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ON A GENERALIZATION OF A COMPLEMENTARY TRIANGLE INEQUALITY IN HILBERT SPACES AND BANACH SPACES[1]

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We study a possible generalization of a complementary triangle inequality in Hilbert spaces and Banach spaces. Our results in the present article improve and generalize some of the earlier results in this context. We also present an operator norm inequality in the setting of Banach spaces, as an application of the present study.

Key words : Triangle inequality; convexity; normed spaces; inner product spaces.

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1. Introduction

The purpose of the present article is to generalize some classical and recent results on a complementary triangle inequality in Hilbert spaces and Banach spaces. The study of such inequalities in various forms has been conducted by several authors [1, 2, 5, 6]. The first instance of such an inequality was obtained in the celebrated paper of Diaz and Metcalf [2], motivated by a geometric inequality for complex numbers obtained by Wilf in [11]. The complementary triangle inequality obtained by Diaz and Metcalf, in Theorem 1 of [2] in the Hilbert space setting, was generalized by Dragomir in Theorem 3 of [6]. This was further generalized by Mansoori *et al.* in Theorem 3.1 of [5]. In the

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present article we generalize all the above results, substantially increasing the scope of applying such a complementary triangle inequality in Hilbert spaces and Banach spaces.

Letters $\mathbb{X}, \mathbb{Y}$ stand for Banach spaces. We reserve the symbol $\mathbb{H}$ for a Hilbert space, and the symbol $\langle , \rangle$ for the inner product on $\mathbb{H}$. We will consider only spaces of dimension strictly greater than 1 throughout the article. Unless otherwise specified, we consider the underlying field to be $\mathbb{C}$, the field of complex numbers. Given a complex number z, we use the notations $Re\ z$, $Im\ z$, and $\bar{z}$ to denote the real part of z, the imaginary part of z, and the complex conjugate of z respectively. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of $\mathbb{X}$ respectively and let $\mathbb{X}^*$ denote the dual space of $\mathbb{X}$. We use the symbol $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ to denote the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. For $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$ denote the norm attainment set of T. Given any $x \in \mathbb{X}$ and any $r > 0$, let $B(x, r) = \{y \in \mathbb{X} : \|y - x\| < r\}$ denote the open ball with radius r and center at x. The symbol θ is used to denote the zero vector of any Banach space, other than the scalar field. Let us first state the complementary inequality obtained by Diaz and Metcalf in Theorem 1 of [2].

***Theorem* 1.1** — *Let $\mathbb{H}$ be a Hilbert space and let $a \in S_{\mathbb{H}}$ be fixed. Suppose the vectors $x_1, \ldots, x_n \in \mathbb{H}$, whenever $x_i \neq 0$, satisfy*

$$0 \leq r \leq Re\ \langle x_i, a \rangle\ ,\ i = 1, \ldots, n.$$

Then

$$r \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i \|,$$

where equality holds if and only if

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) a.$$

We would like to note that the above inequality becomes useless if $\langle x_i, a \rangle = 0$ for any $i = 1, \ldots, n$. In light of this fact, we would like to considerably increase the scope of such a complementary triangle inequality. Indeed, we prove that if only some x_i satisfy the desired inequality stated in the above theorem, with $r > 0$ and the other vectors are of sufficiently small norm, then once again it is possible to have a generalized complementary triangle inequality. As a matter of fact, applying this simple idea, we generalize Theorem 1 of [2], Theorem 3 of [6], and Theorem 3.1 of [5]. We illustrate that the complementary triangle inequality obtained by us can be applied to a larger class of vectors than those considered before. We also study similar complementary triangle inequalities in the setting of Hilbert spaces and Banach spaces. We also give an intrinsic condition, in terms of semi-inner-products (s.i.p.), on the concerned vectors in a strictly convex Banach space, in order to obtain a complementary triangle inequality. For the sake of completeness, let us mention the definitions of s.i.p., strict convexity and smoothness, which are integral to serving our purpose.

Definition 1.1 — Let $\mathbb{X}$ be a Banach over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $[\ ,\ ] : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{K}$ is a semi-inner-product (s.i.p.) if for any $\alpha,\ \beta \in \mathbb{K}$ and for any $x,\ y,\ z \in \mathbb{X}$, it satisfies the following:

(a) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$,

(b) $[x, x] > 0$, whenever $x \neq \theta$,

(c) $|[x, y]|^2 \leq [x, x][y, y]$,

(d) $[x, \alpha y] = \bar{\alpha}[x, y]$.

Definition 1.2 — Let $\mathbb{X}$ be a Banach over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say that $\mathbb{X}$ is strictly convex if every point of $S_{\mathbb{X}}$ is an extreme point of $B_{\mathbb{X}}$. Given a non-zero $x \in \mathbb{X}$, we say that $\mathbb{X}$ is smooth at the point x if there exists a unique supporting hyperplane to $B_{\mathbb{X}}$ at the point $\frac{x}{\|x\|}$.

We refer the readers to [3, 4, 10] for more information on s.i.p. and its various applications. In particular, whenever we speak of a s.i.p. $[\ ,\ ]$ in a Banach space $\mathbb{X}$, we implicitly assume that the concerned s.i.p. is compatible with the norm on $\mathbb{X}$, i.e., $[x, x] = \|x\|^2$ for any $x \in \mathbb{X}$. Finally, as an application of the complementary triangle inequality obtained by us in the present paper, we obtain an interesting norm inequality for linear operators on a finite-dimensional real Banach space.

2. Main Results

As mentioned before, the first complementary triangle inequality for vectors in a Hilbert space was obtained in Theorem 1 of [2]. This result was generalized in Theorem 3 of [6], and thereafter in Theorem 3.1 of [5]. Since the equality condition has not been mentioned in Theorem 3.1 of [5] and we require it for our purpose, let us begin with restating Theorem 3.1 of [5] in an equivalent form, along with the equality condition. The proof is omitted as it follows rather trivially from the proofs of the above theorems.

***Theorem* 2.1** — *Let $\mathbb{H}$ be a Hilbert space and let $a \in S_{\mathbb{H}}$ be a fixed vector. Let $x_1, \ldots, x_k \in \mathbb{H} \setminus \{\theta\}$ satisfy the following conditions:*

$$(i)\ \left(\sum_{i=1}^{k} r_1\|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} \mathit{Re}\ \langle x_i, a\rangle\right)^2,\ (ii)\ \left(\sum_{i=1}^{k} r_2\|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} \mathit{Im}\ \langle x_i, a\rangle\right)^2,$$

where $r_1, r_2 \geq 0$. *Then*

$$\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{k}\|x_i\| \leq \|\sum_{i=1}^{k}x_i\|.$$

Moreover, equality holds in the above inequality if and only if

$$(iii)\ \sum_{i=1}^{k}x_i = \left(\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{k}\|x_i\|\right)a,\ \textit{and } (iv) \textit{ inequalities (i) and (ii) are equalities.}$$

We are now ready to state and prove our first complementary triangle inequality in Hilbert spaces. We note that the following theorem generalizes Theorem 1 of [2], Theorem 3 of [6], and Theorem 3.1 of [5]. We would like to remark that the first part of the argument given in the proof of the following theorem is from [5].

Theorem 2.2 — *Let* $\mathbb{H}$ *be a Hilbert space and let* $a \in S_{\mathbb{H}}$ *be fixed. Let* $x_1, \ldots, x_n \in \mathbb{H}\setminus\{\theta\}$ *satisfy the following conditions:* $(i)\ \left(\sum_{i=1}^{k} r_1\|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} \mathit{Re}\ \langle x_i, a\rangle\right)^2,\ (ii)\ \left(\sum_{i=1}^{k} r_2\|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} \mathit{Im}\ \langle x_i, a\rangle\right)^2,\ (iii)\ \sum_{i=k+1}^{n}\|x_i\| \leq \lambda\sum_{i=1}^{k}\|x_i\|$, *where* $r_1, r_2 \geq 0,\ 1 \leq k \leq n,\ 0 < \lambda \leq \frac{\left(r_1^2+r_2^2\right)^{\frac{1}{2}}}{1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}}$. *Then*

$$\left[\left(r_1^2+r_2^2\right)^{\frac{1}{2}} - \lambda\left(1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\right)\right]\sum_{i=1}^{n}\|x_i\| \leq \|\sum_{i=1}^{n}x_i\|.$$

Whenever $k < n$, *the above inequality is strict. Moreover, if* $k = n$, *then the equality* $\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| = \|\sum_{i=1}^{n}x_i\|$ *holds if and only if*

$$(iv)\ \sum_{i=1}^{n}x_i = \left(\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|\right)a,\ \textit{and, } (v) \textit{ inequalities (i), (ii) are equalities.}$$

PROOF : It follows from Theorem 2.1 that $\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{k}\|x_i\| \leq \|\sum_{i=1}^{k}x_i\|$. Therefore, we have,

$$\begin{aligned}
\|\sum_{i=1}^{n}x_i\| &\geq \|\sum_{i=1}^{k}x_i\| - \|\sum_{i=k+1}^{n}x_i\| \\
&\geq \left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{k}\|x_i\| - \sum_{i=k+1}^{n}\|x_i\| \\
&= \left(r_1^2+r_2^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| - \left(1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\right)\sum_{i=k+1}^{n}\|x_i\|
\end{aligned}$$

$$\geq \left(r_1^2+r_2^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| - \lambda\left(1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\right) \sum_{i=1}^{k} \|x_i\|$$
$$\geq \left(r_1^2+r_2^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| - \lambda\left(1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\right) \sum_{i=1}^{n} \|x_i\|$$
$$= \left[\left(r_1^2+r_2^2\right)^{\frac{1}{2}} - \lambda\left(1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}\right)\right] \sum_{i=1}^{n} \|x_i\|$$

This completes the proof of the desired inequality stated in the theorem. We next consider the condition for equality in the above complementary triangle inequality. First, suppose that $k < n$. Since $x_{k+1} \neq \theta$, it follows that $\sum_{i=1}^{k}\|x_i\| < \sum_{i=1}^{n}\|x_i\|$. Consequently, it follows that the above inequality is strict. Let us now assume that $k = n$. It follows from Theorem 2.1 and the above chain of inequalities in the proof of the present theorem, that the equality $\left(r_1^2+r_2^2\right)^{\frac{1}{2}} \sum_{i=1}^{n}\|x_i\| = \|\sum_{i=1}^{n} x_i\|$ holds if and only if the following holds:

$$(iv)\ \sum_{i=1}^{k} x_i = \left(\left(r_1^2+r_2^2\right)^{\frac{1}{2}} \sum_{i=1}^{k}\|x_i\|\right) a, \text{ and } (v) \text{ inequalities (i), (ii) are equalities.}$$

This completes the proof of the theorem.

Our next result illustrates that Theorem 2.2 of the present paper extends the scope of applying a complementary triangle inequality in Hilbert spaces, to a larger class of vectors than those considered in either of Theorem 1 of [2], Theorem 3 of [6], and Theorem 3.1 of [5].

Theorem 2.3 — *Let $\mathbb{H}$ be a Hilbert space and let $x_1, x_2 \in S_{\mathbb{H}}$, with $x_1 \neq -x_2$. Let $n_0 \in \mathbb{N}$ be fixed. Then there exists $r_0 > 0$ such that for any non-zero vectors $y_1, \ldots, y_{n_0} \in B(\theta, r_0) \setminus \{\theta\}$, the following holds:*

$$\frac{1+Re\ \langle x_1, x_2\rangle}{2\|x_1+x_2\|}\left(\|x_1\|+\|x_2\|+\sum_{i=1}^{n_0}\|y_i\|\right) < \|x_1+x_2+\sum_{i=1}^{n_0} y_i\|.$$

PROOF : We first note that since $x_1 \neq -x_2$, it follows that $\frac{1+Re\ \langle x_1,x_2\rangle}{2\|x_1+x_2\|} \neq 0$. We set $r = \frac{1+Re\ \langle x_1,x_2\rangle}{\|x_1+x_2\|} > 0$. Let us choose $\lambda = \frac{r}{2(1+r)} > 0$. Clearly, $r-(1+r)\lambda = \frac{r}{2} > 0$. We define $r_0 = \frac{r(\|x_1\|+\|x_2\|)}{2n_0(1+r)}$. Then for any $y_1, \ldots, y_{n_0} \in B(\theta, r_0)$, we have the following:

$$\sum_{i=1}^{n_0}\|y_i\| \leq n_0 r_0 = \frac{r(\|x_1\|+\|x_2\|)}{2(1+r)} = \lambda(\|x_1\|+\|x_2\|).$$

Let us choose $a = \frac{x_1+x_2}{\|x_1+x_2\|} \in S_{\mathbb{H}}$. We observe that $\langle x_1, a\rangle = \left\langle x_1, \frac{x_1+x_2}{\|x_1+x_2\|}\right\rangle = \frac{1}{\|x_1+x_2\|}\left[\|x_1\|^2 + \langle x_1, x_2\rangle\right].$

In particular, this implies that

$$Re\ \langle x_1, a\rangle = \frac{1}{\|x_1 + x_2\|}\left[1 + Re\ \langle x_1, x_2\rangle\right] = r.$$

Similarly, we have,

$$Re\ \langle x_2, a\rangle = \frac{1}{\|x_1 + x_2\|}\left[1 + Re\ \langle x_2, x_1\rangle\right] = r.$$

Now, taking $r_1 = r$ and $r_2 = 0$ in Theorem 2.2, it is easy to see that $\lambda < \frac{r}{1+r} = \frac{\left(r_1^2+r_2^2\right)^{\frac{1}{2}}}{1+\left(r_1^2+r_2^2\right)^{\frac{1}{2}}}$. Therefore, applying Theorem 2.2, we deduce that

$$[r - \lambda(1+r)]\left(\|x_1\| + \|x_2\| + \sum_{i=1}^{n_0}\|y_i\|\right) \le \|x_1 + x_2 + \sum_{i=1}^{n_0} y_i\|,$$

which is clearly equivalent to the desired inequality, with a possible equality sign. To see that the obtained inequality is actually strict, it is sufficient to observe that $n_0 \geq 1$ and then to apply the equality condition in Theorem 2.2. This establishes the theorem.

As mentioned in [2], Theorem 1 of [2] has an obvious geometric interpretation. The complementary triangle inequality stated in the concerned theorem is valid for a certain set of vectors lying within a cone. In view of this, we present the next figure in connection with Theorem 2.3 of the present article. The vectors $y_1, \ldots, y_n$ can be chosen arbitrarily from the shaded region $B(\theta, r_0)$, whereas the vectors x_1, x_2 can be chosen from $S_{\mathbb{H}}$ so that $x_1 \neq -x_2$. In particular, this illustrates pictorially that Theorem 2.2 of the present article extends the scope of obtaining a complementary triangle inequality in Hilbert spaces.

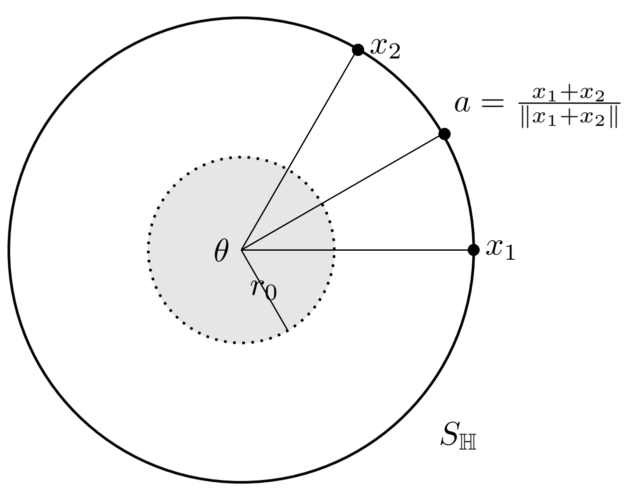

Fig. 1 : Extension of complementary triangle inequality

In Theorem 2 of [2], the authors have extended Theorem 1 of the same article by considering a finite number of orthonormal vectors instead of a single unit vector in a Hilbert space. It is possible to apply the ideas of [5, 6], and the idea developed in the present article, to further generalize Theorem 2 of [2] by removing the orthonormality condition on the unit vectors. We accomplish the goal in the next theorem.

***Theorem* 2.4** — *Let $\mathbb{H}$ be a Hilbert space and let $a_1, \ldots, a_m \in S_{\mathbb{H}}$ be fixed. Let $x_1, \ldots, x_n \in \mathbb{H}\setminus\{\theta\}$ be such that the following conditions hold true for each $l = 1, \ldots, m$: (i) $\left(\sum_{i=1}^{k} r_l \|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} Re \ \langle x_i, a_l\rangle\right)^2$, (ii) $\left(\sum_{i=1}^{k} s_l \|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} Im \, \langle x_i, a_l\rangle\right)^2$,*
(iii) $\sum_{i=k+1}^{n} \|x_i\| \leq \lambda \sum_{i=1}^{k} \|x_i\|$,
where $1 \leq k \leq n$, and for each $l = 1, \ldots, m$, we have that $r_l, s_l \geq 0$ and $0 < \lambda \leq \frac{\left(r_l^2+s_l^2\right)^{\frac{1}{2}}}{1+\left(r_l^2+s_l^2\right)^{\frac{1}{2}}}$.
Then for each $l = 1, \ldots, m$,

$$\left[\left(r_l^2 + s_l^2\right)^{\frac{1}{2}} - \lambda\left(1 + \left(r_l^2 + s_l^2\right)^{\frac{1}{2}}\right)\right] \sum_{i=1}^{n} \|x_i\| \leq \|\sum_{i=1}^{n} x_i\|.$$

Whenever $k < n$, the above inequality is strict. Moreover, if $k = n$, then the equality $\left(r_l^2 + s_l^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| = \|\sum_{i=1}^{n} x_i\|$ holds for some $l = 1, \ldots, m$ if and only if (iv) $\sum_{i=1}^{n} x_i = \left(\left(r_l^2 + s_l^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\|\right) a$, and (v) inequalities (i), (ii) are equalities.

PROOF : For each $l = 1, \ldots, m$, it follows from the Cauchy-Schwarz inequality that $\left|\left\langle a_l, \sum_{i=1}^{k} x_i \right\rangle\right| \leq \|a_l\| \|\sum_{i=1}^{k} x_i\| = \|\sum_{i=1}^{k} x_i\|$. Therefore, we have,

$$\begin{aligned}
\|\sum_{i=1}^{k} x_i\| &\geq \left|\left\langle a_l, \sum_{i=1}^{k} x_i \right\rangle\right| \\
&= \left|\sum_{i=1}^{k} Re \, \langle x_i, a_l\rangle + i \sum_{i=1}^{k} Im \, \langle x_i, a_l\rangle\right| \\
&= \left[\left(\sum_{i=1}^{k} Re \, \langle x_i, a_l\rangle\right)^2 + \left(\sum_{i=1}^{k} Im \, \langle x_i, a_l\rangle\right)^2\right]^{\frac{1}{2}} \\
&\geq \left[r_l^2 \left(\sum_{i=1}^{k} \|x_i\|\right)^2 + s_l^2 \left(\sum_{i=1}^{k} \|x_i\|\right)^2\right]^{\frac{1}{2}} \\
&= \left(r_l^2 + s_l^2\right)^{\frac{1}{2}} \sum_{i=1}^{k} \|x_i\|.
\end{aligned}$$

From here we can follow the arguments given in the proof of Theorem 2.2 to obtain the desired inequality

$$\left[\left(r_l^2+s_l^2\right)^{\frac{1}{2}}-\lambda\left(1+\left(r_l^2+s_l^2\right)^{\frac{1}{2}}\right)\right]\sum_{i=1}^{n}\|x_i\|\leq\|\sum_{i=1}^{n}x_i\|.$$

Now, following the arguments given in the proof of Theorem 2.2, we deduce that the above inequality is strict whenever $k<n$. Moreover, the necessary and sufficient conditions for the equality $\left(r_l^2+s_l^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|=\|\sum_{i=1}^{n}x_i\|$, for some $l=1,\ldots,m$, follows quite easily. This completes the proof of the theorem.

We would like to note that it is possible to obtain a weaker version of the above theorem, completely in the spirit of Theorem 2 of [2]. The next result is stated with only a sketch of the corresponding proof, since it follows easily from the above theorem.

Theorem 2.5 — *Let $\mathbb{H}$ be a Hilbert space and let $a_1,\ldots,a_m\in S_{\mathbb{H}}$ be fixed. Let $x_1,\ldots,x_n\in\mathbb{H}\setminus\{\theta\}$ be such that for each $i=1,\ldots,n$ and for each $k=1,\ldots,m$, the following holds:*

$$0\leq r_k\leq\frac{Re\ \langle x_i,a_k\rangle}{\|x_i\|}.$$

Then

$$\left(\frac{\sum_{k=1}^{m}r_k}{m}\right)\sum_{i=1}^{n}\|x_i\|\leq\|\sum_{i=1}^{n}x_i\|.$$

Moreover, equality holds in the above inequality if and only if for each $k=1,\ldots,m$, and for each $i=1,\ldots,n$,we have that $r_k=\frac{Re\ \langle x_i,a_k\rangle}{\|x_i\|}$ and $\sum_{i=1}^{n}x_i=\left(r_k\sum_{i=1}^{n}\|x_i\|\right)a_k$.

PROOF : For each $k=1,\ldots,m$, we have, $|\langle a_k,\sum_{i=1}^{n}x_i\rangle|\leq\|\sum_{i=1}^{n}x_i\|$. This gives us the following chain of inequalities:

$$\begin{aligned}\|\sum_{i=1}^{n}x_i\|&\geq\left|\left\langle a_k,\sum_{i=1}^{n}x_i\right\rangle\right|\\&\geq Re\ \left\langle a_k,\sum_{i=1}^{n}x_i\right\rangle\\&=\sum_{i=1}^{n}Re\ \langle a_k,x_i\rangle\\&\geq r_k\sum_{i=1}^{n}\|x_i\|.\end{aligned}$$

Since the above inequality is true for each $k = 1, \ldots, m$, adding these relations we obtain:

$$\left(\frac{\sum_{k=1}^{m} r_k}{m}\right) \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i \|.$$

Moreover, equality holds in the above inequality if and only if for each $i = 1, \ldots, n$ and for each $k = 1, \ldots, m$, the following conditions are satisfied:

(i) $\sum_{i=1}^{n} x_i = \mu_k a_k$, where $\mu_k \geq 0$, (ii) $\sum_{i=1}^{n} Im\ \langle x_i, a_k \rangle = 0$, (iii) $r_k = \frac{Re\ \langle x_i, a_k \rangle}{\|x_i\|}$.

The given equality condition in the statement of the theorem follows directly from the above conditions. This completes the proof of the theorem.

Remark 2.1 : We note that the inequality obtained in the above theorem is certainly weaker than the inequality obtained in Theorem 2 of [2]. On the other hand, this is compensated by the fact that we no longer require the orthonormality condition on the unit vectors $a_1, \ldots, a_m$.

We next extend the scope of complementary triangle inequality in Banach spaces by generalizing Theorem 3 of [2]. We would like to mention that Theorem 3 of [2] is valid for both real and complex Banach spaces. On the other hand, the following result is stated only for complex Banach spaces. However, we observe that it can also be applied for real Banach spaces by removing the Condition (ii) in the statement of the theorem and by taking $r_2 = 0$.

***Theorem* 2.6** — *Let $\mathbb{X}$ be a complex Banach space and let $f \in S_{\mathbb{X}^*}$. Suppose the vectors $x_1, \ldots, x_n \in \mathbb{X} \setminus \{\theta\}$ are such that the following conditions are satisfied:* (i) $\left(\sum_{i=1}^{k} r_1 \|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} Re\ f(x_i)\right)^2$, (ii) $\left(\sum_{i=1}^{k} r_2 \|x_i\|\right)^2 \leq \left(\sum_{i=1}^{k} Im\ f(x_i)\right)^2$,
(iii) $\sum_{i=k+1}^{n} \|x_i\| \leq \lambda \sum_{i=1}^{k} \|x_i\|$,
where $r_1, r_2 \geq 0$, $1 \leq k \leq n$, *and* $0 < \lambda \leq \frac{\left(r_1^2 + r_2^2\right)^{\frac{1}{2}}}{1 + \left(r_1^2 + r_2^2\right)^{\frac{1}{2}}}$. *Then*

$$\left[\left(r_1^2 + r_2^2\right)^{\frac{1}{2}} - \lambda \left(1 + \left(r_1^2 + r_2^2\right)^{\frac{1}{2}}\right)\right] \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i \|.$$

Moreover, equality holds in the above inequality if and only if (iv) $k = n$, (v) $f\left(\sum_{i=1}^{n} x_i\right) = \| \sum_{i=1}^{n} x_i \|$, *and* (vi) *the inequalities in* (i) *and* (ii) *are equalities.*

PROOF : We begin the proof with the following chain of inequalities:

$$\begin{aligned}
\left\|\sum_{i=1}^{k} x_i\right\|^2 &\geq \left|f\left(\sum_{i=1}^{k} x_i\right)\right|^2 \\
&= \left|\sum_{i=1}^{k} Re\ (fx_i) + i\sum_{i=1}^{k} Im\ (fx_i)\right|^2 \\
&= \left[\left\{\sum_{i=1}^{k} Re\ (fx_i)\right\}^2 + \left\{\sum_{i=1}^{k} Im\ (fx_i)\right\}^2\right] \\
&\geq \left(r_1^2 + r_2^2\right)\left(\sum_{i=1}^{k} ||x_i||\right)^2 .
\end{aligned}$$

From the above inequalities, we obtain

$$\left\|\sum_{i=1}^{k} x_i\right\| \geq \left(r_1^2 + r_2^2\right)^{\frac{1}{2}} \sum_{i=1}^{k} ||x_i|| .$$

Now we employ the same method, as outlined in the proof of Theorem 2.2 of the present article to obtain the desired inequality

$$\left[\left(r_1^2 + r_2^2\right)^{\frac{1}{2}} - \lambda\left(1 + \left(r_1^2 + r_2^2\right)^{\frac{1}{2}}\right)\right] \sum_{i=1}^{n} \|x_i\| \leq \|\sum_{i=1}^{n} x_i\|.$$

Moreover, it is easy to see that for equality to hold in the above inequality, a necessary and sufficient condition is that $(iv), (v), (vi)$ must hold true. This establishes the theorem. □

It is possible to obtain an intrinsic description of Theorem 2.6 in a strictly convex Banach space, that resembles Theorem 2.5, by using the notion of s.i.p. in a real Banach space instead of referring to bounded linear functionals on the space. In order to achieve this goal, we require the following easy proposition. The proof is omitted as it is rather easy and can be found in Theorem 12 of [7].

Proposition 2.7 — Let $\mathbb{X}$ be a strictly convex Banach space. Let $x \in \mathbb{X}$ be non-zero and let $y \in S_{\mathbb{X}}$. Let [,] be a s.i.p. on $\mathbb{X}$. Then $|[x, y]| = \|x\|$ if and only if $x = \lambda y$, for some $\lambda \in \mathbb{R} \setminus \{0\}$.

We are now in a position to obtain an analogous result of Theorem 2.5 in the setting of Banach spaces.

***Theorem* 2.8** — *Let $\mathbb{X}$ be a strictly convex real Banach space and let [,] be a s.i.p. on $\mathbb{X}$. Let $a \in S_{\mathbb{X}}$ be fixed and let $x_1, \ldots, x_n \in \mathbb{X} \setminus \{\theta\}$ be such that the following conditions are satisfied:*

(i) $\sum_{i=1}^{n} x_i \neq \theta,$ (ii) $r \leq \frac{[x_j,a]}{\|x_j\|}$, for each $j = 1, \ldots, k,$ (iii) $\sum_{i=k+1}^{n} \|x_i\| \leq \lambda \sum_{i=1}^{k} \|x_i\|$, where $r \geq 0$, $1 \leq k \leq n$, and $0 < \lambda \leq \frac{r}{1+r}$. Then

$$\{r - \lambda(1 + r)\} \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i\|.$$

Moreover, equality holds in the above inequality if and only if (iv) $k = n$, (v) $\sum_{i=1}^{n} x_i = \mu a$, for some $\mu \neq 0$, and (vi) each inequality in (ii) is an equality.

PROOF : Since $\|a\| = 1$, using the properties of s.i.p., we obtain the following:

$$\begin{aligned} \left\| \sum_{i=1}^{k} x_i \right\| &\geq \left| \left[\sum_{i=1}^{k} x_i, a \right] \right| \\ &\geq \left[\sum_{i=1}^{k} x_i, a \right] \\ &= \sum_{i=1}^{k} [x_i, a] \\ &\geq \sum_{i=1}^{k} r\|x_i\|. \end{aligned}$$

We now apply the same arguments as given in the proof of Theorem 2.2 of the present article to obtain the desired inequality

$$\{r - \lambda(1 + r)\} \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i\|.$$

Moreover, it follows from the earlier arguments and Proposition 2.7 that a necessary and sufficient condition for equality in the above inequality is that $(iv), (v), (vi)$ must hold true. This completes the proof of the theorem. □

Remark 2.2 : In the above theorem, the inequality $\{r - \lambda(1 + r)\} \sum_{i=1}^{n} \|x_i\| \leq \| \sum_{i=1}^{n} x_i\|$ can be obtained without the strict convexity of $\mathbb{X}$. The equality condition in the above theorem is of course dependent on strict convexity of $\mathbb{X}$.

As an application of the generalized complementary triangle inequalities obtained in the present article, we next present an interesting operator norm inequality in the setting of real Banach spaces as our final result in this article.

Theorem 2.9 — *Let $\mathbb{X}$ be a finite-dimensional real Banach space and $\mathbb{Y}$ be any smooth real Banach space of dimension strictly greater than 2. Let $T_1, T_2 \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be smooth points of unit norm in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $T_1 \neq -T_2$. Then there exists $l > 0$ such that $\|T_1+T_2\| \geq l\,(\|T_1\| + \|T_2\|)$. Moreover, we can choose $l = \min\left\{\frac{[T_1,A]}{\|T_1\|}, \frac{[T_1,A]}{\|T_1\|}\right\}$, where $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is of unit norm, $T_i \not\perp_B A$ $(i = 1, 2)$, and $[\ ,\ ]$ is any s.i.p. on $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Furthermore, given any $n_0 \in \mathbb{N}$, there exists $r_0 > 0$ such that whenever $T_3, \ldots, T_{n_0} \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfy $\|T_i\| < r_0$ for all $i = 3, \ldots, n_0$, it follows that*

$$\frac{l}{2}\sum_{i=1}^{n_0}\|T_i\| < \left\|\sum_{i=1}^{n_0} T_i\right\|.$$

PROOF : Since $T_1 \neq -T_2$, the existence of an l such that $\|T_1 + T_2\| \geq l\,(\|T_1\| + \|T_2\|)$ is guaranteed. We next prove that there exists $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $\|A\| = 1$ and $T_i \not\perp_B A$ $(i = 1, 2)$, where $[\ ,\ ]$ is any s.i.p. on $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. As $T_1, T_2 \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ are smooth, it follows from Theorem 3.3 of [8] that for each $i = 1, 2$, we have that $M_{T_i} = \{\pm u_i\}$, for some $u_i \in S_{\mathbb{Y}}$. Given any $y \in \mathbb{Y}$, let $y^{\perp} = \{z \in \mathbb{Y} : \|y + \lambda z\| \geq \|y\| \textit{ for each scalar } \lambda\}$. Now, it is immediate that y is a smooth point in $\mathbb{Y}$ if and only if $y^{\perp}$ is a subspace of codimension 1 in $\mathbb{Y}$. Since the dimension of $\mathbb{Y}$ is strictly greater than 2, it follows that $\bigcup_{i=1}^{2}(T_i u_i)^{\perp} \subsetneq \mathbb{Y}$. In particular, it is possible to choose $w \in \mathbb{Y} \setminus \bigcup_{i=1}^{2}(T_i u_i)^{\perp}$. Let $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that $Au_i = w,\ i = 1, 2$. It follows from Theorem 2.1 of [9] that $T_i \not\perp_B A$ $(i = 1, 2)$. Now a combination of Remark 2.2 and the norm inequality technique used in the last part of Theorem 2.3 yields the desired result. This establishes the theorem completely. □

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