A \leq B$, is a P -matrix, if a finite collection of matrices in $\mathbb{I}(A, B)$ are P -matrices. Such a finite characterization of interval of P -matrices was first proved by Białas and Garloff [1], formulated in different terms.

In order to prove our results, we also require two basic lemmas. The first is straightforward:

Lemma 2.2. *Let $A, B \in \mathbb{R}^{m \times n}$. Then $I_l, I_u \in \mathbb{I}(A, B)$. If $m = n$, then $I_z \in \mathbb{I}(A, B)$ for all $z \in \{\pm 1\}^n$.*

Remark 2.3. By Lemma 2.2, the necessary condition in the characterizations of the interval hulls below (Theorems 3.6, 3.7, 4.3, 4.5 and 5.1) is obvious.

The next lemma is precisely [16, Theorem 2.1]. We provide the proof for completeness.

Lemma 2.4. *Let $A, B \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$. Let $z \in \{\pm 1\}^n$ such that $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ if $x_i < 0$. If $C \in \mathbb{I}(A, B)$, then*

$$x_i(Cx)_i \geq x_i(I_z x)_i \quad \text{for all } i \in \langle n \rangle.$$

Proof. Let $C \in \mathbb{I}(A, B)$. Then $I_l \leq C \leq I_u$. Since $I_l = I_c - \Delta$ and $I_u = I_c + \Delta$, so

$$I_c - \Delta \leq C \leq I_c + \Delta.$$

Let $x \in \mathbb{R}^n \setminus \{0\}$. For fixed $1 \leq i \leq n$, we have

$$|x_i((C - I_c)x)_i| \leq |x_i|(|C - I_c||x|)_i \leq |x_i|(\Delta|x|)_i.$$

Hence

$$x_i(Cx)_i \geq x_i(I_c x)_i - |x_i|(\Delta|x|)_i.$$

Since $|x| = D_z x$, so $x_i(Cx)_i \geq x_i((I_c - D_z \Delta D_z)x)_i$. $\square$

3. Results for N -matrices

We now characterize the interval hull property for N -matrices (see Definition 1.1). First recall that an N -matrix A is of the *first category* if it has at least one positive entry. Otherwise, A is of the *second category*.

The following result gives a characterization for N -matrices of the second category. This is known as the sign non-reversal property for the N -matrices of the second category.

Theorem 3.1. [14, Theorem 2] *Let $A \in \mathbb{R}^{n \times n}$. Then A is an N -matrix of the second category if and only if $A < 0$ and A does not reverse the sign of any non-unisigned vector, that is, $x \in \mathbb{R}^n$ and $x_i(Ax)_i \leq 0$ for all i imply $x \leq 0$ or $x \geq 0$.*

We next present the analogous characterization for N -matrices of the first category. This requires the following notation.

Definition 3.2. Let $n \geq 1$ be an integer, J be a nonempty proper subset of $\langle n \rangle$, and $A \in \mathbb{R}^{n \times n}$.

1. J^c henceforth denotes $\langle n \rangle \setminus J$.
2. For subsets $I, J \subset \langle n \rangle$ with elements arranged in ascending order, A_{IJ} denotes the submatrix of A whose rows and columns are indexed by I and J , respectively.
3. An N -matrix A is an N -matrix of the first category with respect to J if

$$A_{JJ} < 0, A_{J^c J^c} < 0, A_{JJ^c} > 0 \text{ and } A_{J^c J} > 0. \quad (3.1)$$

Notice, this uniquely determines the set $\{J, J^c\}$.

These definitions are motivated by the following characterization of Mohan and Sridhar.

Theorem 3.3. [13, Theorem 4.3]

1. Every N -matrix of the first category has a representation of the form (3.1) for some (unique) nonempty proper subset $J \subset \langle n \rangle$.
2. Suppose a matrix $A \in \mathbb{R}^{n \times n}$ has a representation (3.1). Then A is an N -matrix of the first category with respect to J if and only if whenever $x_i(Ax)_i \leq 0$ for all $i \in \langle n \rangle$, we have either $x_J \leq 0$ and $x_{J^c} \geq 0$, or $x_J \geq 0$ and $x_{J^c} \leq 0$.

The second result here is known as the sign non-reversal property for N -matrices of the first category with respect to J .

Using these results, we present a necessary condition for the interval hull of N -matrices similar in spirit to that of Theorems 3.1 and 3.3. Notice that the inequality holds uniformly here.

Lemma 3.4. *Let $A, B \in \mathbb{R}^{n \times n}$ such that $\mathbb{I}(A, B)$ is an N -matrix of the second category. Then for each $x \in \mathbb{R}^n \setminus \{0\}$ with $x \not\geq 0$ and $x \not\leq 0$, there exists $i \in \langle n \rangle$ such that*

$$x_i(Cx)_i > 0 \text{ for all } C \in \mathbb{I}(A, B).$$

Proof. Let $x \in \mathbb{R}^n \setminus \{0\}$ with $x \not\geq 0$ and $x \not\leq 0$. Let $z \in \{\pm 1\}^n$ such that $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ if $x_i < 0$. By Lemma 2.2, I_z is N -matrix of the second category. By Theorem 3.1, there exists $i \in \langle n \rangle$ such that $x_i(I_z x)_i > 0$. Thus by Lemma 2.4, $x_i(Cx)_i \geq x_i(I_z x)_i > 0$ for all $C \in \mathbb{I}(A, B)$. $\square$

Remark 3.5. The above result has an analogue for N -matrices of the first category with respect to J for each $x \in \mathbb{R}^n \setminus \{0\}$ with $x_J \not\leq 0$ or $x_{J^c} \not\geq 0$, and $x_J \not\geq 0$ or $x_{J^c} \not\leq 0$. We leave the details to the interested reader.

We now characterize interval hulls of N -matrices by reducing it to a finite set of test matrices, beginning with those of the second category.

Theorem 3.6. *Let $A, B \in \mathbb{R}^{n \times n}$ such that $\max\{a_{ii}, b_{ii}\} < 0$ for all $i \in \langle n \rangle$. Then, $\mathbb{I}(A, B)$ is an N -matrix of the second category if and only if I_z is an N -matrix of the second category for all $z \in \{\pm 1\}^n \setminus \{\pm e\}$.*

Proof. Suppose that I_z is an N -matrix of the second category for all $z \in \{\pm 1\}^n \setminus \{\pm e\}$. Let $C \in \mathbb{I}(A, B)$. First we show that $C < 0$. For $i \in \langle n \rangle$, define $z^i \in \{\pm 1\}^n$ such that $z^i_i = 1$ and $z^i_j = -1$ for $j \neq i$. Then $I_{z^i} < 0$ for all $i \in \langle n \rangle$, since I_{z^i} is an N -matrix of the second category. Thus $(I_u)_{ij} < 0$ for $j \in \langle n \rangle$. Hence $C \leq I_u < 0$. Let $x \in \mathbb{R}^n$ such that $x \not\geq 0$ and $x \not\leq 0$. By Lemma 2.4, there exists $z \in \{\pm 1\}^n \setminus \{\pm e\}$ such that $x_i(Cx)_i \geq x_i(I_z x)_i$, for all $i \in \langle n \rangle$. By Theorem 3.1, there exists $i \in \langle n \rangle$ such that $0 < x_i(I_z x)_i \leq x_i(Cx)_i$, since I_z is N -matrix of the second category. Hence C is an N -matrix of the second category by Theorem 3.1. $\square$

We next characterize the interval hull of N -matrices of the first category with respect to J , where $J \subset \langle n \rangle$.

Theorem 3.7. *Let J be a nonempty proper subset of $\langle n \rangle$, and let $A, B \in \mathbb{R}^{n \times n}$ such that $\max\{a_{ii}, b_{ii}\} < 0$ for all $i \in \langle n \rangle$. Then, $\mathbb{I}(A, B)$ is an N -matrix of the first category with respect to J if and only if I_z is an N -matrix of the first category with respect to J for all $z \in \{\pm 1\}^n \setminus \{\pm e^J\}$.*

Proof. Suppose that I_z is an N -matrix of the first category with respect to J for all $z \in \{\pm 1\}^n \setminus \{\pm e^J\}$. Let $C \in \mathbb{I}(A, B)$. First we show that C has a representation of the form (3.1). For $i \in \langle n \rangle$, define $z^i \in \{\pm 1\}^n$ as follows: $z^i_i = 1$ and if $i \in J$ and $j \neq i$ then

$$z^i_j = \begin{cases} -1 & j \in J, \\ 1 & \text{otherwise.} \end{cases}$$

If $i \notin J$ and $j \neq i$ then

$$z_j^i = \begin{cases} 1 & j \in J, \\ -1 & \text{otherwise.} \end{cases}$$

Since I_{z^i} is an N -matrix of the first category with respect to J for all $i \in \langle n \rangle$, so I_{z^i} has a representation of the form (3.1). Thus for $i \in J$,

$$\begin{aligned} (I_u)_{ij} &< 0 \quad \forall j \in J, \\ (I_l)_{ij} &> 0 \quad \forall j \in J^c. \end{aligned}$$

For $i \in J^c$,

$$\begin{aligned} (I_u)_{ij} &< 0 \quad \forall j \in J^c, \\ (I_l)_{ij} &> 0 \quad \forall j \in J. \end{aligned}$$

Hence $C_{JJ} \leq (I_u)_{JJ} < 0$, $C_{JJ^c} \geq (I_l)_{JJ^c} > 0$ and similarly $C_{J^cJ^c} < 0$, $C_{J^cJ} > 0$. Let $x \in \mathbb{R}^n$ such that $x_J \not\leq 0$ or $x_{J^c} \not\geq 0$, and $x_J \not\geq 0$ or $x_{J^c} \not\leq 0$. By Lemma 2.4, there exists $z \in \{\pm 1\}^n \setminus \{\pm e^J\}$ such that $x_i(Cx)_i \geq x_i(I_z x)_i$, for $i \in \langle n \rangle$. Since I_z is an N -matrix of the first category with respect to J , by Theorem 3.3, there exists $i \in \langle n \rangle$ such that $0 < x_i(I_z x)_i < x_i(Cx)_i$. Thus C is an N -matrix of the first category with respect to J by Theorem 3.3. $\square$

4. Results for almost P -matrices

We now establish the sign non-reversal property for almost P -matrices (see Definition 1.1) and characterize their interval hull. Recall that an $n \times n$ ($n \geq 2$) matrix A is an almost P -matrix if and only if A^{-1} is an N -matrix [9, Lemma 2.4]. Motivated by this result, we classify almost P -matrices into two categories:

Definition 4.1. Let $n \geq 2$ be an integer.

- (i) Let J be a nonempty proper subset of $\langle n \rangle$. An almost P -matrix A is an *almost P -matrix of the first category with respect to J* if A^{-1} is an N -matrix of the first category with respect to J .
- (ii) An almost P -matrix A is an *almost P -matrix of the second category* if A^{-1} is an N -matrix of the second category.

Observe that if A is an almost P -matrix of the second category, then there exists a positive vector x such that $Ax < 0$. Our next result shows a sign non-reversal property for such matrices.

Theorem 4.2. Let $A \in \mathbb{R}^{n \times n}$. Then, A is an almost P -matrix of the second category if and only if the following hold:

- (a) $N(A) \cap \text{int}(\mathbb{R}_+^n) = \emptyset$, where $N(A)$ denotes the null space of A ,
- (b) $x_i(Ax)_i \leq 0$ for all $i \in \langle n \rangle$ imply that $x = 0$, if $x_k = 0$ for some k ; otherwise $x > 0$ or $x < 0$,
- (c) $A(\text{int}(\mathbb{R}_+^n)) \cap \text{int}(\mathbb{R}_-^n) \neq \emptyset$.

Proof. Let A be an almost P -matrix of the second category. Then (a) holds trivially. Let $x \in \mathbb{R}^n$ such that $x_i(Ax)_i \leq 0$ for all i . Suppose that $x_k = 0$ for some k . Let B be the principal submatrix of A obtained by deleting the k -th row and the k -th column of A . Let y be the $(n-1)$ -vector obtained from the vector x by deleting the k -th entry. Then $y_j(By)_j \leq 0$ for all j . Since B is a P -matrix, by Theorem 2.1, $y_j = 0$ for all j . Thus $x = 0$. Suppose that $x_i \neq 0$ for all i . Let $y = Ax$. Then, $y_i(A^{-1}y)_i \leq 0$ for all i . Since A^{-1} is an N -matrix of the second category, so, by Theorem 3.1, either $y \geq 0$ or $y \leq 0$. Note that $A^{-1} < 0$. Hence, all the components of the vector $x = A^{-1}y$ are either positive or negative. As A is an almost P -matrix of the second category, there exists a positive vector x such that $Ax < 0$. Thus (c) holds.

To prove the converse, first let us show that all the proper principal minors are positive. Let B be any $(n-1) \times (n-1)$ principal submatrix of A . Without loss of generality, assume that B is obtained from A by deleting the last row and last column of A . Let $y \in \mathbb{R}^{n-1}$, and define $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$. If $y_i(By)_i \leq 0$ for all i , then $x_i(Ax)_i \leq 0$ for all i . Thus, by the assumption, $x = 0$, and hence $y = 0$. By Theorem 2.1, all the proper principal minors of A are positive. We now claim that A is invertible. Let x be a vector such that $Ax = 0$. Then either $x > 0$ or $x < 0$ or $x = 0$. Since $N(A) \cap \text{int}(\mathbb{R}_+^n) = \emptyset$, so $x = 0$. Also A is not a P -matrix, since $A(\text{int}(\mathbb{R}_+^n)) \cap \text{int}(\mathbb{R}_-^n) \neq \emptyset$, so A reverses the sign of a nonzero vector. Thus $\det A < 0$, hence A is an almost P -matrix. Let $x = A^{-1}e^i$. Note that the i -th entry of the vector x is negative. Now, we have $x_j(Ax)_j \leq 0$ for all $j \in \langle n \rangle$. Thus the vector x is entrywise negative, and hence $A^{-1} < 0$. So A is an almost P -matrix of the second category. $\square$

Using Theorem 4.2, we establish an equivalent condition for an interval hull to be a subset of the set of all almost P -matrices of the second category.

Theorem 4.3. Let $A, B \in \mathbb{R}^{n \times n}$. Then $\mathbb{I}(A, B)$ is an almost P -matrix of the second category if and only if I_u and I_z are almost P -matrices of the second category for all $z \in \{\pm 1\}^n$.

Proof. Let I_u and I_z be almost P -matrices of the second category for all $z \in \{\pm 1\}^n$ and let $C \in \mathbb{I}(A, B)$. First let us show that $N(C) \cap \text{int}(\mathbb{R}_+^n) = \emptyset$. Let $x \in N(C) \cap \text{int}(\mathbb{R}_+^n)$. Then $I_u x \geq 0$, since $I_u \geq C$ and $Cx = 0$. Since $I_u^{-1} < 0$, so $I_u^{-1}(I_u x) = x \leq 0$, a contradiction. Thus $N(C) \cap \text{int}(\mathbb{R}_+^n) = \emptyset$.

Since I_u is an almost P -matrix of the second category, there exists a vector $x > 0$ such that $I_u x < 0$. Thus $Cx < 0$, since $C \leq I_u$. Hence $C(\text{int}(\mathbb{R}_+^n)) \cap \text{int}(\mathbb{R}_-^n) \neq \emptyset$.

Let $x \in \mathbb{R}^n$ be such that $x_i(Cx)_i \leq 0$ for $i \in \langle n \rangle$. By Lemma 2.4, there exists a vector $z \in \{\pm 1\}^n$ such that $0 \geq x_i(Cx)_i \geq x_i(I_z x)_i$ for $i \in \langle n \rangle$. Since I_z is an almost P -matrix of the second category, by Theorem 4.2, $x = 0$, if $x_k = 0$ for some k , otherwise $x < 0$ or $x > 0$. Thus, by Theorem 4.2, C is an almost P -matrix of the second category. $\square$

For a nonempty proper subset J of $\langle n \rangle$, define

$$\mathbb{J} = \mathbb{J}_J := \{x \in \mathbb{R}^n : x_j > 0 \text{ for all } j \in J \text{ and } x_j < 0 \text{ for all } j \notin J\}. \quad (4.1)$$

Note that if A is an almost P -matrix of the first category with respect to J , then there exists $x \in \mathbb{J}$ such that $Ax \in -\mathbb{J}$. Next, we develop the sign non-reversal property for almost P -matrices of the first category with respect to J .

Theorem 4.4. *Let $A \in \mathbb{R}^{n \times n}$ and let J be a nonempty proper subset of $\langle n \rangle$. Then A is an almost P -matrix of the first category with respect to J if and only if the following hold:*

- (a) $N(A) \cap \mathbb{J} = \emptyset$,
- (b) $x \in \mathbb{R}^n$ and $x_i(Ax)_i \leq 0$ for all $i \in \langle n \rangle$ imply $x = 0$, if $x_k = 0$ for some k ; otherwise either $x \in \mathbb{J}$ or $-x \in \mathbb{J}$,
- (c) $A(\mathbb{J}) \cap (-\mathbb{J}) \neq \emptyset$.

Proof. Let A be an almost P -matrix of the first category with respect to J . Then (a) holds trivially and A^{-1} has a representation of the form (3.1). Let $x_i(Ax)_i \leq 0$. If $x_i = 0$ for some i , then, by an argument similar to that of Theorem 4.2, we get $x = 0$. So, let us assume that $x_i \neq 0$ and $x_i(Ax)_i \leq 0$ for each i . Let $y = Ax$. Then $y_i(A^{-1}y)_i \leq 0$ for each i . Hence, by Theorem 3.3, either $y_J \leq 0$, $y_{J^c} \geq 0$ or $y_J \geq 0$, $y_{J^c} \leq 0$. Also, $x = A^{-1}y$ and A^{-1} has a representation of the form (3.1), so $x_J > 0$, $x_{J^c} < 0$ or $x_J < 0$, $x_{J^c} > 0$. As A is an almost P -matrix of the first category with respect to J , so there exists $x \in \mathbb{J}$ such that $Ax \in -\mathbb{J}$. Thus (c) holds.

Conversely, suppose that A satisfies all the three properties as above. Let C be any $(n-1) \times (n-1)$ principal submatrix of A . Without loss of generality, assume C is obtained from A by deleting the last row and last column of A . Let $y \in \mathbb{R}^{n-1}$. Set $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$. If $y_i(Cy)_i \leq 0$ for all i , then $x_i(Ax)_i \leq 0$ for all $i \in \langle n \rangle$. Thus, by assumption $x = 0$, so $y = 0$. By Theorem 2.1, all the proper principal minors of A are positive. Since $N(A) \cap \mathbb{J} = \emptyset$, so A is invertible by a similar argument to that of Theorem 4.2. As, $A(\mathbb{J}) \cap (-\mathbb{J}) \neq \emptyset$, so the matrix A is not a P -matrix by Theorem 2.1. Thus A is an almost P -matrix. Let $x = A^{-1}e^k$. Then $x_j(Ax)_j \leq 0$ for $j \in \langle n \rangle$. If $k \in J$, then $(A^{-1}e^k)_J < 0$ and $(A^{-1}e^k)_{J^c} > 0$, since the k -th entry of x is negative. Otherwise $(A^{-1}e^k)_J > 0$ and $(A^{-1}e^k)_{J^c} < 0$. Hence A is an almost P -matrix of the first category with respect to J . $\square$

For a proper subset J of $\langle n \rangle$, define the diagonal matrix I_J as follows: the i -th diagonal entry is 1, if $i \in J$, and is -1 otherwise. For a nonempty proper $J \subseteq \langle n \rangle$, we define the matrix I_{P_J} as $I_{P_J} := I_c + I_J \Delta I_J$. One can verify that $I_{P_J} \in \mathbb{I}(A, B)$.

In the following theorem, we characterize when an interval hull is a subset of the set of all almost P -matrices of the first category with respect to J .

Theorem 4.5. *Let $A, B \in \mathbb{R}^{n \times n}$. Then $\mathbb{I}(A, B)$ is an almost P -matrix of the first category with respect to J if and only if I_{P_J} and I_z are almost P -matrices of the first category with respect to J for all $z \in \{\pm 1\}^n$.*

Proof. Suppose that the matrices I_{P_J} and I_z are almost P -matrices of the first category with respect to J for all $z \in \{\pm 1\}^n$. From the definition, $(I_{P_J})_{JJ} = (I_u)_{JJ}$, $(I_{P_J})_{J^c J^c} = (I_u)_{J^c J^c}$, $(I_{P_J})_{JJ^c} = (I_l)_{JJ^c}$ and $(I_{P_J})_{J^c J} = (I_l)_{J^c J}$. Let $C \in \mathbb{I}(A, B)$. First let us show that $N(C) \cap \mathbb{J} = \emptyset$. Let $x \in N(C) \cap \mathbb{J}$ and $y = I_{P_J}x$. Then $x_J > 0$ and $x_{J^c} < 0$. Now,

$$\begin{aligned} (I_u)_{JJ}x_J + (I_l)_{JJ^c}x_{J^c} &\geq C_{JJ}x_J + C_{JJ^c}x_{J^c} \\ (I_l)_{J^c J}x_J + (I_u)_{J^c J^c}x_{J^c} &\leq C_{J^c J}x_J + C_{J^c J^c}x_{J^c} \end{aligned} \quad (4.2)$$

and $Cx = 0$ imply that $y_J \geq 0$ and $y_{J^c} \leq 0$. Since $I_{P_J}^{-1}$ has a representation of the form (3.1), so $(I_{P_J}^{-1}y)_J = x_J \leq 0$ and $(I_{P_J}^{-1}y)_{J^c} = x_{J^c} \geq 0$, a contradiction. Thus $N(C) \cap \mathbb{J} = \emptyset$.

Since I_{P_J} is an almost P -matrix of the first category with respect to J , there exists a vector $x \in \mathbb{J}$ such that $I_{P_J}x \in -\mathbb{J}$. By (4.2), $(Cx)_J < 0$ and $(Cx)_{J^c} > 0$. Hence $C(\mathbb{J}) \cap (-\mathbb{J}) \neq \emptyset$.

Let $x \in \mathbb{R}^n$ be such that $x_i(Cx)_i \leq 0$ for $i \in \langle n \rangle$. By Lemma 2.4, there exists a vector $z \in \{\pm 1\}^n$ such that $x_i(I_z x)_i \leq x_i(Cx)_i \leq 0$, for $i \in \langle n \rangle$. Since I_z is an almost P -matrix of the first category with respect to J , by Theorem 4.4, $x = 0$, if $x_k = 0$ for some k , otherwise $x \in \mathbb{J}$ or $x \in -\mathbb{J}$. Hence, by Theorem 4.4, C is an almost P -matrix of the first category with respect to J . $\square$

Remark 4.6. Lemma 3.4 has analogue for almost P -matrices of either category with appropriate choices of x . We leave the details to the interested reader.

5. Results for semipositive matrices

In this section we characterize the interval hull for semipositive and minimally semipositive matrices (see Definition 1.1). In [2], the authors studied the interval hull $\mathbb{I}(A, B)$ of minimally semipositive matrices, where $A \leq B$. In this article, we provide a short and elementary proof.

Theorem 5.1. *Let $A, B \in \mathbb{R}^{m \times n}$. Then we have the following:*

- (a) $\mathbb{I}(A, B)$ is semipositive if and only if I_l is a semipositive matrix.

(b) $\mathbb{I}(A, B)$ is minimally semipositive if and only if I_l is semipositive and I_u is minimally semipositive.

Proof. (a) Let I_l be semipositive. Then, there exists $x \geq 0$ such that $I_l x > 0$. Let $C \in \mathbb{I}(A, B)$. Then $0 < I_l x \leq Cx$. Thus $\mathbb{I}(A, B)$ is semipositive.

(b) Suppose that I_l is semipositive and I_u is minimally semipositive. Then $\mathbb{I}(A, B)$ is semipositive. Suppose $C \in \mathbb{I}(A, B)$ is not minimally semipositive. Then there exists a nonnegative nonzero vector x with at least one zero entry such that $Cx > 0$. Since $I_u \geq C$, so $I_u x > 0$, a contradiction. Thus $\mathbb{I}(A, B)$ is minimally semipositive. $\square$

It is known that a square matrix A is minimally semipositive if and only if A is invertible and $A^{-1} \geq 0$. More generally, an $m \times n$ matrix A is minimally semipositive if and only if A is semipositive and A has a nonnegative left inverse [8]. This leads to the following result:

Theorem 5.2 ([10, Theorem 25.4]). Let $B, C \in \mathbb{R}^{n \times n}$ such that $C \leq B$, B is invertible, and $B^{-1} \geq 0$. Then $C^{-1} \geq 0$ if and only if $\text{int}(\mathbb{R}_+^n) \cap C\mathbb{R}_+^n \neq \emptyset$.

Indeed, in [10], the authors proved the above theorem for any normal and solid cone in $\mathbb{R}^n$. Now, it is clear that part (b) of Theorem 5.1 is an extension of Theorem 5.2 for rectangular matrices. We also observe that our argument provides an alternate, simpler, and elementary proof for Theorem 5.2.

Declaration of competing interest

No competing interest.

Acknowledgements

We thank Jürgen Garloff and Apoorva Khare for a detailed reading of an earlier draft and for providing valuable comments and feedback. P.N. Choudhury was supported by National Post-Doctoral Fellowship (PDF/2019/000275), the SERB, Department of Science and Technology, India, and the NBHM Post-Doctoral Fellowship (0204/11/2018/R&D-II/6437) from DAE (Govt. of India). M.R. Kannan would like to thank the SERB, Department of Science and Technology, India, for financial support through the projects MATRICS (MTR/2018/000986) and Early Career Research Award (ECR/2017/000643).

References

- [1] S. Bialas, J. Garloff, Intervals of P -matrices and related matrices, *Linear Algebra Appl.* 58 (1984) 33–41.
- [2] P.N. Choudhury, M.R. Kannan, K.C. Sivakumar, New contributions to semipositive and minimally semipositive matrices, *Electron. J. Linear Algebra* 34 (2018) 35–53.

- [3] P.N. Choudhury, M.J. Tsatsomeros, Algorithmic detection and construction of N -matrices, *Linear Algebra Appl.* 602 (2020) 46–56.
- [4] Ky Fan, Some matrix inequalities, *Abh. Math. Semin. Univ. Hamb.* 29 (1966) 185–196.
- [5] D. Gale, H. Nikaidô, The Jacobian matrix and global univalence of mappings, *Math. Ann.* 159 (1965) 81–93.
- [6] J. Garloff, M. Adm, J. Titi, A survey of classes of matrices possessing the interval property and related properties, *Reliab. Comput.* 22 (2016) 1–14.
- [7] K-i Inada, The production coefficient matrix and the Stolper-Samuelson condition, *Econometrica* 39 (1971) 219–239.
- [8] C.R. Johnson, M.K. Kerr, D.P. Stanford, Semipositivity of matrices, *Linear Multilinear Algebra* 37 (1994) 265–271.
- [9] M. Kojima, R. Saigal, On the number of solutions to a class of linear complementarity problems, *Math. Program.* 17 (1979) 136–139.
- [10] M.A. Krasnoselskij, Je.A. Lifshits, A.V. Sobolev, *Positive Linear Systems*, Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989, The method of positive operators, from the Russian by Jürgen Appell.
- [11] J.R. Kuttler, A fourth-order finite-difference approximation for the fixed membrane eigenproblem, *Math. Comput.* 25 (1971) 237–256.
- [12] J.M. Miao, Ky Fan’s N -matrices and linear complementarity problems, *Math. Program.* 61 (1993) 351–356.
- [13] S.R. Mohan, R. Sridhar, On characterizing N -matrices using linear complementarity, *Linear Algebra Appl.* 160 (1992) 231–245.
- [14] T. Parthasarathy, G. Ravindran, N -matrices, *Linear Algebra Appl.* 139 (1990) 89–102.
- [15] J. Rohn, Positive definiteness and stability of interval matrices, *SIAM J. Matrix Anal. Appl.* 15 (1994) 175–184.
- [16] J. Rohn, G. Rex, Interval P -matrices, *SIAM J. Matrix Anal. Appl.* 17 (1996) 1020–1024.
- [17] E. Stiemke, Über positive Lösungen homogener linearer Gleichungen, *Math. Ann.* 76 (1915) 340–342.
- [18] J.S. Vandergraft, Applications of partial orderings to the study of positive definiteness, monotonicity, and convergence of iterative methods for linear systems, *SIAM J. Numer. Anal.* 9 (1972) 97–104.