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ORIGINAL PAPER

Orthogonality and norm attainment of operators in semi-Hilbertian spaces

Jeet Sen[1] · Debmalya Sain[2] · Kallol Paul[2]

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Abstract

We study the semi-Hilbertian structure induced by a positive operator A on a Hilbert space $\mathbb{H}$. Restricting our attention to $A-$bounded positive operators, we characterize the norm attainment set and also investigate the corresponding compactness property. We obtain a complete characterization of the $A-$Birkhoff–James orthogonality of $A-$bounded operators under an additional boundedness condition. This extends the finite-dimensional Bhatia-Šemrl Theorem verbatim to the infinite-dimensional setting.

Keywords Semi-Hilbertian structure · Renorming · Positive operators · A-Birkhoff-James orthogonality · Norm attainment set · Compact operators

Mathematics Subject Classification 47C05 · 47L05 · 46B03 · 47A30 · 47B65

1 Introduction

The purpose of the paper was to explore the orthogonality and the norm attainment of bounded linear operators in the context of semi-Hilbertian structure induced by positive operators on a Hilbert space. Such a study was initiated by Krein in [10] and it remains an active and productive area of research till date. We refer the readers

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✉ Kallol Paul
kalloldada@gmail.com

Jeet Sen
senet.jeet@gmail.com

Debmalya Sain
saindebmalya@gmail.com

1 Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India

2 Department of Mathematics, Indian Institute of Science, Bengaluru 560012, Karnataka, India

to [2, 3, 8, 18] and the references therein for more information on this. Let us now mention the relevant notations and the terminologies to be used in the article.

We use the symbol $\mathbb{H}$ to denote a Hilbert space. Finite-dimensional Hilbert spaces are also known as Euclidean spaces. Unless mentioned specifically, we work with both real and complex Hilbert spaces. The scalar field is denoted by $\mathbb{K}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. The underlying inner product and the corresponding norm on $\mathbb{H}$ are denoted by $\langle\ ,\ \rangle$ and $\|\cdot\|$, respectively. In general, inner products on $\mathbb{H}$ are defined as positive definite, conjugate symmetric forms which are linear in the first argument. It should be noted that apart from the underlying inner product $\langle\ ,\ \rangle$ on $\mathbb{H}$, there may be many other inner products defined on $\mathbb{H}$, generating different norms. In order to avoid any confusion, whenever we talk of a topological concept on $\mathbb{H}$, we explicitly mention the norm that generates the corresponding topology. Let $B_{\mathbb{H}} = \{x \in \mathbb{H} : \|x\| \leq 1\}$ and $S_{\mathbb{H}} = \{x \in \mathbb{H} : \|x\| = 1\}$ be the unit ball and the unit sphere of $\mathbb{H}$, respectively. We use the symbol θ to denote the zero vector of any Hilbert space other than the scalar fields $\mathbb{R}$ and $\mathbb{C}$. For any complex number z, $Re(z)$ and $Im(z)$ denote the real part and the complex part of z, respectively. For any set $G \subset \mathbb{H}$, $\overline{G}$ denotes the norm closure of G. Let $\mathbb{L}(\mathbb{H})(\mathbb{K}(\mathbb{H}))$ denote the Banach space of all bounded (compact) linear operators on $\mathbb{H}$, endowed with the usual operator norm. Given any $A \in \mathbb{L}(\mathbb{H})$, we denote the null space of A by $N(A)$ and the range space of A by $R(A)$. The symbol I is used to denote the identity operator on $\mathbb{H}$. For $A \in \mathbb{L}(\mathbb{H})$, A^* denotes the Hilbert adjoint of A. An operator $A \in \mathbb{L}(\mathbb{H})$ can be represented as $A = ReA + iImA$, where $ReA = \frac{1}{2}(A + A^*)$ and $ImA = \frac{1}{2i}(A - A^*)$. Recall that $A \in \mathbb{L}(\mathbb{H})$ is said to be a positive operator if $A = A^*$ and $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$. A positive operator A is said to be positive definite if $\langle Ax, x\rangle > 0$ for all $x \in \mathbb{H} \setminus \{\theta\}$. It is well known [2] that any positive operator $A \in \mathbb{L}(\mathbb{H})$ induces a positive semi-definite sesquilinear form $\langle\ ,\ \rangle_A$ on $\mathbb{H}$, given by $\langle x, y\rangle_A = \langle Ax, y\rangle$, where $x, y \in \mathbb{H}$. It is easy to see that $\langle\ ,\ \rangle_A$ induces a semi-norm $\|\cdot\|_A$ on $\mathbb{H}$, given by $\|x\|_A = \sqrt{\langle Ax, x\rangle}$. Moreover, when A is positive definite, it can be verified that $\langle\ ,\ \rangle_A$ is an inner product on $\mathbb{H}$ and $\|\cdot\|_A$ is a norm on $\mathbb{H}$. In fact, given any $A \in \mathbb{L}(\mathbb{H})$, it is natural to ask when the functions $\langle\ ,\ \rangle_A$ and $\|\cdot\|_A$, defined as above, are an inner product and a norm on $\mathbb{H}$, respectively. We explore this question and some related topics in the first part of our main results. We refer the readers to [1, 4, 7, 11] for some more interesting results in this direction.

Given a Hilbert space $(\mathbb{H}, \|\cdot\|)$ and a positive $A \in \mathbb{L}(\mathbb{H})$, it is clear that $ker\|\cdot\|_A = \{x \in \mathbb{H} : \|x\|_A = 0\}$ is a closed linear subspace of $\mathbb{H}$. Then there is a closed linear subspace $W \subseteq \mathbb{H}$ such that $W \perp ker\|\cdot\|_A$ and $\mathbb{H} = W + ker\|\cdot\|_A$. Let P be the linear projection on W such that $kerP = ker\|\cdot\|_A$. Then it follows from [17] that $\|x\|_A = \|Px\|_A$. In other words, the restriction of $\|\cdot\|_A$ to the subspace W is indeed a norm which satisfies the parallelogram property and so $(W, \|\cdot\|_A)$ is an inner product space. The investigations for the space $\mathbb{H}$ equipped with the seminorm $\|\cdot\|_A$ are very closely connected to the investigations for the inner product space $(W, \|\cdot\|_A)$. Furthermore, we consider A–bounded linear operator $T : \mathbb{H} \longrightarrow \mathbb{H}$. Next, we define linear operator $\hat{T} : W \longrightarrow W$ by $\hat{T}(w) := T(w)$. Now, it is very easy to see that we can think of the A–norm on $\mathbb{L}(\mathbb{H})$ as the classical operator norm in the operator space $\mathbb{L}(W)$. Of course, in this case, W is equipped with the norm $\|\cdot\|_A : W \longrightarrow [0, \infty)$. Recently, Zamani [18] investigated the orthogonality relation induced by a positive linear operator on a Hilbert space and obtained some interesting results. In particular, he generalized Theorem 1.1 of [5], also known as the Bhatia-Šemrl Theorem, that characterizes the Birkhoff-James orthogonality of matrices on Euclidean spaces. Let us now recall some relevant definitions from [2] and [18].

Definition 1.1 Let $\mathbb{H}$ be a Hilbert space. Let $A \in \mathbb{L}(\mathbb{H})$ be positive. An element $x \in \mathbb{H}$ is said to be $A-$orthogonal to an element $y \in \mathbb{H}$, denoted by $x \perp_A y$, if $\langle x, y\rangle_A = 0$.

Note that if $A = I$, then the above definition coincides with the usual notion of orthogonality in Hilbert spaces.

Let $B_{A^{1/2}}(\mathbb{H}) = \left\{T \in \mathbb{L}(\mathbb{H}) : \exists\, c > 0 \text{ such that } \|Tx\|_A \leq c\|x\|_A \ \forall x \in \mathbb{H}\right\}$. The $A-$norm of $T \in B_{A^{1/2}}(\mathbb{H})$ is given as follows:

$$\|T\|_A = \sup_{x\in\mathbb{H}, \|x\|_A=1} \|Tx\|_A = \sup\left\{|\langle Tx, y\rangle_A| : x, y \in \mathbb{H}, \|x\|_A = \|y\|_A = 1\right\}.$$

An operator $T \in \mathbb{L}(\mathbb{H})$ is said to be $A-$bounded if $T \in B_{A^{1/2}}(\mathbb{H})$.

Definition 1.2 $T \in B_{A^{1/2}}(\mathbb{H})$ is said to be $A-$Birkhoff–James orthogonal to $S \in B_{A^{1/2}}(\mathbb{H})$, denoted by $T \perp_A^B S$, if $\|T + \gamma S\|_A \geq \|T\|_A$ for all $\gamma \in \mathbb{C}$.

Note that the above definition gives a generalization of the Birkhoff–James orthogonality of bounded linear operators on a Hilbert space. For more information on Birkhoff–James orthogonality in normed linear spaces, we refer the readers to the pioneering articles [6, 9]. Birkhoff–James orthogonality of bounded linear operators and some related applications have been explored in recent times in [5, 12, 13, 15, 16]. We also make use of the following notations:

Given a positive operator $A \in \mathbb{L}(\mathbb{H})$, let $B_{\mathbb{H}(A)}$ and $S_{\mathbb{H}(A)}$ denote the $A-$unit ball and the $A-$unit sphere of $\mathbb{H}$, respectively, i.e., $B_{\mathbb{H}(A)} = \left\{x \in \mathbb{H} : \|x\|_A \leq 1\right\}$ and $S_{\mathbb{H}(A)} = \left\{x \in \mathbb{H} : \|x\|_A = 1\right\}$. For any $T \in B_{A^{1/2}}(\mathbb{H})$, the $A-$norm attainment set M_A^T of T was considered in [18]:

$$M_A^T = \left\{x \in \mathbb{H} : \|x\|_A = 1, \ \|Tx\|_A = \|T\|_A\right\}.$$

We study the structure of the $A-$norm attainment set of an $A-$bounded operator $T \in \mathbb{L}(\mathbb{H})$ and also explore the corresponding compactness property. As the most important result of the present article, we obtain a complete characterization of the $A-$Birkhoff–James orthogonality of compact and $A-$bounded operators on $\mathbb{H}$ under an additional condition. This extends the Bhatia–Šemrl Theorem to the setting of semi-Hilbertian spaces, induced by a positive operator.

2 Main Results

We begin this section with a characterization of the norm-generating operators on a Hilbert space.

Theorem 2.1 *Let* $\mathbb{H}$ *be a Hilbert space and let* $A \in \mathbb{L}(\mathbb{H})$*. Then* $\|\cdot\|_A$ *is a norm on* $\mathbb{H}$ *if and only if* $\langle Ax, x\rangle > 0$ *for all* $x \in \mathbb{H} \setminus \{\theta\}$*.*

Proof As the necessary part of the theorem follows trivially, we only prove the sufficient part.

Clearly, $\|x+y\|_A^2 = \|x\|_A^2 + \|y\|_A^2 + \langle Ax, y\rangle + \langle Ay, x\rangle$. This shows that $\langle Ax, y\rangle + \langle Ay, x\rangle$ is real. It is easy to see that $Re\langle Ax, y\rangle + Re\langle Ay, x\rangle = \langle (ReA)x, y\rangle + \langle (ReA)y, x\rangle$, where $ReA = \frac{1}{2}(A + A^*)$.

Clearly, $\|\cdot\|_A$ trivially satisfies all the properties for being a norm, except possibly the triangle inequality. The triangle inequality is satisfied if for all $x, y \in \mathbb{H}$,

$$\begin{aligned} \|x+y\|_A &\leq \|x\|_A + \|y\|_A \\ i.e.,\ if, \langle A(x+y), x+y\rangle &\leq \langle Ax, x\rangle + \langle Ay, y\rangle + 2\langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \\ i.e.,\ if, \langle Ax, y\rangle + \langle Ay, x\rangle &\leq 2\langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \\ i.e.,\ if, Re\langle Ax, y\rangle + Re\langle Ay, x\rangle &\leq 2\langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \\ i.e.,\ if, \langle (ReA)x, y\rangle + \langle (ReA)y, x\rangle &\leq 2\langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}. \end{aligned}$$

Note that for all $x \in \mathbb{H}$, $\langle ReAx, x\rangle = \frac{1}{2}(\langle Ax, x\rangle + \langle A^*x, x\rangle)) = \frac{1}{2}(\langle Ax, x\rangle + \overline{\langle Ax, x\rangle}) = \langle Ax, x\rangle$. This proves that ReA is positive definite and so there exists a unique positive operator B on $\mathbb{H}$ such that $ReA = B^2$. Now, we have

$$\begin{aligned} |\langle (ReA)x, y\rangle| &= |\langle B^2x, y\rangle| = |\langle Bx, By\rangle| = \|Bx\|\|By\| \\ &= \langle B^2x, x\rangle^{1/2}\langle B^2y, y\rangle^{1/2} = \langle (ReA)x, x\rangle^{1/2}\langle (ReA)y, y\rangle^{1/2} \\ &= \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}. \end{aligned}$$

Similarly, we can show that $|\langle (ReA)y, x\rangle| \leq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}$. Therefore,

$$\langle (ReA)x, y\rangle + \langle (ReA)y, x\rangle \leq |\langle (ReA)x, y\rangle + \langle (ReA)y, x\rangle| \leq 2\langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}.$$

This completes the proof of the fact that $\|\cdot\|_A$ is a norm on $\mathbb{H}$. □

As mentioned in the introduction, if A is a positive definite operator on a Hilbert space $\mathbb{H}$, then A generates an inner product $\langle\ ,\ \rangle_A$ on $\mathbb{H}$ defined as $\langle x, y\rangle_A = \langle Ax, y\rangle$ for all $x, y \in \mathbb{H}$. On the other hand, suppose that $A \in \mathbb{L}(\mathbb{H})$ is such that $\langle x, y\rangle_A$ is an inner product on $\mathbb{H}$. From the conjugate-symmetry of inner product, it follows that A must be self adjoint and from the positive definiteness of inner product, it follows that A must be positive definite. This is mentioned in the following proposition:

Proposition 2.1 *Let* $\mathbb{H}$ *be a Hilbert space and let* $A \in \mathbb{L}(\mathbb{H})$*. Then* $\langle\ ,\ \rangle_A$ *is an inner product on* $\mathbb{H}$ *if and only if* A *is positive definite.*

Remark 2.1 In view of the above theorem, there is a subtle difference in the description of the norm generating operators, depending on whether the underlying Hilbert space is complex or real. This is illustrated in the following two points:

1. If $\mathbb{H}$ is a complex Hilbert space then $\langle\ ,\ \rangle_A$ and $\|\cdot\|_A$ are inner product and norm on $\mathbb{H}$, respectively, if and only if A is a positive definite operator on $\mathbb{H}$. This is because of the well-known fact that in case of a complex Hilbert space $\mathbb{H}$, if $A \in \mathbb{L}(\mathbb{H})$ is such that $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$, then $A = A^*$.
2. If $\mathbb{H}$ is real, then there may exist $A \in \mathbb{L}(\mathbb{H})$ such that $A \neq A^*$ (and consequently, A is not positive definite) but $\|\cdot\|_A$ is a norm on $\mathbb{H}$. As for example, consider the operator A on the Hilbert space $\ell_2^2(\mathbb{R})$ defined as $A(x, y) = (x - y, x + y)$ for all $(x, y) \in \mathbb{R}^2$. Then it is easy to see that $\langle Ax, x\rangle > 0$ for all $x \neq \theta$ but $A \neq A^*$. A generates a norm given by $\|x\|_A = \langle Ax, x\rangle^{1/2}$ on $\ell_2^2(\mathbb{R})$ but $\langle x, y\rangle_A = \langle Ax, y\rangle$ is not an inner product on $\ell_2^2(\mathbb{R})$. The inner product that induces the norm $\|\cdot\|_A$ is given by $\langle (ReA)x, y\rangle$. In fact, given any $A \in \mathbb{L}(\mathbb{H})$ with $\langle Ax, x\rangle > 0$ for all $x \neq \theta$, the positive definite operator ReA always generates an inner product $\langle x, y\rangle_{ReA} = \langle (ReA)x, y\rangle$ which induces the norm $\|\cdot\|_A$.

Our next theorem guarantees that under a suitable condition, given any inner product on an infinite-dimensional separable Hilbert space $\mathbb{H}$, there exists a unique positive definite operator that generates the given inner product.

Theorem 2.2 *Let $(\mathbb{H}, \langle\ ,\ \rangle)$ be a separable Hilbert space. Let $\langle\ ,\ \rangle_1$ be another inner product on $\mathbb{H}$. Then the following two conditions are equivalent*:

(i) *there exists a positive definite operator A on $\mathbb{H}$ such that $\langle\ ,\ \rangle_1 = \langle\ ,\ \rangle_A$.*
(ii) *there exists $M > 0$ such that $\|x\|_1 \leq M\|x\|$ for all $x \in \mathbb{H}$, where $\|\cdot\|_1$ is the norm induced by the inner product $\langle\ ,\ \rangle_1$ on $\mathbb{H}$.*

Proof (i) $\Rightarrow$ (ii) : Clearly, $\|x\|_1^2 = \langle x, x\rangle_1 = \langle x, x\rangle_A = \langle Ax, x\rangle \leq \|A\|\|x\|^2$.

(ii) $\Rightarrow$ (i) : Since $\|x\|_1 \leq M\|x\|$ for all $x \in \mathbb{H}$, it follows that $\mathbb{H}$ is a separable inner product space with respect to $\langle\ ,\ \rangle_1$. Let $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ be the completion of $(\mathbb{H}, \langle\ ,\ \rangle_1)$. Clearly, $\langle x, y\rangle_{\mathcal{H}} = \langle x, y\rangle_1$ for all $x, y \in \mathbb{H}$. Since $\mathbb{H}$ is separable with respect to $\langle\ ,\ \rangle_1$, it is easy to deduce that $\mathcal{H}$ is separable with respect to $\langle\ ,\ \rangle_{\mathcal{H}}$. Let $B = \{e_1, e_2, e_3, \dots\}$ be an orthonormal basis of $\mathbb{H}$ with respect to $\langle\ ,\ \rangle$ and let $B_1 = \{f_1, f_2, f_3, \dots\}$ be an orthonormal basis of $\mathcal{H}$ with respect to $\langle\ ,\ \rangle_{\mathcal{H}}$. Consider the map $\widetilde{T} : (\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}}) \to (\mathbb{H}, \langle\ ,\ \rangle)$ defined by $\widetilde{T}(\sum_{i=1}^{\infty} a_i f_i) = \sum_{i=1}^{\infty} a_i e_i$ where $a_i \in \mathbb{K}(= \mathbb{R}, \mathbb{C})$ for all $i \in \mathbb{N}$. It can be verified easily that $\widetilde{T}$ is well-defined and linear. Let $T = \widetilde{T}\,|_{(\mathbb{H}, \langle\ ,\ \rangle_1)}$. It is easy to see that $\langle x, y\rangle_1 = \langle Tx, Ty\rangle$ for all $x, y \in \mathbb{H}$. Thus $\|Tx\|^2 = \langle x, x\rangle_1 \leq M^2\|x\|^2$. In particular, T is bounded and, therefore, the adjoint operator $T^* : (\mathbb{H}, \langle\ ,\ \rangle) \longrightarrow (\mathbb{H}, \langle\ ,\ \rangle_1)$ exists. Let $A = T^*T$. Then it is easy to see that A is a positive definite operator on $(\mathbb{H}, \langle\ ,\ \rangle)$ such that $\langle x, y\rangle_1 = \langle Ax, y\rangle$ for all $x, y \in \mathbb{H}$.

The uniqueness of A follows from the fact that if B is any positive definite operator that generates the inner product $\langle\ ,\ \rangle_1$ then $\langle Ax, y\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathbb{H}$ and so $A = B$. □

In light of the above theorem, let us make the following two remarks:

Remark 2.2 In case $\mathbb{H}$ is finite-dimensional, Condition (*ii*) of the above theorem holds true automatically. Therefore, we obtain a complete description of the set of all inner products defined on an Euclidean space, in terms of positive definite operators on $\mathbb{H}$. Following the usual matricial representation of linear operators on Euclidean spaces, it seems convenient to say that every positive definite matrix defines an inner product on $\mathbb{K}^n$ and conversely.

Remark 2.3 We note that if $\langle\ ,\ \rangle_1$ is an inner product on $\mathbb{H}$ such that Condition (*ii*) of the above theorem is satisfied, it is not necessarily true that $(\mathbb{H}, \langle\ ,\ \rangle_1)$ is complete. Such an example will be constructed explicitly in the proof of Theorem 2.3 (*iv*).

The unit ball $B_{\mathbb{H}}$ is convex and bounded with respect to $\|\cdot\|$. Also, it is compact (in the topology induced by $\|\cdot\|$) if and only $\mathbb{H}$ is finite-dimensional. We next study some analogous geometric and topological properties of the A–unit ball $B_{\mathbb{H}(A)}$ with respect to the norm $\|\cdot\|$. We begin with the following proposition, the proof of which is omitted as it follows rather trivially from the convexity of the A–norm and the continuity of the inner product.

Proposition 2.2 *Let* $\mathbb{H}$ *be a Hilbert space and let* $A \in \mathbb{L}(\mathbb{H})$ *be positive. Then* $B_{\mathbb{H}(A)}$ *is convex and closed with respect to* $\|\cdot\|$.

We would like to describe the boundedness properties of the A–unit ball and the A–unit sphere with respect to the norm $\|\cdot\|$. We require the following proposition which is particularly useful in our study. The proof is omitted, as it can be obtained quite easily.

Proposition 2.3 *Let* $\mathbb{H}$ *be a Hilbert space. Let* $A \in \mathbb{L}(\mathbb{H})$ *be positive. Then* $\mathbb{H} = N(A) \oplus \overline{R(A)}$.

We describe the boundedness properties of the A–unit ball and the A–unit sphere in the next theorem.

Theorem 2.3 *Let* $\mathbb{H}$ *be a Hilbert space and let* $A \in \mathbb{L}(\mathbb{H})$ *be positive. Then the following hold true*:

(i) *If* $N(A) \neq \{\theta\}$, *then both* $S_{\mathbb{H}(A)}$ *and* $B_{\mathbb{H}(A)}$ *are unbounded with respect to* $\|\cdot\|$.
(ii) *If* $\mathbb{H}$ *is finite-dimensional, then* $B_{\mathbb{H}(A)} \cap \overline{R(A)} (= B_{\mathbb{H}(A)} \cap R(A))$ *is bounded with respect to* $\|\cdot\|$.
(iii) *If H is finite-dimensional, then* $B_{\mathbb{H}(A)}$ *is bounded with respect to* $\|\cdot\|$ *if and only if* $N(A) = \{\theta\}$.
(iv) *Both (ii) and (iii) fail to hold if* $\mathbb{H}$ *is infinite-dimensional.*

Proof We first observe that

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^2x, x\rangle = \langle Ax, x\rangle_A$$

and so it follows that $\|x\|_A = 0$ if and only if $x \in N(A)$:

(i) Let $x \in N(A)$ be such that $x \neq \theta$. Then $\|x\|_A = 0$ and so $\|\lambda x\|_A = 0$ for all $\lambda \in \mathbb{K}(= \mathbb{R}, \mathbb{C})$. Next we claim that if $z \in S_{\mathbb{H}(A)}$, then $z + \lambda x \in S_{\mathbb{H}(A)}$ for all $\lambda \in \mathbb{K}(= \mathbb{R}, \mathbb{C})$. Clearly, $\|z + \lambda x\|_A \leq \|z\|_A + |\lambda|\|x\|_A = 1$. Again, $\|z + \lambda x\|_A \geq \|z\|_A - |\lambda|\|x\|_A = 1$. Thus $z + \lambda x \in S_{\mathbb{H}(A)}$ for all $\lambda \in \mathbb{K}(= \mathbb{R}, \mathbb{C})$. Therefore, $S_{\mathbb{H}(A)}$ is unbounded with respect to $\|\cdot\|$ and so $B_{\mathbb{H}(A)}$ is also unbounded with respect to $\|\cdot\|$.

(ii) Suppose on the contrary that $B_{\mathbb{H}(A)} \cap R(A)$ is unbounded with respect to $\|\cdot\|$. Then for each $n \in \mathbb{N}$, there exists $v_n \in B_{\mathbb{H}(A)} \cap R(A)$ such that $\|v_n\| \geq n$. Let $w_n = \frac{v_n}{\|v_n\|}$. Then $\|w_n\| = 1$ and $\|w_n\|_A \leq \frac{1}{n}$. Clearly, $\{w_n\} \subseteq S_{\mathbb{H}}$. Since $\mathbb{H}$ is finite-dimensional, $S_{\mathbb{H}}$ is compact. Without loss of generality we may assume that $w_n \longrightarrow w$, where $w \in S_{\mathbb{H}}$. By Proposition 2.2, $B_{\mathbb{H}(A)} \cap R(A)$ is a closed set with respect to $\|\cdot\|$ and hence $w \in B_{\mathbb{H}(A)} \cap R(A)$. It is easy to check that $\|w_n\|_A \longrightarrow \|w\|_A$. Therefore, $\|w\|_A = 0$ and so $w \in N(A)$. This shows that $w \in N(A) \cap R(A)$ and so $w = \theta$, a contradiction to our assumption that $w \in S_{\mathbb{H}}$. Therefore, $B_{\mathbb{H}(A)} \cap R(A)$ is bounded with respect to $\|\cdot\|$.

(iii) As $\mathbb{H}$ is finite-dimensional, $\mathbb{H} = N(A) \oplus R(A)$. Therefore, any $x \in B_{\mathbb{H}(A)}$ can be uniquely written as $x = u + v$, where $u \in B_{\mathbb{H}(A)} \cap N(A)$ and $v \in B_{\mathbb{H}(A)} \cap R(A)$. From (ii), it follows that $B_{\mathbb{H}(A)}$ is bounded with respect to $\|\cdot\|$ if and only if $B_{\mathbb{H}(A)} \cap N(A)$ is bounded with respect to $\|\cdot\|$. From (i), it follows that $N(A) = \{\theta\}$ if $B_{\mathbb{H}(A)}$ is bounded with respect to $\|\cdot\|$. On the other hand, if $N(A) = \{\theta\}$ then $B_{\mathbb{H}(A)} = B_{\mathbb{H}(A)} \cap R(A)$ is bounded with respect to $\|\cdot\|$ by applying (ii).

(iv) Consider the Hilbert space ℓ_2. Let $A \in \mathbb{L}(\ell_2)$ be defined by $A(x_1, x_2, x_3, \ldots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$, where $(x_1, x_2, x_3, \ldots) \in \ell_2$. It is easy to check that A is positive definite and $N(A) = \{\theta\}$. Therefore, $\ell_2 = \overline{R(A)}(\neq R(A))$. Consider the sequence $\{v_n\} \subseteq \ell_2$ where $\{v_n\} = \{\sqrt{n}e_n\}$, where $\{e_n\}$ is the usual orthonormal basis of ℓ_2. Clearly, $\|v_n\|_A^2 = \langle Av_n, v_n\rangle = 1$ for each $n \in \mathbb{N}$ but $\|v_n\| = \sqrt{n}$ for each $n \in \mathbb{N}$.

□

In view of the above theorem, we make the following remark on the geometry of semi-Hilbertian spaces.

Remark 2.4 Let A be a positive operator on a Hilbert space $\mathbb{H}$. If $\|x\|_A = 0$ for some $x \neq \theta$, then by (i) of Theorem 2.3, the $A-$unit sphere of $\mathbb{H}$ contains a straight line. In other words, the semi-normed space $(\mathbb{H}, \|\cdot\|_A)$ is not strictly convex whenever A is not positive definite.

There is another nice way to obtain Remark 2.4. Namely, now suppose that A is positive, but not positive definite. Since $\|\cdot\|_A$ is a seminorm, it follows from

Theorem 3.2 from [17] that any $x \in B_{\mathbb{H}(A)}$ can be uniquely written as $x = u + v$, where $u \in B_W$ and $v \in ker\|\cdot\|_A$. Note that B_W is the closed unit ball in the inner product space $(W, \|\cdot\|_A)$, and $ker\|\cdot\|_A$ is a linear subspace. Therefore, it is easy to see that $A-$unit sphere of $\mathbb{H}$ contains a straight line and the seminormed space $(\mathbb{H}, \|\cdot\|_A)$ is not strictly convex whenever A is not positive definite.

In Theorem 2.2 of [14], the authors studied the norm attainment sets of bounded linear operators on a Hilbert space. In particular, it was proved that in an inner product space $\mathbb{H}$, for any operator $T \in \mathbb{L}(\mathbb{H})$, the norm attainment set M_T is either the empty set ϕ, or, M_T is the unit sphere of some subspace of $\mathbb{H}$. Our next result generalizes this, in case of $A-$bounded operators.

Theorem 2.4 *Let $\mathbb{H}$ be a Hilbert space. Let $A \in \mathbb{L}(\mathbb{H})$ be positive and let $T \in B_{A^{1/2}}(\mathbb{H})$. Then either $M_A^T = \phi$ or $M_A^T \cap \overline{R(A)}$ is the $A-$unit sphere of some subspace of $\mathbb{H}$.*

Proof If $M_A^T = \phi$, then we have nothing to prove. Let us assume that $M_A^T \neq \phi$. Let $x \in M_A^T$. As $\mathbb{H} = N(A) \oplus \overline{R(A)}$, x can be uniquely written as $x = u + v$, where $u \in N(A)$ and $v \in \overline{R(A)}$. Hence $\|u\|_A = 0$ and $\|x\|_A = \|v\|_A$. As $T \in B_{A^{1/2}}(\mathbb{H})$, it follows that $\|Tu\|_A = 0$ and, therefore, $\|Tx\|_A = \|Tv\|_A = \|T\|_A$. This proves that $M_A^T \cap \overline{R(A)} \neq \phi$.

To prove that $M_A^T \cap \overline{R(A)}$ is the $A-$ unit sphere of some subspace of $\mathbb{H}$, it is enough to show that $\frac{\lambda_1 e_1 \pm \lambda_2 e_2}{\|\lambda_1 e_1 \pm \lambda_2 e_2\|_A} \in M_A^T \cap \overline{R(A)}$, whenever $e_1, e_2 \in M_A^T \cap \overline{R(A)}$ and $\lambda_1, \lambda_2 \in \mathbb{K}(= \mathbb{R}, \mathbb{C})$. Let $e_1, e_2 \in M_A^T \cap \overline{R(A)}$, then $\|Te_1\|_A = \|Te_2\|_A = \|T\|_A$ and $\|e_1\|_A = \|e_2\|_A = 1$. First we claim that $\|\cdot\|_A$ satisfies the parallelogram law for all $x, y \in \mathbb{H}$. Let $x, y \in \mathbb{H}$. Then we have

$$\begin{aligned}\|x+y\|_A^2 + \|x-y\|_A^2 &= \langle x+y, x+y\rangle_A + \langle x-y, x-y\rangle_A \\ &= \langle A(x+y), x+y\rangle + \langle A(x-y), x-y\rangle \\ &= 2(\langle Ax, x\rangle + \langle Ay, y\rangle) \\ &= 2(\|x\|_A^2 + \|y\|_A^2).\end{aligned}$$

This proves our claim. Therefore,

$$\begin{aligned}2(|\lambda_1|^2 + |\lambda_2|^2)\|T\|_A^2 &= 2(\|\lambda_1 Te_1\|_A^2 + \|\lambda_2 Te_2\|_A^2) \\ &= \|\lambda_1 Te_1 + \lambda_2 Te_2\|_A^2 + \|\lambda_1 Te_1 - \lambda_2 Te_2\|_A^2 \\ &= \|T(\lambda_1 e_1 + \lambda_2 e_2)\|_A^2 + \|T(\lambda_1 e_1 - \lambda_2 e_2)\|_A^2 \\ &\leq \|T\|_A^2(\|\lambda_1 e_1 + \lambda_2 e_2\|_A^2 + \|\lambda_1 e_1 - \lambda_2 e_2\|_A^2) \\ &= 2(|\lambda_1|^2 + |\lambda_2|^2)\|T\|_A^2.\end{aligned}$$

Hence the above inequality is actually an equality. Since $\|T(\lambda_1 e_1 \pm \lambda_2 e_2)\|_A \leq \|T\|_A \|\lambda_1 e_1 \pm \lambda_2 e_2\|_A$, it follows that

$$\|T(\lambda_1 e_1 \pm \lambda_2 e_2)\|_A = \|T\|_A \|\lambda_1 e_1 \pm \lambda_2 e_2\|_A.$$

This establishes the theorem. □

In the next theorem we study the compactness property of $M_A^T \cap \overline{R(A)}$.

Theorem 2.5 *Let $\mathbb{H}$ be a Hilbert space and let $A \in \mathbb{L}(\mathbb{H})$ be a positive operator such that $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is bounded with respect to $\|\cdot\|$. Let $T \in \mathbb{K}(\mathbb{H}) \cap B_{A^{1/2}}(\mathbb{H})$. Then $M_A^T \cap \overline{R(A)}$ is compact with respect to $\|\cdot\|$.*

Proof Clearly, $\overline{R(A)}$ is a Hilbert space with respect to $\|\cdot\|$. It is easy to see that A is positive definite on $\overline{R(A)}$. Therefore, $\|\cdot\|_A$ is a norm on $\overline{R(A)}$. We claim that $\|\cdot\|_A$ and $\|\cdot\|$ are equivalent norms on $\overline{R(A)}$. Clearly, $\frac{1}{\sqrt{\|A\|}}\|x\|_A \leq \|x\|$ for all $x \in \mathbb{H}$. Let $x \in \overline{R(A)}$. Then $\frac{x}{\|x\|_A} \in B_{\mathbb{H}(A)} \cap \overline{R(A)}$. Since $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is bounded, there exists $M > 0$ such that $\|z\| \leq M$ for all $z \in B_{\mathbb{H}(A)} \cap \overline{R(A)}$. Therefore, $\frac{\|x\|}{\|x\|_A} \leq M$. Thus $\frac{1}{\sqrt{\|A\|}}\|x\|_A \leq \|x\| \leq M\|x\|_A$ for all $x \in \overline{R(A)}$. Thus our claim is established. Therefore, $\overline{R(A)}$ is a Hilbert space with respect to $\|\cdot\|_A$. Next, let $\{v_n\}$ be a sequence in $M_A^T \cap \overline{R(A)}$. We show that $\{v_n\}$ has a convergent subsequence in $M_A^T \cap \overline{R(A)}$ with respect to $\|\cdot\|$. Since $\mathbb{H}$ is reflexive and $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is closed, convex and bounded with respect to $\|\cdot\|$, it follows that $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is weakly compact with respect to $\|\cdot\|$. Thus the sequence $\{v_n\}$ has a weakly convergent subsequence $\{v_{n_k}\}$ with respect to $\|\cdot\|$. Suppose $v_{n_k} \rightharpoonup v$ for some $v \in B_{\mathbb{H}(A)} \cap \overline{R(A)}$ with respect to $\|\cdot\|$. Since $T \in \mathbb{K}(\mathbb{H})$, it follows that $Tv_{n_k} \longrightarrow Tv$ with respect to $\|\cdot\|$. It is easy to see that

$$\|T\|_A^2 = \lim_{k\to\infty} \|Tv_{n_k}\|_A^2 = \lim_{k\to\infty} \langle ATv_{n_k}, Tv_{n_k}\rangle = \langle ATv, Tv\rangle = \|Tv\|_A^2.$$

As $\|v\|_A \leq 1$, we conclude that $v \in M_A^T \cap \overline{R(A)}$ and $1 = \|v_{n_k}\|_A \longrightarrow \|v\|_A = 1$. As $v_{n_k} \rightharpoonup v$ with respect to $\|\cdot\|$, clearly, $v_{n_k} \rightharpoonup v$ with respect to $\|\cdot\|_A$. Since $(\overline{R(A)}, \langle\ ,\ \rangle_A)$ is a Hilbert space, it follows that $v_{n_k} \longrightarrow v$ with respect to $\|\cdot\|_A$. As $\|\cdot\|_A$ and $\|\cdot\|$ are equivalent norms on $\overline{R(A)}$, therefore, $v_{n_k} \longrightarrow v$ with respect to $\|\cdot\|$. This establishes the theorem. □

Remark 2.5 Note that, $M_A^T \cap \overline{R(A)}$ is also compact with respect to $\|\cdot\|_A$ in $\overline{R(A)}$, due to the fact that $\|\cdot\|_A$ and $\|\cdot\|$ are equivalent norms on $\overline{R(A)}$.

In [18], the author has characterized the A−Birkhoff–James orthogonality of A−bounded operators on a Hilbert space with the help of A−norming sequences. In the finite-dimensional case, the Bhatia-Šemrl Theorem follows from the said characterization, as shown in Theorem 2.4 of [18]. The main difference between the characterizations of A−Birkhoff–James orthogonality of operators in the infinite-dimensional case and the finite-dimensional case is that the approximate orthogonality of the images of norming sequences in the former case can be strengthened to the exact orthogonality of the images of a norming vector in the later case. For the convenience of the readers, let us mention the relevant results from [18] and [5].

Theorem 2.6 (Zamani, Theorem 2.2 of [18]). *Let $T, S \in B_{A^{1/2}}(\mathbb{H})$. Then the following conditions are equivalent*:

(i) *there exists a sequence of A–unit vectors $\{x_n\}$ in $\mathbb{H}$ such that* $\lim_{n\to\infty} \|Tx_n\|_A = \|T\|_A$ *and* $\lim_{n\to\infty}\langle Tx_n, Sx_n\rangle_A = 0$.

(ii) $T\perp_A^B S$.

Theorem 2.7 (*Bhatia and Šemrl, Theorem* 1.1 *of* [5]) *A matrix A is orthogonal to a matrix B if and only if there exists a unit vector $x \in \mathbb{H}$ such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx\rangle = 0$.*

In our next theorem, we show that under certain additional conditions, the said strengthening of the A–Birkhoff–James orthogonality of A–bounded operators can be preserved even in the infinite-dimensional case.

Theorem 2.8 *Let $\mathbb{H}$ be a Hilbert space and let $A \in \mathbb{L}(\mathbb{H})$ be positive such that $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is bounded with respect to $\|\cdot\|$. Let $T, S \in \mathbb{K}(\mathbb{H}) \cap B_{A^{1/2}}(\mathbb{H})$. Then $T\perp_A^B S$ if and only if there exists $v \in M_A^T$ such that $Tv\perp_A Sv$.*

Proof The sufficient part of the theorem follows easily. Indeed, suppose that there exists $v \in M_A^T$ such that $Tv\perp_A Sv$. Then

$$\begin{aligned}\|T + \lambda S\|_A &\geq \|Tv + \lambda Sv\|_A\\ &\geq \|Tv\|_A\\ &= \|T\|_A \text{ for all } \lambda \in \mathbb{K}(= \mathbb{R},\ \mathbb{C}).\end{aligned}$$

Let us prove the necessary part of the theorem. By Theorem 2.2 of [18], there exists a sequence $\{x_n\} \subseteq S_{\mathbb{H}(A)}$ such that

$$\lim_{n\to\infty} \|Tx_n\|_A = \|T\|_A \text{ and } \lim_{n\to\infty}\langle Tx_n, Sx_n\rangle_A = 0.$$

Since $\mathbb{H} = N(A) \oplus \overline{R(A)}$, it follows that $x_n = u_n + v_n$ for each $n \in \mathbb{N}$, where $u_n \in N(A)$ and $v_n \in \overline{R(A)}$. Clearly, $\|u_n\|_A = 0$ for all $n \in \mathbb{N}$. Thus $\|x_n\|_A = \|v_n + u_n\|_A \leq \|v_n\|_A + \|u_n\|_A = \|v_n\|_A$. Again, $\|x_n\|_A = \|v_n + u_n\|_A \geq \|v_n\|_A - \|u_n\|_A = \|v_n\|_A$. Therefore, $\|x_n\|_A = \|v_n\|_A$ for each $n \in \mathbb{N}$. As $\{x_n\} \subseteq S_{\mathbb{H}(A)}$, we conclude that $\{v_n\} \subseteq S_{\mathbb{H}(A)} \cap \overline{R(A)}$. Since $T, S \in B_{A^{1/2}}(\mathbb{H})$, $\|Tu_n\|_A = \|Su_n\|_A = 0$ for all $n \in \mathbb{N}$. Hence $\|Tx_n\|_A = \|Tv_n\|_A$ and $\|Sx_n\|_A = \|Sv_n\|_A$ for each $n \in \mathbb{N}$. Since $\mathbb{H}$ is reflexive and $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is closed, convex and bounded with respect to $\|\cdot\|$, therefore, $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is weakly compact with respect to $\|\cdot\|$. Thus the sequence $\{v_n\}$ has a weakly convergent subsequence. Without loss of generality we may assume that $v_n \rightharpoonup v$ with respect to $\|\cdot\|$ on $\mathbb{H}$, for some $v \in B_{\mathbb{H}(A)} \cap \overline{R(A)}$. Since $T, S \in \mathbb{K}(\mathbb{H})$, it follows that $Tv_n \longrightarrow Tv$ and $Sv_n \longrightarrow Sv$ with respect to $\|\cdot\|$ in $\mathbb{H}$. Therefore,

$$\begin{aligned}\|T\|_A^2 &= \lim_{n\to\infty} \|Tx_n\|_A^2 = \lim_{n\to\infty} \|Tv_n\|_A^2\\ &= \lim_{n\to\infty}\langle ATv_n, Tv_n\rangle = \|Tv\|_A^2.\end{aligned}$$

As $\|v\|_A \leq 1$, we conclude that $v \in M_A^T \cap \overline{R(A)}$.

Next we show that $Tv \perp_A Sv$. As $\|Tu_n\|_A = \|Su_n\|_A = 0$, it is immediate that $Tu_n, Su_n \in N(A)$ for all $n \in \mathbb{N}$. Since A is positive, it follows that $N(A) = N(A^{1/2})$. Hence $A^{1/2}(Tu_n) = A^{1/2}(Su_n) = \theta$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle Tx_n, Sx_n\rangle_A = \lim_{n\to\infty} \langle Tu_n + Tv_n, Su_n + Sv_n\rangle_A \\ &= \lim_{n\to\infty} \langle A(Tu_n + Tv_n), Su_n + Sv_n\rangle \\ &= \lim_{n\to\infty} \langle A^{1/2}(Tu_n + Tv_n), A^{1/2}(Su_n + Sv_n)\rangle \\ &= \lim_{n\to\infty} \langle A^{1/2}Tv_n, A^{1/2}Sv_n\rangle = \lim_{n\to\infty} \langle ATv_n, Sv_n\rangle = \langle ATv, Sv\rangle = \langle Tv, Sv\rangle_A. \end{aligned}$$

Thus $Tv \perp_A Sv$. This completes the proof of the theorem. □

We end this article with the following closing remark:

Remark 2.6 Note that in Theorem 2.8, if $\mathbb{H}$ is finite-dimensional and $A = I$, then the Bhatia-Šemrl Theorem (Theorem 1.1 of [5]) follows immediately. In particular, the finite-dimensional Bhatia-Šemrl Theorem can be extended verbatim to the infinite-dimensional setting of semi-Hilbertian spaces, provided certain additional conditions are satisfied. We further observe that Theorem 2.4 of [18] follows as a corollary to Theorem 2.8, since in a finite-dimensional Hilbert space, $B_{\mathbb{H}(A)} \cap \overline{R(A)}$ is bounded with respect to $\|\cdot\|$ and every linear operator is compact.

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