Metric Properties of Parabolic Ample Bundles

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We introduce a notion of admissible Hermitian metrics on parabolic bundles and define
positivity properties for the same. We develop Chern–Weil theory for parabolic bundles
and prove that our metric notions coincide with the already existing algebro-geometric
versions of parabolic Chern classes. We also formulate a Griffiths conjecture in the
parabolic setting and prove some results that provide evidence in its favor for certain
kinds of parabolic bundles. For these kinds of parabolic structures, we prove that the
conjecture holds on Riemann surfaces. We also prove that a Berndtsson-type result
holds and that there are metrics on stable bundles over surfaces whose Schur forms
are positive.

1 Introduction

Given a Hermitian holomorphic vector bundle (E, H) on a complex manifold X, it is said
to be Griffiths (respectively, Nakano) positive if the curvature Θ_H is a positive bilinear
form when tested against $v \otimes s$ where v is a tangent vector and s is a vector from E
(respectively, when tested against all vectors in $TX \otimes E$). Another notion of positivity
is Hartshorne ampleness—a holomorphic vector bundle E is Hartshorne ample if the
tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is ample in the usual sense. It is clear that a
Griffiths positive bundle is Hartshorne ample. The converse is a well-known conjecture
of Griffiths.

The evidence available in favor of Griffiths' conjecture is as follows:

(1) Mori [35] proved Hartshorne's conjecture [25]. This means that a compact complex manifold M whose tangent bundle TM is Hartshorne ample is biholomorphic to $\mathbb{CP}^n$. Since the Fubini–Study metric on $\mathbb{CP}^n$ has positive bisectional curvature, the vector bundle TM is Griffiths positive.

(2) Umemura [44] and, later, Campana and Flenner [18] proved the Griffiths conjecture for Riemann surfaces.

(3) Bloch and Gieseker [15] proved that the Schur polynomials of Hartshorne ample bundles are numerically positive. Griffiths himself proved that c_1 and c_2 of a Griffiths positive metric are positive as forms. Guler [24] and Diverio [21] proved, using a complicated calculation based on an elegant idea of Guler, that the signed Segre forms of Griffiths positive bundles are positive (in particular, on surfaces the Schur polynomials of a Griffiths positive metric are positive pointwise). In [39], it was proven that a Hartshorne ample semistable bundle over a surface admits a metric whose Schur polynomials are positive pointwise. It is still unknown as to whether Schur polynomials of Griffiths positive metrics are pointwise positive; however if so, this would be further indirect evidence.

(4) Demailly [20] proved that if E is Griffiths positive then $E \otimes \det(E)$ is Nakano positive. Berndtsson [3] proved that if E is Hartshorne ample, then $E \otimes \det(E)$ is Nakano positive. Mourougane–Takayama [36] independently proved that $E \otimes \det(E)$ is Griffiths positive if E is Hartshorne ample.

(5) Typically, "good" metrics are produced using flows. Naumann [37] outlined a promising approach to the Griffiths conjecture using the relative Kähler–Ricci flow. If it works, it ought to work just as well in the equivariant context (which is roughly what this paper deals with).

It is but natural to wonder if the same kind of a conjecture can be made for singular Hermitian metrics. Unfortunately, the notion of a singular Hermitian metric on general vector bundles (as opposed to line bundles where a lot of work has been done) is quite subtle and only recently has there been progress on it [4, 5, 19, 26, 29, 38, 41, 42]. A compromise can be made by choosing to work with parabolic bundles, which are essentially vector bundles equipped with flags (and weights) over divisors. Any reasonable notion of a "metric" on a parabolic bundle should degenerate on the divisor, that is, it should be a singular Hermitian metric. The differential geometry of parabolic bundles has been studied reasonably well [7, 30, 40, 43]. The notion of

parabolic Hartshorne ampleness has also been studied [8, 13, 14]. However, to our knowledge, the metric aspects of parabolic ampleness have not received any attention so far. This paper attempts to remedy that situation.

In this paper we prove the following results.

(1) In Section 3 we introduce a notion of admissible Hermitian metrics on parabolic bundles with rational weights over projective manifolds. It is interesting to compare our definition of admissibility with existing ones. We plan on addressing this in future work.

(2) We define a metric notion of Griffiths (and Nakano) positivity for parabolic bundles in Section 4 and formulate a Griffiths conjecture in this context. We prove it for Riemann surfaces (for certain kinds of parabolic structures induced from "good Kawamata covers"). Moreover, we prove that our notion of positivity agrees with the algebro-geometric notion in [8] for line bundles.

(3) In Section 5, we develop Chern–Weil theory for admissible metrics on parabolic bundles. We verify that the Chern classes coincide with the ones defined algebraically in [10], [28] and [12]. We prove that the pushforward of $c_1^k(\mathcal{O}_{\mathbb{P}(E_*)}(1))$ gives (signed) Segre forms of E. This is a parabolic version of some results in [24] and [21]. Our proof has a small technical innovation in terms of generating functions and we hope it generalizes to computing pushforwards for flag bundles. Lastly, we prove a parabolic version of a result (for parabolic structures arising from good Kawamata covers) in [39] concerning the existence of metrics whose Schur forms are positive on stable bundles over surfaces.

(4) In Section 6 we prove a parabolic analog of Berndtsson's theorem (as above, for parabolic structures arising from good Kawamata covers), that is, if E is Hartshorne ample, $E \otimes \det(E)$ is Nakano positive.

2 Preliminaries

2.1 Definition of parabolic vector bundles

Let X be an irreducible smooth complex projective variety and $D \subset X$ a reduced effective simple normal crossing divisor; this means that for the decomposition

$$D = \sum_{i=1}^{\mu} D_i$$

into irreducible components, each component D_i is smooth and they intersect transversally. In this paper we will state and prove results only for the case of $\mu = 1$, that is, for smooth divisors. The general case of simple normal crossings is not such a big leap from our current study.

Definition 2.1.   Let E be a holomorphic vector bundle on X of rank r. A quasi-parabolic structure on E over D is a filtration

$$E|_{D_i} = F_1^i \supsetneq F_2^i \supsetneq \cdots \supsetneq F_{m_i}^i \supsetneq F_{m_{i+1}}^i = 0 , \tag{2.1}$$

where each F_j^i is a subbundle of $E|_{D_i}$ such that they are locally abelian, which means that for every $x \in D$ there is a decomposition of E_x into a direct sum of lines with the property that for any i with $x \in D_i$, the filtration of $E|_{D_i}$ when restricted to x, is given by combinations of these lines. Note that when $\mu = 1$, this condition of being locally abelian is automatically satisfied.

A *parabolic structure* is a quasi-parabolic structure as in (2.1) endowed with *parabolic weights* that are collections of rational numbers

$$0 < \alpha_1^i \leq \alpha_2^i \leq \cdots \leq \alpha_r^i < 1 ,$$

(where α_j^i can be repeated) associated to the subbundles F_k^i, that is, $\alpha_1^i = \ldots = \alpha_{r_{i,1}}^i$ correspond to F_1^i/F_2^i, etc. where $r_{i,j}$ is the rank of F_j^i/F_{j+1}^i; more precisely,

$$\alpha_{1+\sum_{j=1}^a r_{i,j}}^i = \alpha_{2+\sum_{j=1}^a r_{i,j}}^i = \cdots = \alpha_{\sum_{j=1}^{a+1} r_{i,j}}^i$$

correspond to F_{a+1}^i/F_{a+2}^i for all $1 \leq a \leq m_i - 1$, and this common number is called the weight of F_{a+1}^i/F_{a+2}^i. A parabolic vector bundle is one that is equipped with a parabolic structure. For notational convenience, a parabolic vector bundle $(E, \{F_j^i\}, \alpha_j^i)$ will also be denoted as E_*. The divisor D is called the parabolic divisor for E_*.

Remark 2.2.   Note that if all the parabolic weights are zero (the "trivial parabolic structure"), we do not call it a parabolic bundle in this paper.

Take a parabolic vector bundle E_*. Maruyama and Yokogawa associate to E_* a filtration of coherent sheaves $\{E_t\}_{t\in\mathbb{R}}$ parametrized by $\mathbb{R}$ [33]. This filtration encodes the entire parabolic data. We recall from [33] some properties of this filtration:

(1)   the filtration $\{E_t\}_{t\in\mathbb{R}}$ is decreasing as t increases, meaning $E_{t+t'} \subset E_t$ for all $t' > 0$ and t;

(2) it is left-continuous, meaning for all $t \in \mathbb{R}$, there is $\epsilon_t > 0$ such that the above inclusion of E_t in $E_{t-\epsilon_t}$ is an isomorphism;

(3) $E_{t+1} = E_t \otimes \mathcal{O}_X(-D)$ for all t;

(4) the vector bundle E is E_0;

(5) for a finite interval $[a, a']$, the set

$$\{t \in [a, a'] \mid E_{t+\epsilon} \subsetneq E_t \; \forall \; \epsilon > 0\}$$

is finite; and

(6) the filtration $\{E_t\}_{t \in \mathbb{R}}$ has a right jump at t if and only if $t - [t]$ is a parabolic weight for E_*.

Fix a very ample line bundle on X to define *degree* of coherent sheaves on X. The parabolic degree of a parabolic bundle E_* as above is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^{\mu} \sum_{j=1}^{m_i} \text{degree}(F_j^i / F_{j+1}^i) \cdot \text{weight}(F_j^i / F_{j+1}^i).$$

In terms of the filtration $\{E_t\}_{t \in \mathbb{R}}$, we have

$$\text{par-deg}(E_*) = r \cdot \text{degree}(\mathcal{O}_X(D)) + \int_0^1 \text{degree}(E_t) \, dt.$$

Now we will recall the definitions of direct sum, tensor product and dual of parabolic vector bundles.

Let E_* and V_* be parabolic vector bundles with a common parabolic divisor D. The underlying vector bundles for E_* and V_* will be denoted by E and V, respectively. Let

$$\iota : X \setminus D \hookrightarrow X$$

be the inclusion map. Consider the quasi-coherent sheaf $\iota_* \iota^*(E \oplus V)$ on X. The parabolic direct sum $E_* \oplus F_*$ is defined to be the parabolic vector bundle that corresponds to the filtration $\{E_t \oplus F_t\}_{t \in \mathbb{R}}$ of subsheaves of it.

Next, consider the quasi-coherent sheaf $\iota_* \iota^*(E \otimes V)$ on X. For any $t \in \mathbb{R}$, let U_t be the coherent subsheaf of it generated by all $E_s \otimes V_{t-s}$, $s \in \mathbb{R}$. The conditions on $\{E_b\}_{b \in \mathbb{R}}$ and $\{V_b\}_{b \in \mathbb{R}}$ ensure that this U_t is indeed a coherent sheaf. The parabolic tensor product $E_* \otimes F_*$ is defined to be the parabolic vector bundle that corresponds to this filtration $\{U_t\}_{t \in \mathbb{R}}$.

For any $t \in \mathbb{R}$, define E_{t+} to be $E_{t+\epsilon}$, where $\epsilon > 0$ is sufficiently small so that $E_{t+\epsilon}$ is independent of ϵ (recall that the filtration parametrized by $\mathbb{R}$ has finitely many

jumps in each bounded interval so it is constant except for those finitely many jumps). Therefore, $(E_{-t-1+\epsilon})^*$ is a subsheaf of $\iota_*\iota^*E^*$. The parabolic dual E_*^* of E_* is defined by the filtration $\{(E_{-t-1+\epsilon})^*\}_{t\in\mathbb{R}}$. So the underlying vector bundle for the parabolic dual E_*^* is $(E_{\epsilon-1})^*$.

2.2   Parabolic bundles and equivariant bundles

Let Y be a connected smooth complex projective variety and

$$\Gamma \subset \mathrm{Aut}(Y)$$

a finite subgroup of the group of automorphisms of the variety Y. A Γ-linearized vector bundle over Y of rank r is a holomorphic vector bundle V of rank r over Y equipped with a holomorphic action of Γ such that

- the projection $V \longrightarrow Y$ is Γ-equivariant, and
- the action of Γ on V is fiber-wise linear.

In other words, V is an orbifold vector bundle; it is also called an equivariant bundle.

For any point $y \in Y$, let $\Gamma_y \subset \Gamma$ be the isotropy subgroup of y for the action of Γ on Y.

Assume that quotient variety Y/Γ is smooth. Let

$$q : Y \longrightarrow Y/\Gamma$$

be the quotient map. Consider the ramification divisor for q; let

$$D_q \subset Y$$

be the reduced ramification divisor for q. We assume that D_q is a normal crossing divisor of Y.

Take a Γ-linearized vector bundle V on Y. Let $\widetilde{D} \subset D_q$ be the union of all the irreducible components D' of D_q with the property that the isotropy subgroup Γ_z of every point z of D' acts nontrivially on the fiber V_z of V over z. By means of the invariant direct image construction, V produces a parabolic vector bundle E_* on Y/Γ with parabolic structure over the divisor $q(\widetilde{D})$ [9, 16, 17].

Conversely, given a parabolic vector bundle E_* on X with parabolic structure over a simple normal crossing divisor D, there is a triple (Y, Γ, V) as above such that

- $X = Y/\Gamma$, and
- E_* coincides with the parabolic vector bundle over X associated to V [9], [16], [17].

This covering Y is an example of "Kawamata covering" introduced by Kawamata [31, Theorem 17], [32, Theorem 1.1.1] in order to prove what is known as Kawamata–Viehweg vanishing theorem. It should be clarified that the ramification divisor of the above quotient map

$$q : Y \longrightarrow X = Y/\Gamma$$

is in general bigger than D. However on the inverse image $q^{-1}(X \setminus D) \subset Y$, the vector bundle V is canonically Γ-equivariantly identified with the pullback $q^*(E|_{X \setminus D})$ (note that the pulled back bundle $q^*(E|_{X \setminus D})$ has a tautological action of Γ); in other words, $E|_{X \setminus D}$ is the descent of $V|_{q^{-1}(X \setminus D)}$. In particular, for any point $y \in q^{-1}(X \setminus D)$, the action on the fiber V_y, of the equivariant vector bundle V, of the isotropy subgroup Γ_y is trivial.

The above correspondence between the parabolic vector bundles and the orbifold vector bundles is compatible with the operations of direct sum, tensor product, dualization, etc.

In view of the above, we make the following definitions.

Definition 2.3.   Suppose X is a complex manifold and $D \subset X$ is a divisor whose components are smooth and intersect transversally. Assume that (E_*, D) is a parabolic vector bundle on X. A triple (Y, q, V) is called a *Kawamata cover* of (X, E_*, D) if the following two conditions are satisfied:

- $q : Y \longrightarrow X$ is a finite branched cover of X whose ramification divisor

$$D'' = D \cup D'$$

   has smooth transversally intersecting irreducible components, and
- E_* coincides with the bundle obtained by the invariant direct image construction of the equivariant vector bundle V over Y equipped with an action of the covering group $\mathrm{Gal}(q)$ for q.

Such a Kawamata cover (Y, q, V) is called *good* if $D'' = D$. Also, a Kawamata cover (Y, q, V) is called *locally good around a point $p \in X$* if there is a Zariski open neighborhood $p \in U_p \subset X$ such that $D'' \cap U_p = D \cap U_p$. A Kawamata cover of (X, E_*, D) is called *minimal* if its degree is the minimum possible one.

The following lemma is useful for us.

Lemma 2.1.   Let E_* be a parabolic vector bundle over X with parabolic divisor D. Take a point $x \in X$. Then there is Kawamata cover that is good over some Zariski neighborhood of x.

Proof. This follows from the construction of Kawamata cover [31, Theorem 17], [32, Theorem 1.1.1] and the correspondence between parabolic bundles and equivariant bundles [9, 16, 17]. The divisor D' mentioned above moves freely as it can be assumed to be very ample. This allows us to have $D'' \cap U_x = D \cap U_x$ for suitable D' and open neighborhood U_x of x. ∎

It should be clarified that the covering in Lemma 2.1 depends on the point x.

2.3 Parabolic bundles and ramified bundles

One issue with the above correspondence between parabolic bundles and equivariant bundles is that the ramified Galois covering (Y, q) is not uniquely determined by the pair (X, E_*). If (Y', q') is a ramified Galois covering of X that factors through the covering (Y, q), and the map $Y' \longrightarrow Y$ is étale, then there is an equivariant vector bundle on (Y', q') that also corresponds to E_*. More generally, we can introduce extra divisors D' on X such that $D \cup D'$ is still a simple normal crossing divisor, introduce trivial parabolic structure over D', and demand that the covering map ramified over D' also. This nonuniqueness of the covering is addressed by introducing what are known as ramified bundles, which we will briefly recall; more details can be found in [12, Section 2.2], [13, Section 3], and [1].

Take any (Y, q, V) corresponding to E_*, and as before denote $\mathrm{Gal}(q)$ by Γ. Let $\xi : F_V \longrightarrow Y$ be the holomorphic principal $\mathrm{GL}(r, \mathbb{C})$-bundle defined by V, where r as before is the rank of V. So the group $\mathrm{GL}(r, \mathbb{C})$ acts on F_V holomorphically and freely, and each fiber of ξ is an orbit. The key point is that this action of $\mathrm{GL}(r, \mathbb{C})$ commutes with the action of Γ on F_V given by the action of Γ on V. Now consider the quotient

$$\widehat{\xi} : F_V' := F_V / \Gamma \longrightarrow Y / \Gamma = X, \tag{2.2}$$

where $\widehat{\xi}$ is the descent of ξ. Since the action of $\mathrm{GL}(r, \mathbb{C})$ and Γ on F_V commute, the quotient F_V' is equipped with an action of $\mathrm{GL}(r, \mathbb{C})$. This action is free on $\widehat{\xi}^{-1}(X \setminus D)$, because for every $z \in q^{-1}(X \setminus D)$, the action of Γ_z on the fiber V_z is trivial. This makes $F_V'|_{X \setminus D}$ a holomorphic principal $\mathrm{GL}(r, \mathbb{C})$-bundle over $X \setminus D$. However, over $\widehat{\xi}^{-1}(D)$, the action of $\mathrm{GL}(r, \mathbb{C})$ has finite isotropies.

A ramified principal $\mathrm{GL}(r, \mathbb{C})$-bundle over X with ramification over D is defined keeping the above model in mind. More precisely, a ramified principal $\mathrm{GL}(r, \mathbb{C})$-bundle over X with ramification over D consists of a smooth complex variety $E_{\mathrm{GL}(r, \mathbb{C})}$ equipped

with an algebraic right action of $\mathrm{GL}(r,\mathbb{C})$

$$f \;:\; E_{\mathrm{GL}(r,\mathbb{C})} \times \mathrm{GL}(r,\mathbb{C}) \;\longrightarrow\; E_{\mathrm{GL}(r,\mathbb{C})}\,, \tag{2.3}$$

and a surjective map

$$\widehat{\xi} \;:\; E_{\mathrm{GL}(r,\mathbb{C})} \;\longrightarrow\; X \tag{2.4}$$

such that the following conditions hold:

(1) $\widehat{\xi} \circ f \;=\; \widehat{\xi} \circ p_1$, where p_1 is the natural projection of $E_{\mathrm{GL}(r,\mathbb{C})} \times \mathrm{GL}(r,\mathbb{C})$ to $E_{\mathrm{GL}(r,\mathbb{C})}$;

(2) for each point $x \in X$, the action of $\mathrm{GL}(r,\mathbb{C})$ on the reduced fiber $\widehat{\xi}^{-1}(x)_{\mathrm{red}}$ is transitive;

(3) the restriction of $\widehat{\xi}$ to $\widehat{\xi}^{-1}(X \setminus D)$ makes $\widehat{\xi}^{-1}(X \setminus D)$ a principal $\mathrm{GL}(r,\mathbb{C})$-bundle over $X \setminus D$ (note that the 1st condition implies that $\widehat{\xi}^{-1}(X \setminus D)$ is preserved by the action of $\mathrm{GL}(r,\mathbb{C})$);

(4) for each irreducible component $D_i \subset D$, the reduced inverse image $\widehat{\xi}^{-1}(D_i)_{\mathrm{red}}$ is a smooth divisor and

$$\widehat{D} \;:=\; \sum_{i=1}^{\ell} \widehat{\xi}^{-1}(D_i)_{\mathrm{red}}$$

is a normal crossing divisor on $E_{\mathrm{GL}(r,\mathbb{C})}$; and

(5) for any point x of D and any point $z \in \widehat{\xi}^{-1}(x)$, the isotropy group

$$G_z \;\subset\; \mathrm{GL}(r,\mathbb{C})\,, \tag{2.5}$$

for the action of $\mathrm{GL}(r,\mathbb{C})$ on $E_{\mathrm{GL}(r,\mathbb{C})}$, is a finite group, and if x is a smooth point of D, then the natural action of G_z on the quotient line $T_z E_{\mathrm{GL}(r,\mathbb{C})}/T_z\widehat{\xi}^{-1}(D)_{\mathrm{red}}$ is faithful.

The quotient in (2.2) has all the above properties. Conversely, given any ramified principal $\mathrm{GL}(r,\mathbb{C})$-bundle over X with ramification over D, there is a parabolic vector bundle on X of rank r with parabolic structure on D. More precisely, we have an equivalence of categories between the parabolic vector bundles on X of rank r with parabolic structure over D and the ramified principal $\mathrm{GL}(r,\mathbb{C})$-bundle over X with ramification over D.

3 Admissible Hermitian Metric

For the rest of the paper, we assume that D is smooth for the sake of convenience. Our results can be easily generalized to the case of simple normal crossing divisors.

Let F be a C^∞ complex vector bundle of rank r over a complex manifold Z. Let $F_{\mathrm{GL}(r)} \longrightarrow Z$ be the corresponding C^∞ principal $\mathrm{GL}(r, \mathbb{C})$-bundle. Giving a Hermitian structure on F is equivalent to giving a C^∞ reduction of structure group of $F_{\mathrm{U}(r)} \subset F_{\mathrm{GL}(r)}$ to the subgroup $\mathrm{U}(r) \subset \mathrm{GL}(r, \mathbb{C})$.

Let $\widehat{\xi} : E_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$ be a ramified principal $\mathrm{GL}(r, \mathbb{C})$-bundle over X with ramification over D, as in (2.4). A *Hermitian structure* on $E_{\mathrm{GL}(r,\mathbb{C})}$ (cf. [11]) is a C^∞ submanifold

$$E_{\mathrm{U}(r)} \subset E_{\mathrm{GL}(r,\mathbb{C})}$$

satisfying the following three conditions:

(1) for the action of $\mathrm{GL}(r, \mathbb{C})$ in (2.3), the submanifold $E_{\mathrm{U}(r)}$ is preserved by $\mathrm{U}(r) \subset \mathrm{GL}(r, \mathbb{C})$;

(2) for each point $x \in X$, the action of $\mathrm{U}(r)$ on $E_{\mathrm{U}(r)} \bigcap \widehat{\xi}^{-1}(x)$ is transitive; and

(3) for each point $z \in \widehat{\xi}^{-1}(D) \bigcap E_{\mathrm{U}(r)}$, the isotropy subgroup for the action of $\mathrm{GL}(r, \mathbb{C})$ for z is contained in $\mathrm{U}(r)$.

A couple of comments are in order on the above definition. Take any $x \in D$. If the isotropy subgroup of $\mathrm{GL}(r, \mathbb{C})$ for some $z \in \widehat{\xi}^{-1}(x) \bigcap E_{\mathrm{U}(r)}$ is contained in $\mathrm{U}(r)$, then the isotropy subgroup for every point of $\widehat{\xi}^{-1}(x) \bigcap E_{\mathrm{U}(r)}$ is contained in $\mathrm{U}(r)$; this is because any two such isotropy subgroups are conjugate by some element of $\mathrm{U}(r)$. Since the isotropy subgroup for every $z \in \widehat{\xi}^{-1}(D)$ is compact, a conjugate of it is contained in $\mathrm{U}(r)$.

Let E_* be a parabolic vector bundle on X with parabolic structure over D. Let $E_{\mathrm{GL}(r,\mathbb{C})}$ be the corresponding ramified principal $\mathrm{GL}(r, \mathbb{C})$-bundle over X with ramification over D.

Definition 3.1. An admissible Hermitian metric on E_* is a smooth Hermitian metric H on the vector bundle $E|_{X \setminus D}$ such that the C^∞ reduction of structure group

$$E'_{\mathrm{U}(r)} \subset E_{\mathrm{GL}(r,\mathbb{C})}|_{X \setminus D}$$

to the subgroup $\mathrm{U}(r) \subset \mathrm{GL}(r, \mathbb{C})$ over $X \setminus D$ extends to a Hermitian structure on the ramified principal bundle $E_{\mathrm{GL}(r,\mathbb{C})}$.

As above, let E_* be a parabolic vector bundle of rank r with $\widehat{\xi} : E_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$ the corresponding ramified principal $\mathrm{GL}(r,\mathbb{C})$-bundle. Let V be a Γ-equivariant bundle on Y that corresponds to E_*. The Γ-equivariant holomorphic principal $\mathrm{GL}(r,\mathbb{C})$-bundle on Y associated to V will be denoted by $V_{\mathrm{GL}(r,\mathbb{C})}$. Let

$$q_0 : V_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow V_{\mathrm{GL}(r,\mathbb{C})}/\Gamma = E_{\mathrm{GL}(r,\mathbb{C})} \qquad (3.1)$$

be the quotient map.

With the above set up, we have the following simple but useful lemma.

Lemma 3.1.   Every admissible Hermitian metric on E_* is the descent of a unique Γ-invariant Hermitian structure on V.

Conversely, if h is a Γ-invariant Hermitian structure on V such that the corresponding reduction of structure group

$$V_{\mathrm{U}(r)} \subset V_{\mathrm{GL}(r,\mathbb{C})}$$

to $\mathrm{U}(r)$ has the property that the quotient $V_{\mathrm{U}(r)}/\Gamma$ is a C^∞ submanifold of $V_{\mathrm{GL}(r,\mathbb{C})}/\Gamma = E_{\mathrm{GL}(r,\mathbb{C})}$, then

$$(V_{\mathrm{U}(r)}/\Gamma)|_{X \setminus D} \subset E_{\mathrm{GL}(r,\mathbb{C})}|_{X \setminus D}$$

is an admissible Hermitian metric on E_*.

Proof.   Take an admissible Hermitian metric

$$E'_{\mathrm{U}(r)} \subset E_{\mathrm{GL}(r,\mathbb{C})}|_{X \setminus D}$$

on E_*. Let

$$E_{\mathrm{U}(r)} \subset E_{\mathrm{GL}(r,\mathbb{C})}$$

be the Hermitian structure on the ramified principal bundle $E_{\mathrm{GL}(r,\mathbb{C})}$ obtained by extending $E'_{\mathrm{U}(r)}$. Now

$$q_0^{-1}(E_{\mathrm{U}(r)}) \subset V_{\mathrm{GL}(r,\mathbb{C})}$$

is a Γ-invariant Hermitian structure on V, where q_0 is the quotient map in (3.1). Uniqueness of the Γ-invariant Hermitian structure on V is evident.

To prove the converse, take a Γ-invariant Hermitian structure h on V satisfying the condition in the statement of the lemma. Let

$$V_{\mathrm{U}(r)} \subset V_{\mathrm{GL}(r,\mathbb{C})}$$

be the corresponding reduction of structure group to the subgroup $\mathrm{U}(r) \subset \mathrm{GL}(r,\mathbb{C})$. Since h is Γ-invariant, the action of Γ on $V_{\mathrm{GL}(r,\mathbb{C})}$ preserves the submanifold $V_{\mathrm{U}(r)}$. So $V_{\mathrm{U}(r)}/\Gamma$ is equipped with an action of $\mathrm{U}(r)$.

Since $V_{\mathrm{U}(r)}$ is preserved by the action of Γ, for any point $y \in Y$ and any $z \in (V_{\mathrm{U}(r)})_y$, the orbit of z under the action of the isotropy subgroup $\Gamma_y \subset \Gamma$ is contained in $(V_{\mathrm{U}(r)})_y$. This implies that for each point $u \in \widehat{\xi}^{-1}(D) \bigcap E_{\mathrm{U}(r)}$, the isotropy subgroup for the action of $\mathrm{GL}(r,\mathbb{C})$ for u is contained in $\mathrm{U}(r)$. Consequently,

$$(V_{\mathrm{U}(r)}/\Gamma)|_{X \setminus D} \subset E_{\mathrm{GL}(r,\mathbb{C})}|_{X \setminus D}$$

is an admissible Hermitian metric on E_*.   ∎

Recall that the proof of Lemma 2.1 is based on the fact that the divisor D' moves freely. This, combined with the proof of Lemma 3.1, gives the following:

Lemma 3.2.   Let E_* be a parabolic vector bundle on X with parabolic divisor D. Let H be a C^∞ Hermitian metric on $E|_{X \setminus D}$ such that for every Kawamata covering

$$q : Y \longrightarrow X$$

for E_*, there is a $\mathrm{Gal}(q)$-invariant Hermitian structure H' on the $\mathrm{Gal}(q)$-equivariant vector bundle V corresponding to E_* such that H' descends to H. Then H is admissible.

In the next few paragraphs we discuss the above constructions from a concrete differentio-geometric point of view. For ease of exposition, in the rest of this section (unless specified otherwise) we will deal with good Kawamata covers.

As preparation, whenever we talk of a trivialization around a point on D, we consider "adapted frames", that is, frames $\{e_1, e_2, \ldots\}$ on a neighborhood in X of a point in D such that when restricted to D, the collection $e_1, \ldots, e_{r_j}$ is a frame for F_j ($r_j - r_{j+1}$ is called the multiplicity of the weight α_j).

Fix a good Kawamata cover

$$p : Y \longrightarrow X \tag{3.2}$$

such that the parabolic bundle E_* is the invariant direct image of an equivariant vector bundle V over Y. The Galois group $\mathrm{Gal}(p)$ will be denoted by Γ.

Assume that the adapted frame on X is induced from a frame $\widetilde{e}_i$ on Y such that the action of Γ in this frame is diagonal. As mentioned above, for point $y \in p^{-1}(X \setminus D)$, the action of Γ_y on V_y is trivial and $E|_{X \setminus D}$ is the descent of $V|_{p^{-1}(X \setminus D)}$. Therefore, any Γ-invariant Hermitian metric on $V|_{p^{-1}(X \setminus D)}$ descends to a Hermitian metric on $E|_{X \setminus D}$, and

conversely, every Hermitian metric on $E|_{X \setminus D}$ is given by a unique Γ-invariant Hermitian metric on $V|_{p^{-1}(X \setminus D)}$. Therefore, for the construction of a singular Hermitian metric on E singular over D we may pretend that D' is absent. In fact, our singular Hermitian metric on E will be given by a Γ invariant Hermitian metric on V for a covering p with minimal degree of ramification over D.

Cover $X \setminus D$ with coordinate open sets U_γ with coordinates $z_{1,\gamma}, z_{2,\gamma}, \ldots, z_{n,\gamma}$ such that $p^{-1}(U_\gamma)$ is a coordinate open set on Y with coordinates $w_{1,\gamma}, \ldots, w_{n,\gamma}$, the branched cover is given by $w_{1,\gamma} = z_{1,\gamma}^{n_\gamma}$ (and $w_{i,\gamma} = z_{i,\gamma} \ \forall \ i \geq 2$), if $n_\gamma > 1$ the component D of the branching divisor for q is given by $z_1 = 0$, and V is locally trivial over $p^{-1}(U_\gamma)$. Denote neighborhoods intersecting D by $U_{D,\gamma}$. The sheaf E_* on $U_{D,\gamma}$ is generated freely by $z_1^{\alpha_{r-j+1}} \widetilde{e}_j$. Therefore, the transition functions between $U_{D,\gamma}$ and $U_{D,\beta}$ are $g_{\gamma_D \beta_D}(z) = \mathrm{diag}[z_{1,\gamma}^{-\alpha_r} \ldots z_{1,\gamma}^{-\alpha_1}] \widetilde{g}(\widetilde{z})_{\gamma\beta} \mathrm{diag}[z_{1,\beta}^{\alpha_r} \ldots z_{1,\beta}^{\alpha_1}]$, where $\widetilde{g}$ are the transition functions of V and $\widetilde{z}$ is the corresponding element in $p^{-1}(z)$. Likewise, $g_{\gamma_D \beta}(z) = \mathrm{diag}[z_{1,\gamma}^{-\alpha_r} \ldots z_{1,\gamma}^{-\alpha_1}] \widetilde{g}(\widetilde{z})_{\gamma\beta}$.

Definition 3.2. Suppose E_* and G_* are parabolic bundles with parabolic divisor D on a complex projective manifold X, and flags, rational weights, and transition functions $(F_j^E, \alpha_j = \frac{a_j}{N}, g^E)$ and $(F_k^G, \beta_j = \frac{b_j}{N}, g^G)$, respectively, where N is the smallest such integer. Consider the N-fold branched cover $p : Y \longrightarrow X$ ramified over a smooth reduced effective divisor D such that E_* and F_* are invariant direct images of bundles V and W over Y with transition functions $\widetilde{g}^V$ and $\widetilde{g}^W$. For the convenience of differential geometers, we write the transition functions of the aforementioned constructions of parabolic bundles below. On $X \setminus D$ they are given by their usual constructions. Therefore, we mention only $g_{\gamma_D \beta}$ and $g_{\gamma_D \beta_D}$.

Bundle	Flag	Weights	Transition functions
E_*^*	$(F_j^E)^*$	$\alpha^* = 1 - \alpha_i$ if $\alpha_i > 0$ and 0 otherwise.	$g_{\gamma_D \beta_D}^{E^*} = \mathrm{diag}[z_{1,\gamma}^{-\alpha_r^*} \ldots z_{1,\gamma}^{-\alpha_1^*}]$ $\times ((\widetilde{g}_{\gamma\beta})^T)^{-1} \mathrm{diag}[z_{1,\beta}^{\alpha_r^*} \ldots z_{1,\beta}^{\alpha_1^*}]$ $g_{\gamma_D \beta}^{E^*} = \mathrm{diag}[z_{1,\gamma}^{-\alpha_r^*} \ldots z_{1,\gamma}^{-\alpha_1^*}]((\widetilde{g}_{\gamma\beta})^T)^{-1}$.
$E_* \oplus G_*$	$F_j^E \oplus F_k^G$	$\{\alpha_j\} \cup \{\beta_k\}$	$g = g^E \oplus g^G$.
$\det(E_*)$	$\det(E_*) \supsetneq 0$	$\alpha^* = \sum \alpha_i \bmod 1$	$g_{\gamma_D \beta_D}^{\det(E)} = \det(\widetilde{g}_{\gamma\beta}) \frac{z_{1,\beta}^{\alpha^*}}{z_{1,\gamma}^{\alpha^*}}$, and $g_{\gamma_D \beta}^{\det(E)} = \det(\widetilde{g}_{\gamma\beta}) \frac{1}{z_{1,\gamma}^{\alpha^*}}$

Note that the fractional powers z_1^α in the previous definition and the paragraph preceding that are to be taken on a suitably chosen branch of the complex plane.

Take any triple (Y, Γ, V) as above such that the parabolic vector bundle E_* coincides with the parabolic vector bundle associated to the Γ-linearized V. Then the

projectivization $\mathbb{P}(E_*)$ is defined to be the quotient $\mathbb{P}(V)/\Gamma$. Since for any $y \in p^{-1}(X \setminus D)$ the action of the isotropy subgroup $\Gamma_y \subset \Gamma$ on the fiber V_y is trivial, the quotient $\mathbb{P}(V)/\Gamma$ is a projective bundle over $X \setminus D$. More precisely, the pullback of this bundle

$$(\mathbb{P}(V)/\Gamma)|_{X \setminus D} \longrightarrow X \setminus D$$

to $Y \setminus p^{-1}(D)$ is canonically Γ-equivariantly identified with the projective bundle

$$\mathbb{P}(V)|_{Y \setminus p^{-1}(D)} \longrightarrow Y \setminus p^{-1}(D)$$

(any pullback to Y from X is equipped with a tautological action of Γ). The above quotient $\mathbb{P}(V)/\Gamma$ depends only on E_* and it is independent of the choice of (Y, Γ, V). There is a positive integer m such that the isotropy groups, for the action of Γ on $\mathbb{P}(V)$, act trivially on the fibers of $\mathcal{O}_{\mathbb{P}(V)}(m)$. Therefore, $\mathcal{O}_{\mathbb{P}(V)}(m)$ descends to the quotient $\mathbb{P}(V)/\Gamma = \mathbb{P}(E_*)$ as a line bundle. Note that the discussion in this paragraph applies equally well even when Y is a general (not necessarily good) Kawamata cover.

For differentio-geometric purposes, we need to study metrics and connections on both the bundle as well as the base manifold that respect the parabolic structure. For a given E_* there may be coverings as in (3.2) giving E_* such that the degree of the ramification over D is arbitrarily large. We would be interested in coverings for which this degree is minimal. By a minimal branched cover of X for E_* we will mean a covering p as in (3.2) giving E_* such that the degree of the ramification over D is minimal. As in the previous paragraph, the Kawamata cover may be ramified over a divisor $D \cup D'$ but the parabolic structure over D' is trivial.

Now we look at the local picture of an admissible metric. Suppose we choose coordinates $(w_1, \ldots, w_n)$ on a good Kawamata cover Y such that $w_1 = 0$ is the ramification divisor (assumed to be smooth), $z_1 = w_1^N$, $z_i = w_i \; \forall \, i \geq 2$ is the quotient map near D, and a holomorphic frame $\widetilde{e}_1, \ldots, \widetilde{e}_r$ that is compatible with the flag. Note that the invariant direct image E_* is locally freely generated by $e_j = \widetilde{e}_j w_1^{k_{r+1-j}}$ where k_j are the weights of the action of Γ.

Lemma 3.3.   The metric H induced on E_* is locally (near D) of the form

$$H_{ij}(z) = \bar{z}_1^{\alpha_{r+1-i}} \widetilde{H}_{ij}(w) z_1^{\alpha_{r+1-j}}$$

(on a chosen branch of the complex plane). Conversely, if H is a smooth metric on $E_*|_{X\setminus D}$ such that

$$\widetilde{H}_{ij}(w) = \bar{z}_1^{-\alpha_{r+1-i}}H_{ij}(z)z_1^{-\alpha_{r+1-j}}$$

extends smoothly (as a function of w) and positively across the branch cut then H is induced from a Γ-invariant metric $\widetilde{H}$ on V.

Proof. Since $H_{ij}(z) = \langle e_i, e_j\rangle_{\widetilde{H}}$, we see that (using the physicist's convention for inner product)

$$H_{ij}(z) = \bar{z}_1^{\alpha_{r+1-i}}\widetilde{H}_{ij}(w)z_1^{\alpha_{r+1-j}}.$$

Conversely, given a metric H on E_* over $X \setminus D$, it induces an invariant metric $\widetilde{H}$ on V over $Y \setminus \pi^{-1}(D)$. It follows from the above observation that

$$\widetilde{H}_{ij}(w) = \bar{z}_1^{-\alpha_{r+1-i}}H_{ij}(z)z_1^{-\alpha_{r+1-j}}.$$

This clearly defines an invariant metric on V because

$$\widetilde{H}_{ij}(e^{\sqrt{-1}\theta}w_1, w_2, \ldots) = e^{(k_{r+1-i}-k_{r+1-j})\sqrt{-1}\theta}\bar{z}_1^{-\alpha_{r+1-i}}H_{ij}(z)z_1^{-\alpha_{r+1-j}} = e^{(k_{r+1-i}-k_{r+1-j})\sqrt{-1}\theta}\widetilde{H}_{ij}(w)$$

(and hence $\langle a, b\rangle_{\widetilde{H}}$ is invariant). Moreover, the expression clearly does not depend on the branch cut chosen. So the only potentially problematic points occur on $w_1 = 0$. But we know that $\widetilde{H}$ extends smoothly and hence defines a smooth invariant metric on V. ∎

Now we show a way to produce examples of admissible Hermitian metrics in the case where minimal good Kawamata covers exist.

Lemma 3.4. Let H be induced by a Γ_1-invariant metric $\widetilde{H}_1$ on V_1 over a minimal N_1-fold good Kawamata cover Y_1 over $X = Y_1/\Gamma_1$. Let (Y_p, q_p, V_p) be any locally good Kawamata covering of X around p, with Galois group Γ_2 inducing the parabolic bundle E_* on X. Then, $H|_{U_p}$ is induced by a smooth Γ_2-invariant metric $\widetilde{H}_2|_{q_p^{-1}(U_p)}$ on $V_2|_{q_p^{-1}(U_p)}$.

Proof. By minimality, $\frac{N_2}{N_1} = u$ is an integer ≥ 1.

Suppose we choose coordinates z and w near $D : z_1 = 0$ on X and Y_1, respectively (such that $z_1 = w_1^{N_1}$), and an admissible frame e_i for E_* induced from Y_1. Then

$$(\widetilde{H}_1)_{ij}(w) = \bar{w}_1^{-k_{r+1-i}}H_{ij}(z(w))w_1^{-k_{r+1-j}}. \tag{3.3}$$

Let Z and W (such that $Z_1 = W_1^{N_2}$) be coordinates near $D : Z_1 = 0$ on X and Y_2, respectively, and an admissible frame f_I for E_* induced from Y_2. Clearly, $Z_1 = g(z)z_1$ where $g(z) \neq 0$ is a holomorphic function. Therefore, $w_1 = z_1^{1/N_1} = \frac{1}{g(z)^{1/N_1}}W_1^u$, that is, w_1 is a holomorphic function of W_1. Note that H induces an invariant metric $\widetilde{H}_2$ outside the ramification divisor on Y_2. Let $f_I = t_I^j e_j$ (where we used the Einstein summation convention). By definition of admissibility, $t_I^j = 0$ whenever $j > m(I)$ where $m(1), m(2) \ldots, m(r_1) = r_1; m(r_1 + 1), \ldots, m_{r_2} = r_2, \ldots$. The expression for $\widetilde{H}_2$ near D is given by

$$
\begin{aligned}
(\widetilde{H}_2)_{IJ}(W) &= \overline{W}_1^{-k_{r+1-I}} H_{IJ}(Z(W)) W_1^{-k_{r+1-J}} = \overline{Z}_1^{-\alpha_{r+1-I}} H_{IJ}(Z(W)) Z_1^{-\alpha_{r+1-J}} \\
&= \overline{g(z)}^{-\alpha_{r+1-I}} g(z)^{-\alpha_{r+1-J}} \overline{z}_1^{-\alpha_{r+1-I}} H_{IJ}(Z(W)) z_1^{-\alpha_{r+1-J}} \\
&= \overline{g(z)}^{-\alpha_{r+1-I}} g(z)^{-\alpha_{r+1-J}} \overline{z}_1^{-\alpha_{r+1-I}} H_{IJ}(Z(W)) z_1^{-\alpha_{r+1-J}} \\
&= \overline{g(z)}^{-\alpha_{r+1-I}} g(z)^{-\alpha_{r+1-J}} \overline{z}_1^{-\alpha_{r+1-I}} t_I^i \overline{t}_J^j H_{ij}(Z(W)) z_1^{-\alpha_{r+1-J}}.
\end{aligned}
\tag{3.4}
$$

As in Lemma 3.3, $\widetilde{H}_2$ can only be problematic on $W_1 = 0$. Since the weights satisfy $\alpha_I \geq \alpha_J$ if $I \geq J$, and the expression in (3.3) is a smooth function of w (which is in turn a smooth function of W), we see that $\widetilde{H}_2(W)$ is smooth even at the origin. ∎

Remark 3.3.   Lemma 3.3, being a local condition, applies even to locally good Kawamata covers Y_1.

4   Positivity and Ampleness

Suppose (Y, q, V) is a Kawamata cover of (X, E_*, D). Then a Γ-invariant Hermitian metric on V produces a Γ-invariant Hermitian metric on $\mathcal{O}_{\mathbb{P}(V)}(1)$, and hence a Γ-invariant Hermitian metric on every $\mathcal{O}_{\mathbb{P}(V)}(n)$. Therefore, if $\mathcal{O}_{\mathbb{P}(V)}(m)$ descends to $\mathbb{P}(E_*)$, the descended line bundle gets a Hermitian metric.

It is known that the parabolic vector bundle E is Hartshorne ample (respectively, Hartshorne nef) if the vector bundle V is Hartshorne ample (respectively, Hartshorne nef) [8, 13, 14].

Now we define Griffiths positivity of an admissible Hermitian metric.

Definition 4.1.   Suppose H is an admissible Hermitian metric on E_*. Then H is said to be Griffiths (respectively, Nakano) positively curved if for every locally good Kawamata cover (Y_p, q_p, V_p) around an arbitrary point p, the induced Hermitian metric $\widetilde{H}_p$ is so (in the usual sense of positivity) on $q^{-1}(U_p)$.

In particular, if an admissible metric is positively curved, then it is so in the usual sense on $X \setminus D$ because of Lemma 2.1. Moreover, for any Kawamata cover (Y, q, V), it is easy to see that the induced smooth Hermitian metric $\widetilde{H}$ is Griffiths nonnegatively curved in the usual sense with strict positivity outside the branching divisor.

Remark 4.2.   If there is a good Kawamata cover, then it is locally good over every open set and hence by definition the bundle $(V, \widetilde{H})$ is Griffiths positively curved. Therefore, V is Hartshorne ample and hence E is parabolic Hartshorne ample. However, if there is no good Kawamata cover, it is not obvious whether the parabolic Griffiths positivity of (E_*, H) implies the Hartshorne ampleness of V (which in turn implies the parabolic Hartshorne ampleness of E) or not. The best we can say directly is that V is Hartshorne nef because the induced metric is positively curved away from the branching divisor. When there is no good Kawamata cover it is nontrivial (Lemma 4.4) to prove that V is actually Hartshorne ample.

We have the following obvious analogue of the usual Griffiths conjecture.

Parabolic Griffiths conjecture: There is an admissible metric H such that (E_*, H) is Griffiths positively curved if and only if E_* is Hartshorne ample.

Before proceeding further, we define the notion of an admissible metric on X with cone singularities on an effective reduced irreducible divisor D with simple normal crossings.

Definition 4.3.   If (X, D) is a compact projective manifold with an effective smooth reduced divisor D, then an $0 \leq \alpha < 2$-admissible Hermitian (respectively, Kähler) metric on TX is a Hermitian (respectively, Kähler) metric ω on $X \setminus D$ such that near D, in coordinates $z_1, \ldots, z_n$ such that D is $z_1 = 0$,

$$\omega - \left(|z_1|^{-\alpha} dz_1 \wedge d\bar{z}_1 + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i \right) \tag{4.1}$$

is smooth.

Remark 4.4.   Suppose $\alpha = 2 - \frac{2}{N}$ for a positive integer N. If there is a good Kawamata N-fold branched cover Y branched over the divisor D (thus $X = Y/\Gamma$ where Γ is a finite group), then it is not hard to see that an α-admissible Kähler metric in this case is induced by a Γ-invariant Kähler metric on Y and vice versa. In the general case, suppose we cover X by finitely many neighborhoods U_p that admit locally good Kawamata covers

(Y_p, q_p, V_p). Assume that Y_p are endowed with Kähler metrics ω_p. Then on U_p these metrics induce admissible Kähler metrics $\omega_{p,\alpha}$. Using a partitionofunity one can patch these to get an admissible Hermitian metric ω_α. Actually, every (X, D) admits an α-admissible Kähler metric if X is compact Kähler. Indeed, take any smooth Hermitian metric h on the bundle $[D]$. Suppose σ is a defining section of $[D]$ and ω is a Kähler metric on X. Then $k\omega + \sqrt{-1}\partial\bar{\partial}|\sigma|_h^{2-\alpha}$ is an admissible Kähler metric for large enough k. This metric induces Kähler metrics on $q_p^{-1}(U_p)$ for any locally good Kawamata cover (Y_p, q_p).

The following lemma reduces the problem of checking parabolic Griffiths positivity to usual Griffiths positivity on an open set.

Lemma 4.1. Suppose N is the degree of any locally good minimal Kawamata cover of (X, E_*, D). Suppose H is an admissible Hermitian metric. Then (E_*, H) is parabolic Griffiths positive if and only if on $X \setminus D$,

$$\Theta_H \geq C\omega_\alpha$$

in the Griffiths sense where ω_α is any $\alpha = 2 - \frac{2}{N}$ admissible metric and $C > 0$ is a constant.

Proof. We recall that (E_*, H) is parabolic Griffiths positive if and only if the induced metric $\widetilde{H}_p$ is Griffiths positive on U_p (and of course Griffiths nonnegative on Y_p) for every locally good Kawamata cover and every point $p \in X$. Now, the Hermitian metric $\widetilde{H}_p$ is Griffiths positive if and only if $\Theta_{\widetilde{H}_p} \geq C\omega_p$ in the Griffiths sense where ω_p is any Kähler metric on Y_p. Using Lemma 3.3, we compute the curvature Θ_H of H in U_p where $p \in D$ as follows:

$$(\Theta_H)_{ij}(z) = z_1^{-\alpha_{r+1-i}}(\Theta_{\widetilde{H}})_{ij}(w(z))z_1^{\alpha_{r+1-j}}. \tag{4.2}$$

Now ω_p induces an α-admissible metric $\omega_{p,\alpha}$ on X. The inequality $\Theta_{\widetilde{H}_p} \geq C\omega_p$ when written in the z coordinates is equivalent to

$$z_1^{-\alpha_{r+1-i}}(\Theta_H)_{ij}(z)z_1^{\alpha_{r+1-j}} \geq C\omega_{p,\alpha}.$$

Elsewhere this inequality is obvious. Therefore, $\Theta_H \geq C\omega_\alpha$ where ω_α is constructed by patching together $\omega_{p,\alpha}$. But since this holds for one admissible metric, it is easy to see that it does so for all. The converse follows by retracing the arguments above. ∎

The next lemma shows that the curvature of an admissible metric is a current for line bundles.

Lemma 4.2.   If (L_*, h) is a parabolic line bundle with an admissible metric on X, then the curvature Θ_h extends to a closed current on X that agrees with a smooth form outside D. Moreover, $c_1(h) + \alpha[D]$ is an L^1 form (smooth outside D).

Proof.   Obviously Θ_h is smooth outside D. In a neighborhood of a point in D, let $(Y_p, q_p, V_p, \widetilde{h}_p)$ be a locally good Kawamata cover. Then $h(z) = \widetilde{h}_p(w)|z_1|^{2\alpha}$. Therefore,

$$
c_1(h) = -\alpha[D] + \frac{\sqrt{-1}}{2\pi} \overline{\partial}_z \partial_z \ln \widetilde{h}_p(w)
$$

$$
= -\alpha[D] + \frac{\sqrt{-1}}{2\pi} \Theta_p(w(z))_{ij} dw^i \wedge dw^{\bar{j}}
$$

$$
= -\alpha[D] + \frac{\sqrt{-1}}{2\pi} \left[\frac{1}{N^2} |z_1|^{2/N-2} \Theta(w(z))_{11} dz^1 \wedge d\bar{z}^1 + \sum_{i=2}^{n} \frac{1}{N} z_1^{1/N-1} \Theta(w(z))_{1i} dz^1 \wedge d\bar{z}^i \right.
$$

$$
+ \sum_{i=2}^{n} \frac{1}{N} \bar{z}_1^{1/N-1} \Theta(w(z))_{1i} dz^i \wedge d\bar{z}^1 + \left. \sum_{i,j=2}^{n} \Theta(w(z))_{ij} dz^i \wedge d\bar{z}^{\bar{j}} \right]
$$

$$
\Rightarrow c_1(h) + \alpha[D] = \frac{\sqrt{-1}}{2\pi} \left[\frac{1}{N^2} |z_1|^{2/N-2} \Theta(w(z))_{11} dz^1 \wedge d\bar{z}^1 \right.
$$

$$
+ \sum_{i=2}^{n} \frac{1}{N} z_1^{1/N-1} \Theta(w(z))_{1i} dz^1 \wedge d\bar{z}^i
$$

$$
+ \sum_{i=2}^{n} \frac{1}{N} \bar{z}_1^{1/N-1} \Theta(w(z))_{1i} dz^i \wedge d\bar{z}^1 + \left. \sum_{i,j=2}^{n} \Theta(w(z))_{ij} dz^i \wedge d\bar{z}^{\bar{j}} \right]. \tag{4.3}
$$

It is easy to see that $c_1(h) + \alpha[D]$ is an L^1 form. It is clearly closed away from D. As a current, suppose f is a smooth compactly supported $(n-1, n-1)$-form in a coordinate neighborhood B of D; if B_ϵ is everything in B outside a tubular neighborhood of D of size ϵ, then

$$
\int_{B_\epsilon} \partial \overline{\partial} f(c_1(h) + \alpha[D]) = \frac{\sqrt{-1}}{2\pi} \int_{\partial B_\epsilon} \partial \overline{\partial} f \partial \ln \widetilde{h}(w(z)). \tag{4.4}
$$

The latter is easily seen to go to 0 as $\epsilon \to 0$. Therefore, $c_1(h) + \alpha[D]$, and hence, $c_1(h)$ are closed currents. ∎

Now we prove that for line bundles, our metric notion of parabolic ampleness coincides with the algebro-geometric one in [8].

Lemma 4.3.   A parabolic line bundle (L_*, H) is ample in the metric sense above for some admissible metric H if and only if L_* is parabolic ample in the algebro-geometric sense.

Proof.   A parabolic line bundle is algebro-geometrically parabolic ample if and only if $L + \alpha[D]$ is a Kähler class [8].

Suppose $L + \alpha D$ is a Kähler class $[\omega]$ and assume that $h_{L,0}$ and $h_{D,0}$ are metrics on L and $[D]$, respectively. Then

$$c_1(h_{L,0}) + \alpha c_1(h_{D,0}) = \omega + \frac{1}{2\pi}\sqrt{-1}\partial\bar{\partial}\phi.$$

Defining a new metric $h_L = h_{L,0}e^\phi$ we see that

$$c_1(h_L) + \alpha c_1(h_D) = \omega.$$

Now define a singular metric on L as $h_{par,L} = h_L|\sigma|^\alpha_{h_{D,0}}$ where σ is the canonical section of $[D]$. Away from D, we have $c_1(h_{par,L}) = \omega$. If Y_p is any locally good minimal branched cover of X around $p \in D$ and we choose coordinates z on X and w on Y so that near p, $w_1^N = z_1$, $w_i = z_i \ \forall \ i \geq 2$, then $\sigma = z_1$ and $|z_1|^{-2\alpha}h_{par,L} = h_L h_{D,0}^\alpha$, which is a smooth positive function of z and hence of w in any neighborhood of $w_1 = 0$. Thus, by Lemma 3.3 $(L_*, h_{par,L})$ is parabolic ample in the metric sense.

Conversely, suppose (L_*, H) is parabolic ample in the metric sense. Then by Lemma 4.1 $\Theta_H \geq K\omega_\alpha$ away from D. Putting a singular metric $\frac{1}{|\sigma|^2}$ on $[D]$, we see that $L + \alpha[D]$ is represented by the current $\widetilde{\omega} = \Theta_H + \alpha[D]$. This current is a positive measure from Lemma 4.2. It is easy to see (using the Bedford–Taylor [2] definition of products of currents) that the Nakai–Moishezon criterion is verified, that is, that $(L + \alpha[D])^k.C > 0$ for all k-dimensional subvarieties C. Indeed, if C is not contained in D, then it is trivial because the current $\widetilde{\omega} > 0$ and in L^1. If C is contained in D, then locally $C = D \cap E$ where E is another subvariety. Now approximating the currents $2\pi[D] = \sqrt{-1}\partial\bar{\partial}\ln|z_1|^2, [E], \widetilde{\omega}$ by $D_\epsilon, E_\epsilon, \widetilde{\omega}_\epsilon$ (in the case of $[D]$ and $\widetilde{\omega}$, simply replacing $|z_1|^2$ by $|z_1|^2 + \epsilon^2$) and taking limits we see that $(L + \alpha D)^k.C > 0$. (The approximations weakly converge by Bedford–Taylor theory.) Hence $L + \alpha[D]$ is a Kähler class. ∎

The next lemma shows that parabolic Griffiths positivity implies parabolic Hartshorne ampleness in the metric sense. This is the "easy" direction of the Griffiths conjecture.

Lemma 4.4.   Suppose (E_*, H) is a parabolic bundle on X with an admissible metric H. This induces an admissible metric h on the parabolic bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1)_*$ over $\mathbb{P}(E_*)$. Moreover, if H is parabolic Griffiths positive, E is parabolic Hartshorne ample.

Proof.   Suppose (Y, q, V) is a Kawamata cover of (X, E_*, D). By definition, H is the descent of a smooth Hermitian metric $\widetilde{H}$ on V. Since

$$\mathbb{P}(E_*) \,=\, \mathbb{P}(V)/\Gamma$$

and $\mathcal{O}_{\mathbb{P}(E_*)}(1)_*$ is the invariant direct image of $\mathcal{O}_{\mathbb{P}(V)}(1)$, the smooth metric $\widetilde{h}$ that $\widetilde{H}$ induces on $\mathcal{O}_{\mathbb{P}(V)}(1)$ induces a unique metric h downstairs on $\mathcal{O}_{\mathbb{P}(E_*)}(1)_*$. Since the cover (Y, q, V) is arbitrary, h is admissible (by Lemma 3.1). Moreover, since $\widetilde{h}$ is positively curved whenever $\widetilde{H}$ is, by Lemma 3.1 and Lemma 4.1, h is positively curved. Hence, by Lemma 4.3 we are done.   ∎

Finally, we recall the notion of stability (and semistability) in the parabolic case and recall the existence of Hermite–Einstein admissible metrics in the parabolic setting when the weights $\alpha_i = \frac{k_i}{N}$ are rational.

Take a parabolic bundle E_*. For any coherent subsheaf $F \subset E$, the parabolic structure on E induces a parabolic structure on F. This induced parabolic structure will be denoted by F_*. A parabolic bundle E_* is called *parabolic stable* (respectively, *parabolic semistable*) if for any coherent subsheaf $0 \neq F \subsetneq F$ with E/F torsion-free, we have

$$\frac{\text{par-deg}(E_*)}{\text{rank}(E)} \,<\, \frac{\text{par-deg}(F_*)}{\text{rank}(F)} \quad \left(\text{respectively,} \quad \frac{\text{par-deg}(E_*)}{\text{rank}(E)} \,\leq\, \frac{\text{par-deg}(F_*)}{\text{rank}(F)}\right).$$

A parabolic vector bundle E_* is called *polystable* if

(1) it is parabolic semistable and
(2) it is a direct sum of parabolic stable bundles.

Mehta and Seshadri proved that a parabolic vector bundle over a Riemann surface admits a unitary flat connection if and only if it is polystable of parabolic degree zero [34]. Biquard proved that a parabolic vector bundle over a Riemann surface admits a Hermite–Einstein connection if and only if it is polystable [6]. This was generalized to parabolic bundles on higher dimensional varieties in [7] and [11].

Let $q : Y \longrightarrow X$ be a good Kawamata cover with Galois group Γ and $V \longrightarrow Y$ be a Γ-equivariant bundle such that E_* corresponds to V.

We will show that E_* is semistable if and only if V is semistable. First assume that E_* is not semistable. Let $F \subset E$ be a subsheaf that violates the semistability condition. Then the subsheaf of V generated by q^*F such that the quotient of V by it is torsion-free contradicts the semistability condition for V. Conversely, if V is not semistable, consider the 1st term W of the Harder–Narasimhan filtration of V. From the uniqueness of the Harder–Narasimhan filtration, it follows that the subsheaf W is preserved by the action of Γ on V. Hence, the invariant direct image $(p_*W)^\Gamma$ is a subsheaf of E. This subsheaf violates the semistability condition for E_*.

By an identical argument, it follows that E_* is polystable if and only if V is polystable; we just need to replace the Harder–Narasimhan filtration by the socle filtration (see [27, p. 23, Lemma 1.5.5] for the socle filtration).

If E_* is polystable, then consider the Hermite–Einstein connection on the polystable vector bundle V. From the uniqueness of the Hermite–Einstein connection it follows that it is preserved by the action of Γ. Hence it descends to a Hermite–Einstein connection on E_*. Conversely, a Hermite–Einstein connection on E_* produces a Hermite–Einstein connection on V, which implies that V is polystable. Hence E_* is polystable. So E_* is polystable if and only if it admits an Hermite–Einstein structure.

4.1   Parabolic Griffiths conjecture for Riemann surfaces

Take a short exact sequence of parabolic bundles

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0. \qquad (4.5)$$

We will show that if A_* and C_* are Nakano positive then so is B_*. To prove this it suffices to show that if

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

is a short exact sequence of equivariant vector bundles, and both A' and C' admit equivariant Hermitian structures that are Nakano positive, then B' also admits a Hermitian structure that is Nakano positive. To prove that B' admits a Hermitian structure that is Nakano positive, observe that Lemma 2.2 of [44] extends to equivariant set up without any change. Therefore, we conclude that B_* is Nakano positive if A_* and C_* are so.

Armed with this observation about short exact sequences, we may prove the parabolic version of Griffiths' conjecture for Riemann surfaces.

Theorem 4.5.   Take a parabolic vector bundle E_* on a Riemann surface arising from a good Kawamata cover. Then the following two are equivalent:

(1) E_* is parabolic ample.
(2) E_* is Nakano positive.

Proof.   First assume that E_* is Nakano positive. Then every quotient of E_* has positive parabolic degree. This implies that E_* is parabolic ample [8, Theorem 3.1].

Now assume that E_* is parabolic ample. Therefore, every quotient of E_* has positive parabolic degree [8, Theorem 3.1]. Consider the Harder–Narasimhan filtration of E_*. Every semistable parabolic vector bundle on a Riemann surface admits a filtration of subbundles such that each successive quotient is stable of same parabolic slope. Using it, we construct a finer filtration of the Harder–Narasimhan filtration of E_* such that each successive quotient is stable and the parabolic slopes are decreasing (now the slope is no longer strictly decreasing). Next, observe that a stable parabolic bundle of positive parabolic degree is Nakano positive because the Hermitian–Einstein metric on it is Nakano positive. On the other hand, E_* is a successive extensions by stable parabolic bundles of positive degree starting with a stable parabolic bundle of positive degree. Therefore, from the earlier observation that B_* in (4.5) is Nakano positive if A_* and C_* are so, we now conclude that E_* is Nakano positive.   ■

5   Chern–Weil Theory for Parabolic Bundles

In this section, we develop Chern–Weil theory for admissible Hermitian metrics and provide further evidence for Griffiths' conjecture by proving that the pushforward of powers of the 1st Chern form of the tautological bundle on the parabolic projectivization are the Segre forms downstairs. (Our proof is a calculation involving generating functions and is hence technically slightly different from the analogous ones for usual bundles by [21] and [24].) In addition, for surfaces, akin to [39] we prove that parabolic ample stable bundles (arising from good Kawamata covers) admit metrics whose Schur forms are positive.

First, we prove a lemma to the effect that invariant closed forms and cohomology classes descend from Y to X.

Lemma 5.1.   Suppose Y is branched cover of X ramified over a smooth effective reduced divisor $D \subset X$. If $\widetilde{\eta}$ is a smooth closed invariant k-form on Y, then it descends to a closed current η on X, which is a smooth form away from D. If $\widetilde{\eta} = d\widetilde{\gamma}$ where $\widetilde{\gamma}$ is smooth and invariant, then $\eta = d\gamma$ where γ is a current on X that is smooth away from D.

Proof.   By invariance, $\widetilde{\eta}$ induces a smooth closed form η away from D on X. (Likewise, if $\widetilde{\eta} = d\widetilde{\gamma}$, away from D, we have a smooth form γ.) Suppose $\widetilde{\eta}$ is a (p,q)-form locally (near D) given by

$$\widetilde{\eta}(w) = \widetilde{\eta}_{IJ}(w)dw^I \wedge d\overline{w}^J,$$

where I, J are multi-indices and w are coordinates on Y chosen such that $z_1 = w_1^N$ where z are coordinates on X. Define

$$\eta(z) = \sum_{1 \notin I, 1 \notin J} \widetilde{\eta}_{IJ}(w(z))dz^I d\overline{z}^J + \sum_{J=(1,J^o),1 \notin I} \frac{\widetilde{\eta}_{I1J^o}(\overline{z}^1)^{1/N-1}}{N}(w(z))dz^I d\overline{z}^1 d\overline{z}^J$$

$$+ \sum_{I=(1,I^o),1 \notin J} \frac{\widetilde{\eta}_{1I^o J}(w(z))(z_1)^{1/N-1}}{N}dz^1 dz^{I^o} d\overline{z}^J$$

$$+ \sum_{I=(1,I^o),J=(1,J^o)} \widetilde{\eta}_{1I^o J}(w(z))\frac{|z^1|^{2/N-2}}{N^2}dz^1 dz^{I^o} d\overline{z}^1 d\overline{z}^{J^o}. \tag{5.1}$$

Suppose f is a smooth $(n-p, n-q)$-form with compact support in the given coordinate neighborhood B of D. Denote the region in B outside an ϵ-tubular neighborhood of B by B_ϵ

$$\int_{B_\epsilon} \mathrm{d}f \wedge \eta = \int_{\partial B_\epsilon} f \wedge \eta \to 0 \tag{5.2}$$

as $\epsilon \to 0$. Therefore $\mathrm{d}\eta = 0$ as a current. If $\widetilde{\eta} = \mathrm{d}_w \widetilde{\gamma}$, and hence $\eta = \mathrm{d}_z \gamma$ away from D (technically, away from a branch cut, but this will play no role because γ is well defined as a smooth form on $X \setminus D$ by invariance of $\widetilde{\gamma}$). Now

$$\int_{B_\epsilon} f \wedge \eta = -\int_{B_\epsilon} \mathrm{d}f \wedge \gamma + \int_{\partial B_\epsilon} f \wedge \gamma \to -\int_{B_\epsilon} \mathrm{d}f \wedge \gamma \tag{5.3}$$

as $\epsilon \to 0$ because by invariance, γ is given by similar expression as (5.1).   ■

Remark 5.1.   Note that actually Lemma 5.1 applies locally as well, that is, when X, Y are non-compact.

It is easy to see that the Chern–Weil forms of an invariant metric on an equivariant bundle V over Y are invariant differential forms. Now, we may define the parabolic Chern–Weil forms of an admissible metric.

Definition 5.2.   Suppose (E_*, D, H) is a parabolic bundle on a smooth projective variety X with an admissible metric H and (Y_p, q_p, V_p) is a locally good Kawamata cover around $p \in X$. Let $\widetilde{H}_p$ be the smooth metric on the equivariant bundle V_p descending to H. Then, given any invariant polynomial Φ acting on matrices, the parabolic Chern–Weil form of $\Phi(E_*, H)(p)$ at the point p is the form $\Phi_{par}(\Theta_H)(p)$ induced from the invariant forms $\Phi(\Theta_{\widetilde{H}_p})(p)$.

We will prove the following lemma.

Lemma 5.2.   The parabolic Chern–Weil currents depend only on H and not on the locally good Kawamata covers Y_p. Moreover, the cohomology classes of parabolic Chern–Weil currents $[\Phi_{par}(\Theta_H)]$ are independent of the admissible metric chosen to define them.

In order to use Lemma 5.1 in this situation, we need to prove that a Bott–Chern form of an invariant metric is an invariant form.

Lemma 5.3.   Suppose a finite group Γ acts by biholomorphisms on a complex manifold Y and its action lifts to an action on a holomorphic vector bundle V over Y of rank r. Assume that $\widetilde{H}_1$ and $\widetilde{H}_2$ are two smooth Hermitian metrics on Y. Also assume that Φ is an invariant polynomial on matrices. Then there exists an invariant Bott–Chern form $\widetilde{\Phi}(\widetilde{H}_2, \widetilde{H}_1)$ satisfying

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \widetilde{\Phi}(\widetilde{H}_2, \widetilde{H}_1) = \Phi(\widetilde{H}_2) - \Phi(\widetilde{H}_1).$$

Proof.   The construction we use here is due to Gillet–Soulé [22]. Consider the vector bundle $\widetilde{V} = \pi_1^* V \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ over $Y \times \mathbb{P}^1$. If

$$i_p : Y \longrightarrow Y \times \mathbb{P}^1$$

is the inclusion map $x \longmapsto (x, p)$, then $i_p^*(\widetilde{V}) \equiv V$ for all $p \in \mathbb{P}^1$. Extend the action of Γ to $Y \times \mathbb{P}^1$ by making it act trivially on the 2nd factor. This also lifts to an action to $\widetilde{V}$. Take an affine open cover U_0, U_1 of $\mathbb{P}^1$. Since this is a trivializing open cover for $\mathcal{O}_{\mathbb{P}^1}(1)$, we may define a Hermitian metric

$$\widetilde{H} = (1 - \rho)\widetilde{H}_1 + \rho \widetilde{H}_2$$

on $\widetilde{V}$, where ρ, $1 - \rho$ is a partition of unity subordinate to the open cover such that $\rho = 1$ on a neighborhood of 0. This is clearly an invariant metric. Therefore,

$$\widetilde{\Phi}(H_2, H_1) = \int_{\mathbb{P}^1} \Phi(\widetilde{V}, \widetilde{H}) \ln |z|^2$$

is the desired Bott–Chern form that is also invariant.   ∎

Now we are in a position to prove Lemma 5.2.

Proof.   of Lemma 5.2 According to (4.2), away from a branch cut near D, we have $\Theta_H = z_1^{-\alpha} \Theta_{\widetilde{H}_p} (z_1^\alpha)^{-1}$ and hence

$$\Phi_{par}(\Theta_H)(p) = \Phi(\Theta_{\widetilde{H}_p})(p).$$

Elsewhere this equality is obvious. Therefore, the parabolic Chern–Weil currents depend only on H and not the specific cover used. If H_1 and H_2 are two admissible metrics induced from $\widetilde{H}_{1,p}$ and $\widetilde{H}_{2,p}$ on V_p over Y_p, Lemma 5.3 shows that

$$\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \widetilde{\Phi}_p(\widetilde{H}_2, \widetilde{H}_1)(q) = \Phi(\widetilde{H}_2)(q) - \Phi(\widetilde{H}_1)(q) \ \forall \ q \in U_p$$

and that the Bott–Chern form $\widetilde{\Phi}_p$ can be chosen to be invariant. The proof of Lemma 5.3 shows that $\widetilde{\Phi}_p$ thus constructed is actually independent of p. Therefore, using Lemma 5.1 we see that the induced parabolic Chern–Weil currents have a unique cohomology class.   ∎

This allows us to define the parabolic Chern classes as the cohomology classes $[c_{par}(\Theta_H)]$ where H is any admissible metric. These cohomology classes coincide with the ones defined in [10], [28], and [12]. The 1st step to proving this is the following theorem (which in the usual case was proven in [24] and [21]).

Theorem 5.3.   Suppose (E_*, D, H) is a parabolic bundle on X with an admissible metric H. Let h be the induced admissible metric on the parabolic bundle $L_* = \mathcal{O}_{\mathbb{P}(E_*)}(1)_*$ over $\mathbb{P}(E_*)$. Let $s(E_*, H)$ and $s(L_*, h)$ be the Segre polynomial currents (inverses of the Chern

polynomials) of (E_*, H) and (L_*, h), respectively. Then, we have the following inequality of smooth forms on $X \setminus D$:

$$\pi_* s(L_*, h) = s(E_*, H),\tag{5.4}$$

where π_* is the pushforward (fiber integral) of forms. Moreover, if f is a smooth form with compact support on X, then

$$\int_X f \wedge s(E_*, H) = \int_{\mathbb{P}(E_*)} \pi^* f \wedge s(L_*, h).\tag{5.5}$$

Proof.   Suppose (Y_p, q_p, V_p) is a locally good Kawamata cover around $p \in X \setminus D$. Let $(\widetilde{L} = \mathcal{O}_{\mathbb{P}(V_p)}(1), \widetilde{h})$ be a Hermitian holomorphic line bundle over $\mathbb{P}(V_p)$ inducing (L_*, h) over X. The results of [24] and [21] show that equation (5.4) holds for $\widetilde{h}$ near $q_p^{-1}(p)$. By equivariance, this implies that the equation holds true for h as well. We shall give a proof of this result of [24] and [21] here. A small technical advantage of this proof is that it uses generating functions and hence has the potential to produce more such formulae (in the context of general flag varieties). In what follows, we denote the projection map from $\mathbb{P}(V)$ to Y as π. Moreover, we choose a trivialization $U_p \times \mathbb{P}^r$ of $\mathbb{P}(V)$ around p such that U_p is a coordinate neighborhood and the frame is a normal frame, that is, the metric at p is Euclidean up to 2nd order. We also evaluate the fiber integrals over the affine chart $w_i = \frac{X_i}{X_0}$:

$$\int_{\pi^{-1}(p) \subset \mathbb{P}(V)} s(\widetilde{L}, \widetilde{h}) = \int_{\pi^{-1}(p) \subset \mathbb{P}(V)} \frac{1}{1 + c_1(\widetilde{L}, \widetilde{h})}$$

$$= \int_{\mathbb{C}^{r-1}} \frac{1}{1 + \frac{\sqrt{-1}}{2\pi}\left(\sum_{i,j} \frac{((1+|\vec{w}|^2)\delta_{ij} - \overline{w}_i w_j)dw_i \wedge d\overline{w}_j}{(1+|\vec{w}|^2)^2} + \frac{\Theta(\widetilde{H})_{00} + \sum_i (\Theta_{i0}(\widetilde{H})w_i + \Theta_{0i}(\widetilde{H})\overline{w}_i) + \sum_{i,j}(\Theta(\widetilde{H}))_{ij} w_i \overline{w}_j}{1 + \sum_i |w_i|^2}\right)},$$

$$= f(\Theta)\tag{5.6}$$

where it is easy to see that $f(\Theta)$ is a universal polynomial (does not depend on Θ) with rational coefficients in the entries of the matrix of 2-forms Θ. Therefore, we may assume without loss of generality that actually, Θ is simply a skew-Hermitian matrix of complex numbers. It is also easy to see (a change of trivialization does not change the Chern forms) that f is an invariant polynomial. Hence, we may (without loss of generality) assume that $\Theta = \mathrm{diag}(a_0, a_1, a_2, \ldots, a_r)$ where $a_i = \sqrt{-1}b_i$ are purely

imaginary numbers. Now we may evaluate $f(\Theta)$ easily:

$$f(\Theta) = \int_{\mathbb{C}^{r-1}} \frac{1}{1 + \frac{\sqrt{-1}}{2\pi}\left(\sum_{i,j}\frac{((1+|\vec{w}|^2)\delta_{ij} - \overline{w}_i w_j)dw_i \wedge d\overline{w}_j}{(1+|\vec{w}|^2)^2} + \frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)}$$

$$= \int_{\mathbb{C}^{r-1}} \frac{1}{1 + \frac{\sqrt{-1}}{2\pi}\left(\frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)} \frac{1}{1 + \frac{\frac{\sqrt{-1}}{2\pi}\left(\sum_{i,j}\frac{((1+|\vec{w}|^2)\delta_{ij} - \overline{w}_i w_j)dw_i \wedge d\overline{w}_j}{(1+|\vec{w}|^2)^2}\right)}{1 + \frac{\sqrt{-1}}{2\pi}\left(\frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)}}$$

$$= (-1)^{r-1}\int_{\mathbb{C}^{r-1}} \frac{1}{1 + \frac{\sqrt{-1}}{2\pi}\left(\frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)} \left(\frac{\frac{\sqrt{-1}}{2\pi}\left(\sum_{i,j}\frac{((1+|\vec{w}|^2)\delta_{ij} - \overline{w}_i w_j)dw_i \wedge d\overline{w}_j}{(1+|\vec{w}|^2)^2}\right)}{1 + \frac{\sqrt{-1}}{2\pi}\left(\frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)}\right)^{r-1}$$

$$= (-1)^{r-1}(r-1)!\left(\frac{\sqrt{-1}}{2\pi}\right)^{r-1}\int_{\mathbb{C}^{r-1}} \left(\frac{1}{1 + \frac{\sqrt{-1}}{2\pi}\left(\frac{a_0 + \sum_i a_i|w_i|^2}{1+|\vec{w}|^2}\right)}\right)^r \left(\frac{1}{1+|\vec{w}|^2}\right)^r dw_1 \wedge d\overline{w}_1 \ldots$$

$$= (-1)^{r-1}(r-1)!\left(\frac{\sqrt{-1}}{2\pi}\right)^{r-1}\int_{\mathbb{C}^{r-1}} \left(\frac{1}{c_0 + \sum_i c_i|w_i|^2}\right)^r dw_1 \wedge d\overline{w}_1 \ldots, \tag{5.7}$$

where $c_i = \frac{\sqrt{-1}}{2\pi}a_i + 1$. We evaluate the last integral as follows:

$$f(\Theta) = \frac{(-1)^{r-1}(r-1)!}{c_0^r}\left(\frac{\sqrt{-1}}{2\pi}\right)^{r-1}\int_{\mathbb{C}^{r-1}} \left(\frac{1}{1 + \sum_i \frac{c_i}{c_0}|w_i|^2}\right)^r dw_1 \wedge d\overline{w}_1 \ldots$$

$$= \frac{1}{c_0 c_1 c_2 \ldots c_{r-1}} = s(V, \widetilde{H}), \tag{5.8}$$

where the 2nd-to-last equality follows from a simple change of variables and the fact that $c_1(\mathbb{P}^{r-1})^{r-1} = 1$.

Now suppose f is a smooth compactly supported form and X_ϵ is everything in X outside of an ϵ-tubular neighborhood of D. Then

$$\int_X f \wedge s(E_*, H) = \lim_{\epsilon \to 0}\int_{X_\epsilon} f \wedge s(E_*, H)$$

$$= \lim_{\epsilon \to 0}\int_{\pi^*(X_\epsilon)\subset\mathbb{P}(E_*)} \pi^*f \wedge s(L_*, h)$$

$$= \int_{\mathbb{P}(E_*)} \pi^*f \wedge s(L_*, h), \tag{5.9}$$

where the 2nd-to-last equality follows from (5.4) and the limits follow from the proofs of Lemmas 5.1 and 5.2. Indeed, the Chern–Weil currents are actually L^1 forms. ∎

Let E_* be a parabolic vector bundle on X. Let Y be a Kawamata cover of (X, E_*, D) and V an equivariant vector bundle on Y, such that E_* corresponds to V. The i-th parabolic Chern class of E_* is the pushforward of the i-th Chern class of V [10], [28], [12]. Therefore, Theorem 5.3 has the following corollary:

Corollary 5.4.   The parabolic Chern classes defined earlier using admissible metrics coincide with the parabolic Chern classes defined in [10], [28], and [12].

Remark 5.5.   The usual proof (see [23] for instance) that $c_1(\Theta), c_2(\Theta) > 0$ for Griffiths positive bundles shows that this holds even for parabolic Griffiths positive bundles (as weakly positive currents). The proof in [24] shows that positivity is preserved under fiber integral and hence the signed parabolic Segre forms are positive. Hence, on surfaces, if E_* is parabolic Griffiths positive, then $c_{1,par}, c_{2,par}, c_{1,par}^2 - c_{2,par} > 0$ as currents.

Given Remark 5.5, it is but natural to ask whether a parabolic Hartshorne ample bundle admits an admissible metric whose Schur polynomial currents are weakly positive. In the usual (non-parabolic) case, this was proven for semistable bundles on surfaces. Here, we prove an analogous result for parabolic stable bundles induced from *good* Kawamata covers. Prior to that we define the notion of an admissible form:

Definition 5.6.   Given an integer N, suppose Y is an N-fold branched cover of X branched over a divisor $D \subset X$. An N-admissible (k,k)-form η is the L^1 current induced from a smooth form $\widetilde{\eta}$ on Y. It is said to be weakly positive if $\widetilde{\eta}$ is weakly positive on Y. Note that this definition is applicable locally too.

Remark 5.7.   Lemma 3.3 shows that the notion of weak positivity does not depend on the cover Y chosen and that an admissible (n,n)-form is $\geq C\omega_{2-2/N}^N$ on $X \setminus D$, where $\omega_{2-2/N}$ is induced from a smooth Kähler form of a Hermitian metric ω on Y.

Theorem 5.8.   If E_* induced from a minimal good Kawamata cover (Y, q, V) is parabolic Hartshorne ample and parabolic stable with respect to an admissible Kähler class $[\omega_{2-2/N}]$ on a compact complex surface X, then E_* admits an admissible metric G such that

$$c_{1,par}(G) > 0, \ \ c_{2,par}(G) > 0, \ \ c_{1,par}^2 - c_{2,par} > 0.$$

Proof.   Fix an admissible Kähler metric ω_α arising from a smooth Kähler metric on a minimal N-fold Kawamata cover Y (with covering group Γ).

In [10] a result akin to Bloch–Gieseker [15] was proven for parabolic Chern classes of parabolic ample bundles. Therefore, $\int_X (c_{1,par}^2 - c_{2,par}) > 0$. This means that the cohomology class $[c_{1,par}^2 - c_{2,par}]$ admits an admissible positive representative $\eta \geq C\omega_\alpha^2$, that is, η arises from an invariant smooth nonnegative form $\widetilde{\eta}$ on Y. Since E_* is stable with respect to the $(2 - 2/N)$-admissible Kähler class $[\omega_{2-2/N}]$, it admits an admissible Hermite–Einstein metric H. Now H induces a smooth invariant Hermite–Einstein (with respect to the induced Kähler metric $\widetilde{\omega}$) $\widetilde{H}$ on V over Y.

We wish to find a smooth invariant function $\widetilde{\phi}$ on V such that $\widetilde{G} = \widetilde{H}e^{-\phi}$ satisfies

$$c_1(\widetilde{G}) > 0,$$

and

$$(c_1^2 - c_2)(\widetilde{G}) = \widetilde{\eta}. \tag{5.10}$$

If we manage to do so, then the calculations in [39] show that $c_2(\widetilde{G}) > 0$. This would complete the proof of Theorem 5.8. Indeed, just as in [39], equation reduces to the following Monge–Ampère equation.

$$\frac{r(r+1)}{2}\left(\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\widetilde{\phi} + \frac{c_1(\widetilde{H})}{r}\right)^2 = \eta + \frac{2rc_2(\widetilde{H}) - (r-1)c_1^2(\widetilde{H})}{2r} \tag{5.11}$$

The right-hand side of (5.11) is positive owing to the Kobayashi–Lübke inequality (see [39] for details). Moreover, $[c_1(V)]$ is a Kähler class admitting an invariant Kähler metric (indeed take the pushforward of $c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^r$ over $\mathbb{P}(V)$). Thus, (5.11) admits an invariant smooth solution $\widetilde{\phi}$. It admits a smooth solution thanks to Yau's proof of the Calabi conjecture [45]. The uniqueness of the solution makes it invariant because the Kähler class and the right-hand side are invariant under the action of Γ. This induces a continuous function ϕ on X smooth outside D such that the induced metric G on E_* is admissible and satisfies the desired properties.   ∎

6   Curvature of Direct Images

In this section, we prove an analog of Berndtsson's theorem [3] for equivariant (i.e., parabolic) bundles. In particular, this implies that if E_* is Hartshorne ample, then $E_* \otimes \det(E)_*$ is Nakano positive.

Theorem 6.1. Suppose X is a compact complex manifold equipped with a holomorphic action of a finite group Γ. Assume that the action of Γ lifts to an action on a holomorphic line bundle L over X and that h is a Γ-invariant Hermitian metric on L with semipositive curvature. Suppose X admits a holomorphic submersion $\pi : X \longrightarrow \widetilde{X}$ to a compact complex manifold $\widetilde{X}$ such that $\widetilde{X}$ admits an action of Γ that commutes with the action on X.

There exists a holomorphic vector bundle E over $\widetilde{X}$ with a lift of the action of Γ such that for each n-dimensional fiber $X_t = \pi^{-1}(t)$, where $t \in \widetilde{X}$, the fiber E_t of E over t is the vector space of holomorphic sections of $L|_{X_t} \otimes K_{X_t}$ on X_t. Moreover, the following metric H on E is invariant under Γ and has Nakano semipositive curvature:

$$|u_t|^2 = \int_{X_t} \sqrt{-1}^{n^2} u_t \wedge \overline{u}_t h = \int_{X_t} \sqrt{-1}^{n'} u_t \wedge \overline{u}_t h, \qquad (6.1)$$

where n' is 0 or 1 depending on whether n is even or odd. Moreover, if h has strictly positive curvature in a neighborhood of $\pi^{-1}(p)$ where $p \in \widetilde{X}$, then H is Nakano (strictly) positively curved near p.

Proof. The fact that E exists as a vector bundle and that H has Nakano semipositive curvature follows from Theorem 1.2 in [3]. The proof in [3] also shows the strict positivity statement. All we need to do is to check that the action of Γ lifts to E and that H is invariant under the same.

Firstly, the action of Γ extends to (p, q)-forms by means of pullback, that is, $g.\omega = (f_{g^{-1}})^*\omega$. By equivariance, $L_t \otimes K_{X_t}$ is taken to $L_{g.t} \otimes K_{X_{g.t}}$. Thus, Γ acts on sections of $L_t \otimes K_{X_t}$ over X_t and takes to them to sections of $L_{g.t} \otimes K_{X_{g.t}}$ over $X_{g.t}$. Consequently, E admits a lift of the action of Γ. By invariance of h, it is easy to see that the metric 6.1 is an invariant metric. ∎

Theorem 6.1 implies the following result. This result provides further evidence for the parabolic version of Griffiths' conjecture.

Corollary 6.2. A parabolic Hartshorne ample bundle E_* arising from a minimal good Kawamata cover (Y, q, V) over a projective manifold X admits an admissible metric H that induces an admissible parabolic Nakano positively curved metric on $E_* \otimes \det(E_*)$.

Proof. By parabolic Hartshorne ampleness, $\mathcal{O}_{\mathbb{P}(V)}(1)$ admits a positively curved invariant metric. Thus, V admits an invariant Hermitian metric $\widetilde{H}$ such that the Hermitian

metric on $V \otimes \det(V)$ induced by $\widetilde{H}$ is Nakano positive. The induced metric on $E_* \otimes \det(E_*)$ is admissible because Y is a minimal good Kawamata cover (Lemma 3.4). It is of course Nakano positively curved in the parabolic sense. ∎

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