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On the lacunary spherical maximal function on the Heisenberg group

Pritam Ganguly, Sundaram Thangavelu *

Department of Mathematics, Indian Institute of Science, 560 012 Bangalore, India

A R T I C L E   I N F O

A B S T R A C T

In this paper we investigate the L^p boundedness of the lacunary maximal function $M^{lac}_{\mathbb{H}^n}$ associated to the spherical means $A_r f$ taken over Koranyi spheres on the Heisenberg group. Closely following an approach used by M. Lacey in the Euclidean case, we obtain sparse bounds for these maximal functions leading to new unweighted and weighted estimates. The key ingredients in the proof are the L^p improving property of the operator $A_r f$ and a continuity property of the difference $A_r f - \tau_y A_r f$, where $\tau_y f(x) = f(xy^{-1})$ is the right translation operator.

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1. Introduction and the main results

The study of spherical means has received considerable attention in the last few decades. In 1976, Stein first considered the spherical maximal function on $\mathbb{R}^n$ defined by

* Corresponding author.
    E-mail addresses: pritamg@iisc.ac.in (P. Ganguly), veluma@iisc.ac.in (S. Thangavelu).

$$Mf(x) = \sup_{r>0} |f * \mu_r(x)| = \sup_{r>0} \left| \int_{|y|=r} f(x-y) d\mu_r(y) \right|$$

where μ_r is the normalised measure on the Euclidean sphere of radius r. In [18], for $n \geq 3$ he proved that

$$\|Mf\|_p \leq C\|f\|_p \text{ if and only if } p > \frac{n}{n-1}.$$

Later Cowling-Mauceri [7] revisited this and proved Stein's result using completely different arguments. In 1986, Bourgain [3] settled the case $n = 2$. On the other hand, as proved by C. P. Calderon [4], the lacunary spherical maximal function

$$M_{lac}f(x) = \sup_{j \in \mathbb{Z}} |f * \mu_{2^j}(x)|$$

turned out to be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Recently, M. Lacey has come up with a new idea to prove these two results and much more. In [13] he has obtained sparse bounds for both M and M_{lac} leading to new weighted norm estimates.

Our aim in this paper is to prove sparse bounds for the lacunary spherical maximal functions on the Heisenberg group. Recall that the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ is equipped with the group operation

$$(z,t).(w,s) := \left(z + w, t + s + \frac{1}{2}\Im(z.\bar{w})\right), \ \forall(z,t), (w,s) \in \mathbb{H}^n.$$

On this group, for every $r > 0$, we have a family of non-isotropic dilations defined by $\delta_r(z,t) := (rz, r^2t), \ \forall(z,t) \in \mathbb{H}^n$ These δ_r are automorphisms of the group $\mathbb{H}^n$. The Koranyi norm of (z,t) in $\mathbb{H}^n$ is defined by

$$|(z,t)| := (|z|^4 + 16t^2)^{\frac{1}{4}}$$

and it is easy to see that $|\delta_r(z,t)| = r|(z,t)|$, i.e., the norm is homogeneous of degree 1 relative to these non-isotropic dilations. The Haar measure on $\mathbb{H}^n$ is simply the Lebesgue measure $dzdt$. As in the Euclidean case, this Haar measure has a polar decomposition. Let $S_K := \{(z,t) \in \mathbb{H}^n : |(z,t)| = 1\}$ be the unit sphere with respect to the Koranyi norm. Then there is a unique Radon measure σ on S_K such that for every integrable function on $\mathbb{H}^n$ we have

$$\int_{\mathbb{H}^n} f(z,t)dzdt = \int_0^\infty \int_{S_K} f(\delta_r\omega) r^{Q-1} d\sigma(\omega)dr \tag{1.1}$$

where $Q = (2n+2)$ is known as the homogeneous dimension of $\mathbb{H}^n$.

Let us fix some more notations first. Given f on $\mathbb{H}^n$, its dilation $\delta_r f$ is simply defined by $\delta_r f(z,t) = f(\delta_r(z,t)), \ \forall(z,t) \in \mathbb{H}^n$. More generally, if U is a distribution on $\mathbb{H}^n$,

then its dilation $\delta_r U$ is defined by $\langle \delta_r U, \phi \rangle := \langle U, \delta_r \phi \rangle$, $\forall \phi \in C_c^\infty(\mathbb{H}^n)$. We let $\sigma_r = \delta_r \sigma$ and define the spherical mean value operator A_r by

$$A_r f(x) = f * \sigma_r(x) = \int_{|y|=1} f(x.\delta_r y^{-1}) d\sigma(y).$$

The associated spherical maximal function $M_{\mathbb{H}^n} f(x) = \sup_{r>0} |A_r f(x)|$ was studied by M. Cowling in [5] where he established the following result.

Theorem 1.1. *The spherical maximal function $M_{\mathbb{H}^n}$ is bounded on $L^p(\mathbb{H}^n)$ for all $p > \frac{2n+1}{2n}$.*

This is the Heisenberg analogue of Stein's theorem on $\mathbb{R}^n$ but Cowling used an entirely different approach in proving this. He used a very clever idea of expressing σ_r as a weighted integral of Riesz potentials which are easy to handle, see Theorem 3.2 below. Other proofs of the above theorem are also available in the literature. By adapting Stein's second proof of the Euclidean case [19] (Corollary 3 in XI, section 3), Oliver Schmidt, in his still unpublished work [17], studied maximal functions on stratified groups associated to hyper-surfaces with a non vanishing rotational curvature, giving another proof of the above theorem of Cowling. Also in [10], using a square function argument, Fischer proved the boundedness of the spherical maximal operator on functions on free step two nilpotent Lie groups which includes the above result of Cowling.

In this paper we are interested in establishing an analogue of Calderon's theorem for the Heisenberg group. Though the methods used in [4] are adaptations of the original idea of Stein [18], it is unlikely that a proof along similar lines is possible. This is mainly because the group Fourier transform of the measure σ_r is not known explicitly. To state our main result, we fix $\delta \in (0, 1/96)$ and let

$$M_{\mathbb{H}^n}^{lac} f(x) = \sup_{k \in \mathbb{Z}} |A_{\delta^k} f(x)|$$

stand for the lacunary spherical maximal function on $\mathbb{H}^n$. We prove:

Theorem 1.2. *Let $n \geq 2$. The lacunary spherical maximal function $M_{\mathbb{H}^n}^{lac}$ is bounded on $L^p(\mathbb{H}^n)$ for all $1 < p < \infty$.*

This is the analogue of Calderon's theorem for the spherical averages on Koranyi spheres. In view of the work by Fischer [10], it is natural to ask if the above theorem can be proved in the more general context of free step two nilpotent Lie groups. For reasons we just mentioned before the statement of the theorem, it is going to be very difficult to prove such a result using spectral methods. In proving the above theorem for the Heisenberg group, we combine the idea of Cowling along with a recently formulated strategy of Lacey [13] in the Euclidean context that has turned out to be fruitful in

handling several maximal functions. We believe that our proof can be suitably adapted to handle at least the case of H-type groups, we restrain ourselves from doing so in order to keep the proofs elegant. We plan to take up the general case of free step two groups in a future work.

As mentioned above, we establish Theorem 1.2 by closely following Lacey [13]. The main idea of Lacey in reproving Calderon's theorem on $\mathbb{R}^n$ is to establish a sparse domination for the spherical means. In order to do so, he required two main ingredients: (i) the L^p improving property and (ii) the continuity property, of the spherical means. These ingredients are then used to control the spherical maximal function in terms of certain dyadic maximal functions.

Once a sparse domination for the lacunary spherical maximal function is obtained, it is then a routine matter to deduce weighted and unweighted L^p bounds. Theorem 1.2, as well as certain weighted versions that are stated in Section 5 are easy consequences of the sparse bound stated in Theorem 1.3, which is the main result of this paper. In order to state the result we need to set up some more notation, see [13].

As in the case of $\mathbb{R}^n$, there is a notion of dyadic grids on $\mathbb{H}^n$, the members of which are called (dyadic) cubes. A collection of cubes $\mathcal{S}$ in $\mathbb{H}^n$ is said to be η-sparse if there are sets $\{E_S \subset S : S \in \mathcal{S}\}$ which are pairwise disjoint and satisfy $|E_S| > \eta|S|$ for all $S \in \mathcal{S}$. For any cube Q and $1 < p < \infty$, we define

$$\langle f \rangle_{Q,p} := \left(\frac{1}{|Q|} \int\limits_Q |f(x)|^p dx \right)^{1/p}, \qquad \langle f \rangle_Q := \frac{1}{|Q|} \int\limits_Q |f(x)| dx.$$

In the above, $x = (z,t) \in \mathbb{H}^n$ and $dx = dzdt$ is the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$, which, as we have already mentioned, is the Haar measure on the Heisenberg group. Following Lacey [13], by the term (p,q)-sparse form we mean the following:

$$\Lambda_{\mathcal{S},p,q}(f,g) = \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q}.$$

Theorem 1.3. *Assume $n \geq 2$ and fix $0 < \delta < \frac{1}{96}$. Let $1 < p, q < \infty$ be such that $(\frac{1}{p}, \frac{1}{q})$ belong to the interior of the triangle joining the points $(0,1),(1,0)$ and $(\frac{2n}{2n+1}, \frac{2n}{2n+1})$. Then for any pair of compactly supported bounded functions (f,g) there exists a (p,q)-sparse form such that $\langle M_{\mathbb{H}^n}^{lac} f, g \rangle \leq C \Lambda_{\mathcal{S},p,q}(f,g)$.*

We remark that as in the Euclidean case we can take any $\delta > 0$ in Theorem 1.3, in particular we can choose $\delta = 1/2$. In order to prove the main ingredients in our case, we make use of the idea of Cowling along with analytic interpolation. Once we have these two ingredients in our hand, the method of Lacey can be adapted to the Heisenberg group setting to obtain sparse domination. This has been already done in a joint work of the second author [1] where a different kind of spherical maximal function is studied on the Heisenberg group.

The spherical maximal function mentioned in the previous paragraph is associated to the spherical means taken over complex spheres $S_{r,0} = \{(z,0) \in \mathbb{H}^n : |z| = r\}$ which were initiated in the work of Nevo and Thangavelu. In [16], they showed that the corresponding maximal function is bounded on $L^p(\mathbb{H}^n)$ for $p > \frac{2n-1}{2n-2}$. Later Narayanan-Thangavelu in [15] and independently Müller-Seeger in [14] improved that result and proved that the maximal function is bounded on $L^p(\mathbb{H}^n)$ if and only if $p > \frac{2n}{2n-1}$. Recently, Bagchi et al. [1] have proved the analogue of Calderon's theorem for the associated lacunary spherical maximal function by obtaining a sparse bound as in Lacey [13].

Let μ_r stand for the normalised surface measure on $S_r = \{z \in \mathbb{C}^n : |z| = r\} \subset \mathbb{C}^n$ which can be considered as a measure on $\mathbb{H}^n$. The associated spherical maximal function is then given by $\sup_{r>0} |f * \mu_r(z,t)|$. Even though the measure μ_r is more singular than σ_r, its group Fourier transform is explicitly known, given in terms of the spherical functions associated to the action of the unitary group $U(n)$ on $\mathbb{H}^n$. These spherical functions are given by Laguerre functions for which very precise estimates are available in the literature. Thus the spectral method turns out to be useful in studying the maximal function $\sup_{r>0} |f * \mu_r(z,t)|$ as can be seen from all the three works [16], [15] and [1] (see also [21]). Having said this, we remark that it may be interesting to see if Calderon's approach can be adapted to study the lacunary maximal function associated to the spherical means $f * \mu_r$. In [1] the authors were interested in proving sparse bounds which immediately gives Calderon's theorem as a corollary. Consequently, they did not make any attempt to prove Calderon's theorem using spectral methods. However, the spectral theory of the spherical means cannot be avoided completely. Even in our case, we will see that information on the Fourier transform of σ_r plays an important role in establishing L^p improving and continuity properties of the spherical means $f * \sigma_r$.

We conclude the introduction by briefly describing the plan of the paper. After collecting some relevant results from the harmonic analysis on Heisenberg groups in Section 2 we establish the L^p improving properties of the spherical averages in Section 3. In Section 4 we study the continuity properties of the same. Finally in Section 5 we sketch the proof of the sparse domination (i.e. proof of Theorem 1.3) and deduce weighted and unweighted inequalities.

2. Preliminaries

2.1. Fourier transform on $\mathbb{H}^n$

In this section we collect some basic results from the harmonic analysis on Heisenberg groups that will play important roles in the study of the spherical maximal function. General references for this section are Folland [11] and Thangavelu [23]. The Heisenberg group $\mathbb{H}^n$ introduced in the previous section is a nilpotent Lie group which is non-commutative and yet with a very simple representation theory. For each non zero real number λ we have an infinite dimensional representation π_λ realised on the Hilbert space $L^2(\mathbb{R}^n)$. These are explicitly given by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$. These representations are known to be unitary and irreducible. Moreover, upto unitary equivalence these account for all the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ (see [11]). As the finite dimensional representations of $\mathbb{H}^n$ do not contribute to the Plancherel measure we will not describe them here.

The Fourier transform of a function $f \in L^1(\mathbb{H}^n)$ is the operator valued function obtained by integrating f against π_λ:

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

Note that $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. It is known that when $f \in L^1 \cap L^2(\mathbb{H}^n)$ its Fourier transform is actually a Hilbert-Schmidt operator and one has

$$\int_{\mathbb{H}^n} |f(z,t)|^2 dzdt = (2\pi)^{-n-1}\int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda.$$

The above allows us to extend the Fourier transform as a unitary operator between $L^2(\mathbb{H}^n)$ and the Hilbert space of Hilbert-Schmidt operator valued functions on $\mathbb{R}$ which are square integrable with respect to the Plancherel measure $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$.

2.2. The Heisenberg Lie algebra

We let $\mathfrak{h}_n$ stand for the Heisenberg Lie algebra consisting of left invariant vector fields on $\mathbb{H}^n$. A basis for $\mathfrak{h}_n$ is provided by the $2n + 1$ vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t}, \ Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}, \ j = 1, 2, ..., n$$

and $T = \frac{\partial}{\partial t}$. These correspond to certain one parameter subgroups of $\mathbb{H}^n$. Recall that given such a subgroup $\Gamma = \{\gamma(s) : s \in \mathbb{R}\}$ one associates the left invariant vector field

$$Xf(g) = \frac{d}{ds}\big|_{s=0}f(g\gamma(s)).$$

Associated to each such X we also have a right invariant vector field $\widetilde{X}$ defined by

$$\widetilde{X}f(g) = \frac{d}{ds}\big|_{s=0}f(\gamma(s)g).$$

It then follows that any $(x, y, t) \in \mathbb{H}^n$ can be written as $(x, y, t) = \exp(x \cdot X + y \cdot Y + tT)$ where $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_n)$ and exp is the exponential map taking $\mathfrak{h}_n$ into $\mathbb{H}^n$. From the above definition right invariant vector fields can be explicitly calculated

$$\tilde{X}_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t} \quad \tilde{Y}_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t};$$

they agree with the left invariant ones at the origin. The representations π_λ of $\mathbb{H}^n$ give rise to the derived representations $d\pi_\lambda$ of the Lie algebra $\mathfrak{h}_n$. These are given by

$$d\pi_\lambda(X)\varphi = \frac{d}{ds}\Big|_{s=0} \pi_\lambda(\exp sX)\varphi.$$

For reasonable functions f and any right invariant vector field X, it is known that

$$\pi_\lambda(\tilde{X} f) = d\pi_\lambda(\tilde{X})\pi_\lambda(f)$$

where $d\pi_\lambda$ is the derived representation of the Heisenberg lie algebra corresponding to π_λ. It is also well-known that $d\pi_\lambda(\tilde{X}_j) = i\lambda\xi_j$ and $d\pi_\lambda(\tilde{Y}_j) = \frac{\partial}{\partial\xi_j}$. Now writing $\tilde{Z}_j = \tilde{X}_j + i\tilde{Y}_j$, $\overline{\tilde{Z}_j} = \tilde{X}_j - i\tilde{Y}_j$ and using the above results we have

$$\pi_\lambda(\tilde{Z}_j f) = iA_j(\lambda)\pi_\lambda(f) \ and \ \pi_\lambda(\overline{\tilde{Z}_j} f) = iA_j^*(\lambda)\pi_\lambda(f)$$

where $A_j(\lambda)$ and $A_j^*(\lambda)$ are the annihilation and creation operators given by

$$A_j(\lambda) = \left(-\frac{\partial}{\partial\xi_j} + i\lambda\xi_j \right), \quad A_j^*(\lambda) = \left(\frac{\partial}{\partial\xi_j} + i\lambda\xi_j \right).$$

We make use of these relations in the proof of the continuity property of the spherical means, see Section 4.

2.3. The measure on the Koranyi sphere

Let $S_K = \{(z,t) \in \mathbb{H}^n : |(z,t)| = 1\}$ be the Koranyi sphere of radius 1. Then it is well known that there is a Radon measure σ on K which gives rise to the following polar decomposition of the Haar measure on the Heisenberg group:

$$\int_{\mathbb{H}^n} f(x)dx = \int_0^\infty \left(\int_{S_K} f(\delta_r y)d\sigma(y) \right) r^{Q-1} dr$$

where $Q = 2n+2$ is known as the homogeneous dimension of $\mathbb{H}^n$. For any $r > 0$ we define the measure $\sigma_r = \delta_r\sigma$ and note that it is supported on $K_r = \{(z,t) \in \mathbb{H}^n : |(z,t)| = r\}$. We also have another polar decomposition of the Haar measure given by

$$\int_{\mathbb{H}^n} f(z,t)dzdt = \int_{-\infty}^\infty \int_0^\infty \left(\int_{|\omega|=1} f(r\omega,t)d\mu(\omega) \right) r^{2n-1} dr dt$$

where μ is the surface measure on the unit sphere in $\mathbb{C}^n$. If we let μ_r stand for the surface measure on the sphere $S_r = \{(z, 0) \in \mathbb{H}^n : |z| = r\}$ and δ_t for the Dirac measure on $\mathbb{R}$ supported at the point t then the measure $\mu_{r,t} = \mu_r * \delta_t$ is supported on the set $S_{r,t} = \{(z, t) \in \mathbb{H}^n : |z| = r\}$. The measure σ_r can be expressed in terms of the measures $\mu_{r,t}$ as follows (see Faraut-Harzallah [9]):

$$\sigma_r = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \int\limits_{-\pi/2}^{\pi/2} \mu_{r\sqrt{\cos\theta},\frac{1}{4}r^2\sin\theta} \ (\cos\theta)^{n-1} d\theta.$$

2.4. Fourier transforms of radial measures

The unitary group $U(n)$ has a natural action on $\mathbb{H}^n$ given by $k.(z, t) = (k.z, t), k \in U(n)$ which induces an action on functions and measures on the Heisenberg group. We say that a function f (measure μ) is radial if it is invariant under the action of $U(n)$. It is well known that the subspace of radial functions in $L^1(\mathbb{H}^n)$ forms a commutative Banach algebra under convolution. So is the space of finite radial measures on $\mathbb{H}^n$. The Fourier transforms of such measures are functions of the Hermite operator $H(\lambda) = -\Delta + \lambda^2|x|^2$.

In fact, if $H(\lambda) = \sum_{k=0}^{\infty}(2k+n)|\lambda|P_k(\lambda)$ stands for the spectral decomposition of this operator, then for a radial measure μ we have

$$\hat{\mu}(\lambda) = \sum_{k=0}^{\infty} R_k(\lambda, \mu)P_k(\lambda).$$

More explicitly, $P_k(\lambda)$ stands for the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the kth eigenspace spanned by scaled Hermite functions Φ_α^λ for $|\alpha| = k$. The coefficients $R_k(\lambda, \mu)$ are explicitly given by

$$R_k(\lambda, \mu) = \frac{k!(n-1)!}{(k+n-1)!} \int\limits_{\mathbb{H}^n} e^{i\lambda t}\varphi_k^\lambda(z)d\mu(z, t).$$

In the above formula, φ_k^λ are Laguerre functions of type $(n-1)$:

$$\varphi_k^\lambda(z) = L_k^{n-1}(\frac{1}{2}|\lambda||z|^2)e^{-\frac{1}{4}|\lambda||z|^2}$$

where L_k^{n-1} denotes the Laguerre polynomial of type $(n-1)$. We refer the reader to [22] for the definition and properties. In particular, for the measures σ_r we have

$$R_k(\lambda, \sigma_r) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \frac{k!(n-1)!}{(k+n-1)!} \int\limits_{-\pi/2}^{\pi/2} \varphi_k^\lambda(r\sqrt{\cos\theta})e^{i\lambda\frac{1}{4}r^2\sin\theta} \ (\cos\theta)^{n-1}d\theta.$$

Though the above integral cannot be evaluated in a closed form, it can be used to study the maximal function associated to the spherical means $f * \sigma_r$. See the work of Fischer [10] where the spherical maximal function in a slightly general context has been studied.

3. L^p -improving property of the spherical means

In this section we prove certain $L^p - L^q$ bounds for the spherical means operator A_r. In order to prove the required estimates we embed A_r into an analytic family of operators and then appeal to Stein's analytic interpolation theorem. First we obtain the following representation of the measure σ_r as a superposition of certain operators which can be handled easily. In what follows we let $P_t(x) = c_Q\, t\, (t^2 + |x|^2)^{-\frac{Q+1}{2}}$ and $I_\gamma(x) := C(Q,\gamma)|x|^{-Q+i\gamma}$, $x \in \mathbb{H}^n, t > 0, \gamma \in \mathbb{R}$ where c_Q is defined by the condition

$$c_Q \int_{\mathbb{H}^n} (1 + |x|^2)^{-\frac{Q+1}{2}} dx = 1$$

and $C(Q,\gamma) = c_n \Gamma\left(\frac{Q-i\gamma}{4}\right)^2 / \Gamma\left(\frac{i\gamma}{2}\right)$. In the proof of the next theorem which gives a representation of σ_t in terms of P_t and I_γ we make use of the following simple lemma.

Lemma 3.1. *Let $F(t) = c_Q(1+e^{2t})^{-\frac{Q+1}{2}} e^{Qt}$ and let $\hat{F}(\gamma)$ stands for the Euclidean Fourier transform of F. Then*

$$\sqrt{2\pi}\hat{F}(\gamma) = \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{1}{2}\right)}.$$

Proof. By the definition of the Fourier transform

$$\hat{F}(\gamma) = \frac{c_Q}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 + e^{2t})^{-\frac{Q+1}{2}} e^{Qt} e^{-it\gamma} dt$$

which after the change of variables $e^t = r$ leads to

$$\sqrt{2\pi}\hat{F}(\gamma) = c_Q \int_0^{\infty} (1 + r^2)^{-\frac{Q+1}{2}} r^{Q-1-i\gamma} dr.$$

Another change of variables $(1 - t) = (1 + r^2)^{-1}$ converts the above integral into the Beta integral

$$\frac{1}{2} \int_0^1 (1 - t)^{\frac{1+i\gamma}{2}-1} t^{\frac{Q-i\gamma}{2}-1} dt = \frac{1}{2} \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q+1}{2}\right)}.$$

Consequently we obtain

$$\sqrt{2\pi}\hat{F}(\gamma) = \frac{1}{2}c_Q \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q+1}{2}\right)}.$$

Observe that, by the definition of c_Q we have

$$1 = \sqrt{2\pi}\hat{F}(0) = \frac{1}{2}c_Q \frac{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{Q+1}{2}\right)}$$

which leads to the conclusion $\sqrt{2\pi}\hat{F}(\gamma) = \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{1}{2}\right)}$ completing the proof. □

The following result is the analogue of a theorem by Cowling and Mauceri [7] proved in the context of $\mathbb{R}^n$. We provide the details in the Heisenberg setting for the convenience of the reader. We define another constant $d(Q,\gamma)$ by the requirement

$$d(Q,\gamma)C(Q,\gamma) = 1 - \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{3.1}$$

Theorem 3.2. *For $t > 0$ the following representation holds in the sense of distributions:*

$$\sigma_t = P_t + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d(Q,\gamma)t^{-i\gamma}I_\gamma \; d\gamma.$$

Proof. Let $u \in C_c^\infty(1-\delta < |x| < 1+\delta)$. Then by using polar decomposition of the Haar measure,

$$\int\limits_{\mathbb{H}^n} u(x)|x|^{-Q+i\gamma}dx = \int\limits_{0}^{\infty}\int\limits_{S_K} u(\delta_r\omega)r^{i\gamma-1}d\sigma(\omega)dr.$$

By defining $\bar{u}(r) = \int_{S_K} u(\delta_r\omega)d\sigma(\omega)$ we rewrite the above as

$$\int\limits_{\mathbb{H}^n} u(x)|x|^{-Q+i\gamma}dx = \int\limits_{0}^{\infty} \bar{u}(r)r^{i\gamma-1}dr$$

But by a change of variables we have

$$\int\limits_{0}^{\infty} \bar{u}(r)r^{i\gamma-1}dr = \int\limits_{-\infty}^{\infty} \bar{u}(e^t)e^{i\gamma t}dt.$$

Hence by the Fourier inversion formula we obtain

$$\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{H}^n}u(x)|x|^{-Q+i\gamma}dx\Big)d\gamma = 2\pi\bar{u}(1) = 2\pi\langle\sigma,u\rangle.\tag{3.2}$$

We now consider the following equation:

$$\langle\int\limits_{-\infty}^{\infty}d(Q,\gamma)I_\gamma d\gamma, u\rangle = \int\limits_{\mathbb{H}^n}\Big(\int\limits_{-\infty}^{\infty}d(Q,\gamma)C(Q,\gamma)|x|^{-Q+i\gamma}d\gamma\Big)u(x)dx.$$

Now changing the order of the integration and using (3.1) and (3.2) we get

$$\langle\int\limits_{-\infty}^{\infty}d(Q,\gamma)I_\gamma d\gamma, u\rangle = 2\pi\langle\sigma,u\rangle - \int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{H}^n}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}u(x)|x|^{-Q+i\gamma}dx\Big)d\gamma.\tag{3.3}$$

Now we simplify the second integral in the above equation. Using polar decomposition we have

$$\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{H}^n}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}u(x)|x|^{-Q+i\gamma}dxd\gamma$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{S_K}\int\limits_{0}^{\infty}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}r^{i\gamma-1}u(\delta_r\omega)drd\sigma(\omega)d\gamma$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{S_K}\int\limits_{-\infty}^{\infty}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}e^{i\gamma t}u(\delta_{e^t}\omega)dtd\sigma(\omega)d\gamma.$$

By Fubini's theorem, changing the order of the integration we obtain

$$\int\limits_{-\infty}^{\infty}\int\limits_{\mathbb{H}^n}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}u(x)|x|^{-Q+i\gamma}dxd\gamma$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{S_K}\Big(\int\limits_{-\infty}^{\infty}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}e^{i\gamma t}d\gamma\Big)u(\delta_{e^t}\omega)d\sigma(\omega)dt.$$

We now make use of Lemma 3.1 and obtain

$$\int\limits_{-\infty}^{\infty}\int\limits_{S_K}\Big(\int\limits_{-\infty}^{\infty}\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}e^{i\gamma t}d\gamma\Big)u(\delta_{e^t}\omega)d\sigma(\omega)dt$$

$$= \sqrt{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{S_K}\Big(\int\limits_{-\infty}^{\infty}\hat{F}(\gamma)e^{it\gamma}d\gamma\Big)u(\delta_{e^t}\omega)d\sigma(\omega)dt$$

$$= (2\pi) \int_{-\infty}^{\infty} \int_{S_K} F(t)u(\delta_{e^t}\omega)d\sigma(\omega)dt$$

$$= (2\pi)c_Q \int_{S_K} \int_0^{\infty} (1+r^2)^{-\frac{Q+1}{2}} u(\delta_r\omega)r^{Q-1}drd\sigma(\omega).$$

As the last integral in the above chain of equations is nothing but $(2\pi)\int_{\mathbb{H}^n} u(x)P_1(x)dx$ we have proved

$$\int_{-\infty}^{\infty} \int_{\mathbb{H}^n} \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)\Gamma\left(\frac{1+i\gamma}{2}\right)}{\Gamma\left(\frac{Q}{2}\right)\Gamma\left(\frac{1}{2}\right)} u(x)|x|^{-Q+i\gamma}dxd\gamma = (2\pi)\langle P_1, u\rangle. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain the following equality which holds in the sense of distributions:

$$\sigma = P_1 + (2\pi)^{-1} \int_{-\infty}^{\infty} d(Q,\gamma)I_\gamma d\gamma.$$

As σ_t is obtained from σ by dilation, the theorem is proved. $\square$

We would like to embed the spherical means A_r into an analytic family of operators. As in the Euclidean case, this is achieved by observing that the distributions given by the functions

$$\phi_{r,\alpha}(x) = 2\frac{r^{-Q}}{\Gamma(\alpha)}\left(1 - \frac{|x|^2}{r^2}\right)_+^{\alpha-1}, \quad \Re(\alpha) > 0$$

converge to σ_r as $\alpha \to 0$. In the Euclidean case the Fourier transform of $\phi_{r,\alpha}(x)$ is known explicitly, given in terms of Bessel functions, which allows immediate extension as a homomorphic family of distributions. In the case of the Heisenberg group we do not have a useful formula for the (group) Fourier transform of $\phi_{r,\alpha}$. Hence we make use of the following representation similar to the one proved for σ_r in the preceding theorem in holomorphically extending the operator $f * \phi_{r,\alpha}$.

Proposition 3.3. *Let $r > 0$, $Re(\alpha) > 0$. Then for any Schwartz function f on $\mathbb{H}^n$, we have*

$$f * \phi_{r,\alpha}(x) = 2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^r \left(1 - \frac{t^2}{r^2}\right)^{\alpha-1} t^{Q-1}f * P_t(x)dt$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} d(Q,\gamma)r^{-i\gamma} \frac{\Gamma\left(\frac{Q-i\gamma}{2}\right)}{\Gamma\left(\alpha + \frac{Q-i\gamma}{2}\right)} f * I_\gamma(x)d\gamma.$$

Proof. By definition of convolution on Heisenberg group we have

$$f * \phi_{r,\alpha}(x) = \int_{\mathbb{H}^n} f(x.y^{-1})\phi_{r,\alpha}(y)dy.$$

As $\phi_{r,\alpha}$ is radial, integrating in polar coordinates we get

$$f * \phi_{r,\alpha}(x) = 2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^\infty \Big(\int_{S_K} f(x.\delta_t\omega^{-1})\big(1 - \frac{t^2}{r^2}\big)_+^{\alpha-1} d\sigma(\omega)\Big) t^{Q-1} \, dt$$

$$= 2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^r \big(1 - \frac{t^2}{r^2}\big)^{\alpha-1} t^{Q-1} f * \sigma_t(x)dt.$$

Making use of the representation

$$f * \sigma_t = f * P_t + \frac{1}{2\pi} \int_{-\infty}^\infty d(Q,\gamma)t^{-i\gamma} f * I_\gamma d\gamma$$

proved in the previous theorem we get $f * \phi_{r,\alpha} = S_{r,\alpha}f + T_{r,\alpha}f$ where

$$S_{r,\alpha}f(x) = 2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^r \big(1 - \frac{t^2}{r^2}\big)^{\alpha-1} t^{Q-1} f * P_t(x)dt$$

and

$$T_{r,\alpha}f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \Big(2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^r \big(1 - \frac{t^2}{r^2}\big)^{\alpha-1} t^{Q-1} t^{-i\gamma} dt\Big) d(Q,\gamma) f * I_\gamma(x)d\gamma.$$

The inner integral can be explicitly calculated:

$$r^{-Q} \int_0^r \big(1 - \frac{t^2}{r^2}\big)^{\alpha-1} t^{Q-1} t^{-i\gamma} dt = r^{-i\gamma} \int_0^1 (1 - t^2)^{\alpha-1} t^{Q-1-i\gamma} dt$$

which reduces to a beta integral and yields

$$2\frac{r^{-Q}}{\Gamma(\alpha)} \int_0^r \big(1 - \frac{t^2}{r^2}\big)^{\alpha-1} t^{Q-1} t^{-i\gamma} dt = \frac{\Gamma(\alpha)\Gamma(\frac{Q-i\gamma}{2})}{\Gamma(\alpha + \frac{Q-i\gamma}{2})}.$$

Consequently, we obtain the representation

$$T_{r,\alpha}f := \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d(Q,\gamma) r^{-i\gamma} \frac{\Gamma(\frac{Q-i\gamma}{2})}{\Gamma(\alpha + \frac{Q-i\gamma}{2})} f * I_\gamma(x) d\gamma,$$

proving the theorem. $\square$

If we define $A_{r,\alpha}f = f * \phi_{r,\alpha}$, then by the above proposition we have $A_{r,\alpha}f = S_{r,\alpha}f + T_{r,\alpha}f$ where the above holds under the assumption that $\Re(\alpha) > 0$. But both $S_{r,\alpha}$ and $T_{r,\alpha}$ have analytic continuation to a larger domain of α. Indeed, $T_{r,\alpha}$ can be extended to the whole of $\mathbb{C}$ as an entire function and $S_{r,\alpha}$ extends holomorphically to the region $\Re(\alpha) > -n$. Thus $A_{r,\alpha}$ is an analytic family of operators and when α goes to 0 we recover $f * \sigma_r$. We remark that the L^p-improving property of this generalised spherical mean $A_{r,\alpha}f$ on the Euclidean space has been studied by Strichartz [20].

In order to study the L^p improving property of the spherical mean value operator $f \to f * \sigma_r$ we use analytic interpolation. It is enough to prove an L^p-improving property for the operator $T_{r,0}$ by studying the family $T_{r,\alpha}$. We shall show that for $\alpha = 1 + i\beta$, the operator $T_{r,\alpha}$ is bounded from $L^{1+\delta}$ to L^∞ for any $\delta > 0$ and for some negative value of $\Re(\alpha)$, it is bounded on $L^2(\mathbb{H}^n)$. By a dilation argument, we can assume that $r = 1$ and hence we deal with $T_\alpha := T_{1,\alpha}$. To handle the L^2 boundedness, we need the following Fourier transform computation.

Proposition 3.4. *For $\lambda \neq 0$, the Heisenberg group Fourier transform of the distribution $|.|^{-Q+i\gamma}$ is given by*

$$\widehat{|.|^{-Q+i\gamma}}(\lambda) = (2\pi)^{n+1} |\lambda|^{-i\gamma/2} \frac{\Gamma(\frac{i\gamma}{2})}{\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})^2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{2k+n}{2} + \frac{2-i\gamma}{4})}{\Gamma(\frac{2k+n}{2} + \frac{2+i\gamma}{4})} P_k(\lambda).$$

This has been proved in the work of Cowling and Haagerup [6]. From the above proposition it is now easy to prove the following:

Proposition 3.5. *Assume that $n \geq 1$. Then for any $\alpha \in \mathbb{C}$ with $Re(\alpha) > -n + \frac{1}{2}$ we have*

$$\|T_\alpha f\|_2 \leq C(\Im(\alpha)) \|f\|_2$$

where $C(\Im(\alpha))$ has admissible growth.

Proof. Note that if we write $a(Q,\gamma) := d(Q,\gamma) C(Q,\gamma)$ we have

$$T_\alpha f(x) = \int\limits_{-\infty}^{\infty} a(Q,\gamma) r^{-i\gamma} \frac{\Gamma(\frac{Q-i\gamma}{2})}{\Gamma(\alpha + \frac{Q-i\gamma}{2})} f * |.|^{-Q+i\gamma}(x) d\gamma.$$

It is therefore enough to show that

$$\int_{-\infty}^{\infty} |a(Q,\gamma)| \frac{|\Gamma(\frac{Q-i\gamma}{2})|}{|\Gamma(\alpha + \frac{Q-i\gamma}{2})|} b(Q,\gamma)\, d\gamma \le C(\Im(\alpha))$$

where $b(Q,\gamma)$ is the norm of the operator $f \to f * |\cdot|^{-Q+i\gamma}$ on $L^2(\mathbb{H}^n)$ so that we have the inequality

$$\|f * |\cdot|^{-Q+i\gamma}\|_2 \le b(Q,\gamma)\|f\|_2.$$

In view of Plancherel theorem for the group Fourier transform on $\mathbb{H}^n$ we have the estimate

$$b(Q,\gamma) \le C \sup_{\lambda} \||\widehat{.|^{-Q+i\gamma}}(\lambda)\|.$$

Using the computation in the previous proposition, we have

$$\||\widehat{.|^{-Q+i\gamma}}(\lambda)\| \le C \frac{|\Gamma(\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})|^2} \sup_k \frac{|\Gamma(\frac{2k+n}{2} + \frac{2-i\gamma}{4})|}{|\Gamma(\frac{2k+n}{2} + \frac{2+i\gamma}{4})|} = C \frac{|\Gamma(\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})|^2}.$$

Thus we only need to show that

$$\int_{-\infty}^{\infty} |a(Q,\gamma)| \frac{|\Gamma(\frac{Q-i\gamma}{2})|}{|\Gamma(\alpha + \frac{Q-i\gamma}{2})|} \frac{|\Gamma(\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})|^2}\, d\gamma \le C(\Im(\alpha)).$$

In order to prove the above, we first recall that

$$a(Q,\gamma) = \left(1 - \frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}\right)$$

and hence $a(Q,\gamma)$ has a zero at $\gamma = 0$. Consequently,

$$\int_{-1}^{1} \frac{|a(Q,\gamma)|}{|\gamma|} \frac{|\Gamma(\frac{Q-i\gamma}{2})|}{|\Gamma(\alpha + \frac{Q-i\gamma}{2})|} \frac{|\Gamma(1+\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})|^2}\, d\gamma \le C_1(\Im(\alpha))$$

as long as $\Re(\alpha) > -n - 1$. To prove the integrability away from the origin we make use of the following asymptotic formula for the gamma function: for $|\nu|$ large

$$\Gamma(\mu + i\nu) \sim \sqrt{2\pi}|\nu|^{\mu-1/2} e^{-\frac{1}{2}\pi|\nu|}.$$

So using this formula, a simple calculation shows that for $|\gamma| \ge 1$

$$\frac{|a(Q,\gamma)|}{|\gamma|} \frac{|\Gamma(\frac{Q-i\gamma}{2})|}{|\Gamma(\alpha + \frac{Q-i\gamma}{2})|} \frac{|\Gamma(1+\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4} - \frac{i\gamma}{4})|^2} \le C_2(\Im(\alpha))|\gamma|^{-\Re(\alpha)-n-1/2}.$$

Therefore, it follows that

$$\int\limits_{|\gamma|\geq 1} \frac{|a(Q,\gamma)|}{|\gamma|} \frac{|\Gamma(\frac{Q-i\gamma}{2})|}{|\Gamma(\alpha+\frac{Q-i\gamma}{2})|} \frac{|\Gamma(1+\frac{i\gamma}{2})|}{|\Gamma(\frac{Q}{4}-\frac{i\gamma}{4})|^2} \, d\gamma \leq C_2(\Im(\alpha))$$

for all $\Re(\alpha) > -n+1/2$. This completes the proof of the proposition. $\quad\square$

Proposition 3.6. *For any $\beta \in \mathbb{R}$ and $p > 1$ we have,*

$$\|T_{1+i\beta}f\|_\infty \leq C(\beta)\|f\|_p.$$

Proof. For any $p > 1$, to prove $L^p \to L^\infty$ estimate, first note that in view of the Proposition 3.3, for any $\beta \in \mathbb{R}$ we have

$$T_{1+i\beta}f(x) = f * \phi_{1,1+i\beta}(x) - \frac{1}{\Gamma(1+i\beta)}\int\limits_0^1 \left(1-t^2\right)^{i\beta} t^{Q-1}f * P_t(x)dt. \qquad (3.5)$$

So for any $x \in \mathbb{H}^n$ we have

$$|T_{1+i\beta}f(x)| \leq |f * \phi_{1,1+i\beta}(x)| + \frac{1}{|\Gamma(1+i\beta)|}\int\limits_0^1 t^{Q-1}|f * P_t(x)|dt. \qquad (3.6)$$

Now see that

$$|f * \phi_{1,1+i\beta}(x)| \leq \frac{1}{|\Gamma(1+i\beta)|}|f| * \chi_{B_1}(x) \leq C(\beta)\|f\|_p.$$

As we have $P_t(x) = t^{-Q}P_1(\delta_t^{-1}x)$ it follows that for any $p > 1$,

$$|f * P_t(x)| \leq t^{-Q}\|f\|_p\Big(\int\limits_{\mathbb{H}^n} P_1(\delta_t^{-1}x)^{p'} dx\Big)^{1/p'} \leq Ct^{-Q/p}\|f\|_p.$$
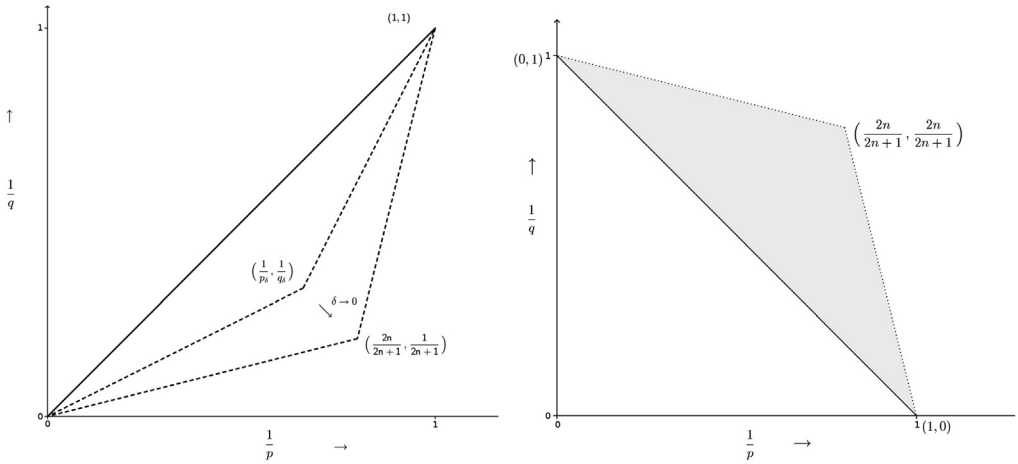
Consequently, the integral in (1.2) is bounded by $\|f\|_p$. Finally, these two estimates together with (3.6) we get

$$\|T_{1+i\beta}f\|_\infty \leq C(\beta)\|f\|_p. \quad\square$$

By using the above two propositions and an analytic interpolation argument, we now prove the following result.

Proposition 3.7. *For any $0 < \delta < 1$, we have $T_0 : L^{p_\delta}(\mathbb{H}^n) \to L^{q_\delta}(\mathbb{H}^n)$ where $p_\delta = (1+\delta)\frac{2n+1-\delta}{2n}$ and $q_\delta = (2n+1-\delta)$.*

Fig. 1. Triangle $\mathbf{L}'_n$, on the left side, shows the region for $L^p - L^q$ estimates for A_1. The dual triangle $\mathbf{L}_n$ is on the right.

Proof. Given $\delta > 0$ we consider the holomorphic family of operators $T_{L(z)}$, where $L(z) = (-n + \frac{1+\delta}{2})(1-z) + z$. In view of the Propositions 3.5 and 3.6, Stein's interpolation theorem gives

$$T_{L(u)} : L^{p(u)}(\mathbb{H}^n) \to L^{q(u)}(\mathbb{H}^n)$$

where $\frac{1}{p(u)} = \frac{1-u}{2} + \frac{u}{1+\delta}$ and $\frac{1}{q(u)} = \frac{1}{2}(1-u)$. Solving for $L(u) = 0$ we get $u = \frac{2n-1-\delta}{2n+1-\delta}$ and simplifying we get $p(u) = p_\delta = (1+\delta)\frac{2n+1-\delta}{2n}$ and $q(u) = q_\delta = (2n+1-\delta)$. □

The following result which follows from the above end point estimates by means of analytic interpolation describes the L^p improving properties of the spherical mean operator A_r.

Theorem 3.8. *Assume that $n \geq 1$. Then we have*

$$\|A_r f\|_q \leq C r^{Q(\frac{1}{q} - \frac{1}{p})} \|f\|_p$$

whenever $(\frac{1}{p}, \frac{1}{q})$ lies in the interior of the triangle joining the points $(0,0), (1,1)$, and $(\frac{2n}{2n+1}, \frac{1}{2n+1})$ as well as the straight line joining the points $(0,0), (1,1)$.

Proof. An easy calculation shows that $\delta_r^{-1} A_1 \delta_r = A_r$ and hence we can assume that $r = 1$. With p_δ and q_δ as in the proof of Theorem 3.7, we first show that

$$A_r : L^{p_\delta}(\mathbb{H}^n) \to L^{q_\delta}(\mathbb{H}^n). \tag{3.7}$$

Recall that from the Proposition 3.3 we have

$$f * \phi_{1,\alpha}(x) = \frac{2}{\Gamma(\alpha)} \int_0^1 (1-t^2)^{\alpha-1} t^{Q-1} f * P_t(x) dt + T_\alpha f(x).$$

Note that A_1 is recovered from $f * \phi_{1,\alpha}$ by letting α tend to 0. As the first term converges to $f * P_1(x)$ we have the equation $A_1 f(x) = f * P_1(x) + T_0 f(x)$. Since $P_1 \in L^p$ for any $p \geq 1$ in view of Young's inequality we get $\|f * P_1\|_{q_\delta} \leq C\|f\|_{p_\delta}$. Hence using Proposition 3.7 we see that the same is true of A_1. Since A_1 is bounded on $L^p(\mathbb{H}^n)$ for any $1 \leq p \leq \infty$ Marcinkiewicz interpolation theorem shows $\|A_1 f\|_q \leq \|f\|_p$ whenever $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle Δ_δ with vertices at $(0,0), (1,1)$ and $(\frac{2n}{(1+\delta)(2n+1-\delta)}, \frac{1}{(2n+1-\delta)})$. The theorem is proved by letting δ go to 0 (Fig. 1). $\square$

Remark. It is possible to use the explicit formula for the measure σ_r stated in the preliminaries to get $L^p - L^q$ estimates for the spherical means. However, we only get the estimates for $(\frac{1}{p}, \frac{1}{q})$ coming from a smaller triangle.

4. The continuity property

Our aim in this section is to prove the following theorem which is known as the continuity property of the spherical mean operator A_r. For $\mathbf{a} \in \mathbb{H}^n$ let $\tau_{\mathbf{a}} f(x) = f(x\mathbf{a}^{-1})$ be the right translation of f by $\mathbf{a}$. We prove:

Theorem 4.1. *Assume that $n \geq 2$. Then for any $\mathbf{a} \in \mathbb{H}^n$ with $|\mathbf{a}| \leq r$ and for $(1/p, 1/q)$ lying in the interior of the triangle joining the points $(0,0), (1,1)$ and $(\frac{2n}{2n+1}, \frac{1}{2n+1})$, we have*

$$\|A_r - A_r \tau_{\mathbf{a}}\|_{L^p \to L^q} \leq C r^{Q(1/q-1/p)} |\delta_r^{-1} \mathbf{a}|^\eta$$

for some $0 < \eta < 1$.

In view of the relation $A_r - A_r \tau_{\mathbf{a}} = \delta_r^{-1}(A_1 - A_1 \tau_{\delta_r^{-1} \mathbf{a}}) \delta_r$ we can assume $r = 1$ without loss of generality. Also in view of Riesz-Thorin interpolation theorem it is enough to prove the case $p = 2$. It is therefore natural to use the Plancherel theorem for the Fourier transform. Since we do not know the group Fourier transform of σ_r explicitly, we proceed as in the case of the proof of the L^p improving property by making use of the integral representation for the measure. For the purpose of proving continuity it is convenient to work with the following family rather than $A_{r,\alpha}$ used earlier.

For $\Re(\alpha) > 0$ and $r > 0$ we define

$$\Phi_{r,\alpha}(x) := 4 \frac{r^{-Q}}{\Gamma(\alpha)} \left(1 - \frac{|x|^4}{r^4}\right)_+^{\alpha-1}, \quad x \in \mathbb{H}^n$$

which converges to σ_r as α goes to 0. Given $\mathbf{a} \in \mathbb{H}^n$ consider the equation

$$\tau_{\mathbf{a}}f * \Phi_{r,\alpha}(x) - f * \Phi_{r,\alpha}(x) = \int_{\mathbb{H}^n} f(xy^{-1}\mathbf{a}^{-1})\Phi_{r,\alpha}(y)dy - \int_{\mathbb{H}^n} f(xy^{-1})\Phi_{r,\alpha}(y)dy.$$

By making a change of variables and using fundamental theorem of calculus we can rewrite the above as

$$\int_{\mathbb{H}^n} f(xy^{-1})\Phi_{r,\alpha}((\mathbf{a})y)dy - \int_{\mathbb{H}^n} f(xy^{-1})\Phi_{r,\alpha}(y)dy = \int_{\mathbb{H}^n} \Big(\int_0^1 \frac{d}{ds}\Phi_{r,\alpha}((\delta_s\mathbf{a})y)ds\Big)dy.$$

We are thus led to calculate the derivative of $\Phi_{r,\alpha}((\delta_s\mathbf{a})y)$ which is given in the following lemma. Let

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j\frac{\partial}{\partial t}, \ Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j\frac{\partial}{\partial t}, \ j = 1, 2, ..., n$$

and $T = \frac{\partial}{\partial t}$ be the $(2n+1)$ left invariant vector fields giving an orthonormal basis for the Heisenberg Lie algebra $\mathfrak{h}_n$. It then follows that any $(x, y, t) \in \mathbb{H}^n$ can be written as $(x, y, t) = \exp(x \cdot X + y \cdot Y + tT)$ where $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ and exp is the exponential map taking $\mathfrak{h}_n$ into $\mathbb{H}^n$. We can then check that

$$\frac{d}{ds}\Big|_{s=0}\varphi(\exp(\delta_s(a,b,c)(x,y,t))) = (a \cdot \widetilde{X} + b \cdot \widetilde{Y} + cT)\varphi(x,y,t)$$

where $\widetilde{X} = (\widetilde{X}_1, ..., \widetilde{X}_n)$ and $\widetilde{Y} = (\widetilde{Y}_1, ..., \widetilde{Y}_n)$, with

$$\widetilde{X}_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \ \widetilde{Y}_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}.$$

We remark that these are the right invariant vector fields agreeing with the left invariant ones at the origin.

Lemma 4.2. *Let* $\mathbf{a} = (a, b, c) \in \mathbb{H}^n$ *be fixed. Then for any* $r > 0$ *and* $\Re(\alpha) > 1$ *we have*

$$\tau_{\mathbf{a}}f * \Phi_{r,\alpha}(x) - f * \Phi_{r,\alpha}(x) = \int_0^1 \Big(\int_{\mathbb{H}^n} \tau_{\delta_s\mathbf{a}}f(xy^{-1})(a \cdot \widetilde{X} + b \cdot \widetilde{Y} + 2csT)\Phi_{r,\alpha}(y)dy\Big)ds$$

Proof. In view of the foregoing calculations we have

$$\tau_{\mathbf{a}}f * \Phi_{r,\alpha}(x) - f * \Phi_{r,\alpha}(x) = \int_{\mathbb{H}^n} f(xy^{-1})\Big(\int_0^1 \frac{d}{ds}\Phi_{r,\alpha}((\delta_s\mathbf{a})y)ds\Big)dy.$$

We note that

$$\frac{d}{ds}\Phi_{r,\alpha}((\delta_s\mathbf{a})y) = \frac{d}{du}\Big|_{u=0}\Phi_{r,\alpha}((\delta_{s+u}\mathbf{a})y),$$

$$(\delta_{s+u}\mathbf{a})y = (ua, ub, (u^2 + 2su)c)(sa, sb, s^2c)y$$

which can also be written in the form

$$\exp\left(u(a \cdot X + b \cdot Y) + (u^2 + 2su)cT\right)(\delta_s\mathbf{a})y.$$

Therefore, taking derivative with respect to u at 0 and making use of the calculations done before the lemma we get the result. $\quad\square$

In order to obtain a usable expression for $(a \cdot \widetilde{X} + b \cdot \widetilde{Y} + 2csT)\Phi_{r,\alpha}(y)$ we make use of the following simple lemma.

Lemma 4.3. *Let φ_0 and ψ_0 be smooth functions on $\mathbb{R}$ and let $\varphi(x) = \varphi_0(|x|^4)$ and $\psi(x) = \psi_0(|x|^4)$. Then for any vector field Z on $\mathbb{H}^n$ we have $Z\varphi(x) = \frac{\varphi_0'(|x|^4)}{\psi_0'(|x|^4)}Z\psi(x)$. In particular,*

$$Z(|x|I_\gamma(x)) = \frac{1}{4}(-Q + 1 + i\gamma)|x|^{-3}Z(|x|^4)I_\gamma(x).$$

Taking $\varphi_0(u) = \frac{4r^{-Q}}{\Gamma(\alpha-1)}(1 - \frac{u^4}{r^4})_+^{\alpha-1}$ and $\psi_0(u) = u$ in the above lemma we get the following.

Lemma 4.4. *Assuming that $\Re(\alpha) > 1$ we have*

$$(a \cdot \widetilde{X} + b \cdot \widetilde{Y} + 2csT)\Phi_{r,\alpha}(y) = cr^{-3}\left(2cs\psi_0(x) + \sum_{j=1}^{n}(a_j\varphi_j(x) + b_j\psi_j(x))\right)\Phi_{r,\alpha-1}(x)$$

where φ_j and ψ_j are homogeneous of degree three whereas ψ_0 is of degree two.

Indeed, we only have to take

$$\varphi_j(x) = \widetilde{X}_j(|x|^4), \ \psi_j(x) = \widetilde{Y}_j(|x|^4), \ \psi_0(x) = T(|x|^4)$$

and note that as $\widetilde{X}_j$ and $\widetilde{Y}_j$ are homogeneous of degree one and T is homogeneous of degree two these functions have the right homogeneity stated in the lemma.

We want to estimate the norm of $A_r\tau_\mathbf{a}f - A_rf$ which will be obtained as the limit of $\tau_\mathbf{a}f * \Phi_{r,\alpha}(x) - f * \Phi_{r,\alpha}(x)$ as α goes to 0. Since the expression in the above lemma is proved under the assumption that $\Re(\alpha) > 1$ we need to holomorphically extend the integral

$$\int_{\mathbb{H}^n} g(xy^{-1})\Phi_{r,\alpha-1}(y)(a \cdot \widetilde{X} + b \cdot \widetilde{Y} + 2csT)(|y|^4) \, dy$$

for smaller values of α before passing to the limit. Assuming $r = 1$ we consider the operator

$$\int_{\mathbb{H}^n} f(xy^{-1})\Phi_{1,\alpha-1}(y)\varphi_j(y)dy.$$

We now obtain the following representation for the above integral which allows holomorphic extension.

Proposition 4.5. *For any* α *with* $\Re(\alpha) > 1$ *we have*

$$\int_{\mathbb{H}^n} f(xy^{-1})\Phi_{1,\alpha-1}(y)\varphi_j(y)dy = S_{\alpha,j}f(x) + T_{\alpha,j}f(x).$$

Here $T_{\alpha,j}$ *is entire and* $S_{\alpha,j}$ *has a holomorphic extension to the region* $\Re(\alpha) > -\frac{Q-1}{4}+1$.

Proof. Integrating in polar coordinates we have

$$\int_{\mathbb{H}^n} f(xy^{-1})\Phi_{1,\alpha-1}(y)\varphi_j(y)dy$$

$$= \frac{4}{\Gamma(\alpha-1)} \int_0^1 \int_{S_K} f(x.\delta_u\omega^{-1})(1-u^4)^{\alpha-2}u^{Q+2}\varphi_j(\omega)d\sigma(\omega)du$$

$$= \frac{4}{\Gamma(\alpha-1)} \int_0^1 (1-u^4)^{\alpha-2}u^{Q+2}f*(\varphi_j\sigma)_u(x)du$$

Making use of the representation of the representation of σ_u given in Theorem 3.2, we have

$$f*(\varphi_j\sigma)_u(x) = f*P_{u,j}(x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} d(Q,\gamma)u^{-i\gamma}f*I_{\gamma,j}(x)d\gamma$$

where $P_{u,j}(x) = P_u(x)\varphi_j(x)|x|^{-3}$ and $I_{\gamma,j}(x) = I_\gamma(x)\varphi_j(x)|x|^{-3}$. Using this expression we have

$$\int_{\mathbb{H}^n} f(xy^{-1})\Phi_{1,\alpha-1}(y)\varphi_j(y)dy = S_{\alpha,j}f(x) + T_{\alpha,j}f(x)$$

where $S_{\alpha,j}f$ is given by

$$S_{\alpha,j}f(x) = \frac{4}{\Gamma(\alpha-1)} \int\limits_0^1 \left(1-u^4\right)^{\alpha-2} u^{Q+2} f * P_{u,j}(x) du$$

and $T_{\alpha,j}f$ by the integral

$$T_{\alpha,j}f(x) = \frac{4}{\Gamma(\alpha-1)} \int\limits_0^1 \left(1-u^4\right)^{\alpha-2} u^{Q+2} \left(\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d(Q,\gamma)u^{-i\gamma} f * I_{\gamma,j}(x) d\gamma\right) du.$$

Holomorphic extension of $S_{\alpha,j}f$ is a routine matter: we integrate by parts by writing

$$S_{\alpha,j}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^1 u^{Q-1} f * P_{u,j}(x) \frac{d}{du}\left(1-u^4\right)^{\alpha-1} du.$$

Under the assumption that $\Re(\alpha) > 1$ the above leads to the formula

$$S_{\alpha,j}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^1 \left(1-u^4\right)^{\alpha-1} \frac{d}{du}\left(u^{Q-1} f * P_{u,j}(x)\right) du.$$

The above procedure can be repeated as long as $\Re(\alpha) > -\frac{Q-1}{4} + 1$. For example, one more integration by parts gives

$$S_{\alpha,j}f(x) = \frac{1}{4\Gamma(\alpha+1)} \int\limits_0^1 \left(1-u^4\right)^{\alpha} \frac{d}{du}\left(u^{-3}\frac{d}{du}\left(u^{Q-1} f * P_{u,j}(x)\right)\right) du,$$

valid for $\Re(\alpha) > 0$. Having taken care of $S_{\alpha,j}f$ we now turn our attention towards the other term, namely $T_{\alpha,j}f$. Interchanging the order of integration in the defining integral of $T_{\alpha,j}$ we get

$$T_{\alpha,j}f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d(Q,\gamma)\left(\frac{4}{\Gamma(\alpha-1)} \int\limits_0^1 \left(1-u^4\right)^{\alpha-2} u^{Q+2} u^{-i\gamma} du\right) f * I_{\gamma,j}(x) d\gamma.$$

A simple calculation shows that

$$\frac{4}{\Gamma(\alpha-1)} \int\limits_0^1 \left(1-u^4\right)^{\alpha-2} u^{Q+2} u^{-i\gamma} du = \frac{\Gamma(\frac{Q+3-i\gamma}{4})}{\Gamma(\alpha-1+\frac{Q+3-i\gamma}{4})}.$$

Thus we have proved the representation

$$T_{\alpha,j}f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d(Q,\gamma) \frac{\Gamma(\frac{Q+3-i\gamma}{4})}{\Gamma(\alpha-1+\frac{Q+3-i\gamma}{4})} f * I_{\gamma,j}(x)d\gamma.$$

Observe that the above is well defined and holomorphic on the whole of $\mathbb{C}$ as $\frac{1}{\Gamma}$ is an entire function. $\square$

As we have already remarked, in proving Theorem 4.1 we assume $p = q = 2$ and $r = 1$. In view of Lemma 4.2 and Proposition 4.5 we are led to show that the operators $S_{0,j}$ and $T_{0,j}$ are bounded on $L^2(\mathbb{H}^n)$. Note that these operators are defined using the function φ_j. We also need to prove the L^2 boundedness of operators defined in terms of ψ_j and ψ_0. As the proofs are similar we only treat $S_{0,j}$ and $T_{0,j}$.

The boundedness of $S_{0,j}$ is something very easy to check. Recall that

$$S_{0,j}f(x) = \frac{1}{4} \int\limits_0^1 \frac{d}{du}\left(u^{-3}\frac{d}{du}\left(u^{Q-1}f * P_{u,j}(x)\right)\right)du$$

$$= \frac{1}{4} \int\limits_0^1 \frac{d}{du}\left((Q-1)u^{Q-5}f * P_{u,j}(x) + u^{Q-4}f * \frac{d}{du}P_{u,j}(x)\right)$$

which under the assumption that $n \geq 2$ reduces to

$$S_{0,j}f(x) = \frac{1}{4}\left((Q-1)f * P_{1,j}(x) + f * K_j(x)\right)$$

where $K_j(x) = \frac{d}{du}|_{u=1}P_{u,j}(x)$ is an integrable function. It is now clear that $S_{0,j}$ is bounded on $L^2(\mathbb{H}^n)$. We use the Fourier transform to prove the L^2 boundedness of $T_{0,j}$. The following formula has been proved by Cowling and Haagerup [6].

Proposition 4.6. *For all $0 < \Re(s) < (n+1)$ the Fourier transform of the function $|x|^{-Q+2s}$ is given by*

$$c_n|\lambda|^{-s}\frac{\Gamma(s)}{\Gamma(\frac{n+1-s}{2})^2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{2k+n+1-s}{2})}{\Gamma(\frac{2k+n+1+s}{2})}P_k(\lambda).$$

Proposition 4.7. *For all $n \geq 2$ and $j = 1, 2, ..., n$ the operator $T_{0,j}$ is bounded on $L^2(\mathbb{H}^n)$.*

Proof. In view of the representation

$$T_{0,j}f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} d(Q,\gamma) \frac{\Gamma(\frac{Q+3-i\gamma}{4})}{\Gamma(\frac{Q-1-i\gamma}{4})} f * I_{\gamma,j}(x)d\gamma$$

we only have to show that

$$\int\limits_{-\infty}^{\infty} |d(Q,\gamma)| \frac{|\Gamma(\frac{Q+3-i\gamma}{4})|}{|\Gamma(\frac{Q-1-i\gamma}{4})|} \|I_{\gamma,j}\| d\gamma < \infty$$

where $\|I_{\gamma,j}\|$ stands for the norm of the operator $f \to f * I_{\gamma,j}$ on $L^2(\mathbb{H}^n)$. In order to estimate the norm of this operator we use Plancherel theorem for the Fourier transform, namely

$$\|f\|_2^2 = (2\pi)^{-(n+1)} \int\limits_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda$$

where $\hat{f}(\lambda) = \pi_\lambda(f)$ the Fourier transform of f at λ. In view of the relation $\pi_\lambda(f * g) = \hat{f}(\lambda)\hat{g}(\lambda)$ it follows that

$$\|f * I_{\gamma,j}\|_2 \leq \sup_{\lambda \neq 0} \|\widehat{I_{\gamma,j}}(\lambda)\|$$

where $\|\widehat{I_{\gamma,j}}(\lambda)\|$ is the operator norm of $\widehat{I_{\gamma,j}}(\lambda)$. In order to calculate the Fourier transform of $I_{\gamma,j}$ we recall that $I_{\gamma,j}(x) = |x|^{-3}\varphi_j(x)I_\gamma(x)$ where $\varphi_j(x) = \widetilde{X}_j(|x|^4)$. In view of Lemma 4.3 we have

$$I_{\gamma,j}(x) = \frac{4}{(-Q+1+i\gamma)}\widetilde{X}_j(|x|I_\gamma(x)) = \frac{4C(Q,\gamma)}{(-Q+1+i\gamma)}\widetilde{X}_j(|x|^{-Q+1+i\gamma}).$$

This allows us to calculate the Fourier transform of $I_{\gamma,j}$ in terms of the Fourier transform of $|x|^{-Q+1+i\gamma}$ which is known explicitly.

For reasonable functions f it is known that $\pi_\lambda(\widetilde{X}_j f) = d\pi_\lambda(\widetilde{X}_j)\pi_\lambda(f)$ where $d\pi_\lambda$ is the derived representation of the Lie algebra $\mathfrak{h}_n$ corresponding to π_λ. It is also known that $d\pi_\lambda(\widetilde{X}_j)\varphi(\xi) = i\lambda\xi_j\varphi(\xi)$ where $\varphi \in L^2(\mathbb{R}^n)$. Rewriting the above in terms of the annihilation and creation operators $A_j(\lambda)^* = \frac{\partial}{\partial\xi_j} + \lambda\xi_j$, $A_j(\lambda) = -\frac{\partial}{\partial\xi_j} + \lambda\xi_j$ we have

$$\widehat{I_{\gamma,j}}(\lambda) = \frac{2iC(Q,\gamma)}{(-Q+1+i\gamma)}(A_j(\lambda) + A_j(\lambda)^*)\pi_\lambda(|x|^{-Q+1+i\gamma}).$$

If we let $H(\lambda)$ stand for the (scaled) Hermite operator $-\Delta + \lambda^2|\xi|^2$ on $\mathbb{R}^n$ then it is well known that the operators $A_j(\lambda)H(\lambda)^{-1/2}$ and $A_j^*(\lambda)H(\lambda)^{-1/2}$ are bounded on $L^2(\mathbb{R}^n)$. Hence we are led to check the boundedness of the operator $H(\lambda)^{1/2}\pi_\lambda(|x|^{-Q+1+i\gamma})$. From Lemma we infer that

$$\pi_\lambda(|x|^{-Q+1+i\gamma}) = c_n|\lambda|^{-(1+i\gamma)/2}\frac{\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q-(1+i\gamma)}{4})^2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{2k+n}{2} + \frac{1-i\gamma}{4})}{\Gamma(\frac{2k+n}{2} + \frac{3+i\gamma}{4})}P_k(\lambda).$$

As $H(\lambda)^{1/2}P_k(\lambda) = ((2k+n)|\lambda|)^{1/2}P_k(\lambda)$ the operator norm of $\widehat{I_{\gamma,j}}(\lambda)$ is a constant multiple of

$$\frac{|C(Q,\gamma)|}{|(-Q+1+i\gamma)|}\frac{|\Gamma(\frac{1+i\gamma}{2})|}{|\Gamma(\frac{Q-(1+i\gamma)}{4})|^2}\sup_{k\in\mathbb{N}}\left((2k+n)^{1/2}\frac{|\Gamma(\frac{2k+n}{2}+\frac{1-i\gamma}{4})|}{|\Gamma(\frac{2k+n}{2}+\frac{3+i\gamma}{4})|}\right).$$

In view of Stirling's formula for large k we have

$$\left|\frac{\Gamma\left(\frac{2k+n}{2}+\frac{1-i\gamma}{4}\right)}{\Gamma\left(\frac{2k+n}{2}+\frac{3+i\gamma}{4}\right)}\right| = \left|\frac{\Gamma\left(\frac{2k+n}{2}+\frac{1-i\gamma}{4}\right)/\Gamma\left(\frac{2k+n}{2}-\frac{i\gamma}{4}\right)}{\Gamma\left(\frac{2k+n}{2}+\frac{3+i\gamma}{4}\right)/\Gamma\left(\frac{2k+n}{2}+\frac{i\gamma}{4}\right)}\frac{\Gamma\left(\frac{2k+n}{2}-\frac{i\gamma}{4}\right)}{\Gamma\left(\frac{2k+n}{2}+\frac{i\gamma}{4}\right)}\right|$$

$$\leq C\frac{|\left(\frac{2k+n}{2}-\frac{i\gamma}{4}\right)|^{1/4}}{|\left(\frac{2k+n}{2}+\frac{i\gamma}{4}\right)|^{3/4}}.$$

From the above it is clear that

$$\sup_{k\in\mathbb{N}}\left((2k+n)^{1/2}\frac{|\Gamma(\frac{2k+n}{2}+\frac{1-i\gamma}{4})|}{|\Gamma(\frac{2k+n}{2}+\frac{3+i\gamma}{4})|}\right)\leq C$$

and consequently the operator norm of $\widehat{I_{\gamma,j}}(\lambda)$ is bounded by a constant multiple of

$$\frac{|C(Q,\gamma)|}{|(-Q+1+i\gamma)|}\frac{|\Gamma(\frac{1+i\gamma}{2})|}{|\Gamma(\frac{Q-(1+i\gamma)}{4})|^2}.$$

Finally we are left with checking the finiteness of the following integral:

$$\int\limits_{-\infty}^{\infty}|d(Q,\gamma)|\frac{|\Gamma(\frac{Q+3-i\gamma}{4})|}{|\Gamma(\frac{Q-1-i\gamma}{4})|}\frac{|C(Q,\gamma)|}{|(-Q+1+i\gamma)|}\frac{|\Gamma(\frac{1+i\gamma}{2})|}{|\Gamma(\frac{Q-(1+i\gamma)}{4})|^2}d\gamma.$$

Recall that

$$C(Q,\gamma)d(Q,\gamma)=a(Q,\gamma)=\left(1-\frac{\Gamma(\frac{Q-i\gamma}{2})\Gamma(\frac{1+i\gamma}{2})}{\Gamma(\frac{Q}{2})\Gamma(\frac{1}{2})}\right)$$

We now make use of the following asymptotic formula for the gamma function: for $|\nu|$ tending to infinity

$$\Gamma(\mu+i\nu)\sim\sqrt{2\pi}|\nu|^{\mu-1/2}e^{-\frac{1}{2}\pi|\nu|}.$$

In view of this it follows that $a(Q,\gamma)$ is a bounded function of γ and

$$\frac{|\Gamma(\frac{Q+3-i\gamma}{4})|}{|\Gamma(\frac{Q-1-i\gamma}{4})|}\leq C_1|\gamma|,\quad\frac{|\Gamma(\frac{1+i\gamma}{2})|}{|\Gamma(\frac{Q-(1+i\gamma)}{4})|^2}\leq C_2|\gamma|^{-n+\frac{1}{2}}$$

as $|\gamma|$ tends to infinity. Consequently, the integral under consideration converges under the assumption that $n\geq 2$. $\square$

5. Sparse bounds and boundedness of the maximal functions

Our aim in this section is to sketch a proof of the sparse bounds for the lacunary spherical maximal function stated in Theorem 1.3. In doing so we closely follow Bagchi et al. [1]. We equip $\mathbb{H}^n$ with a metric induced by the Koranyi norm which makes it a homogeneous space. It is well known that on such spaces there is a well defined notion of dyadic cubes and grids with properties similar to their counter parts in the Euclidean setting. However, we need to be careful with the metric we choose since the group is non-commutative.

Recall that the Koranyi norm on $\mathbb{H}^n$ is homogeneous of degree one with respect to the non-isotropic dilations. Since we are considering $f * \sigma_r$ it is necessary to work with the left invariant metric $d_L(x,y) = |x^{-1}y| = d_L(0, x^{-1}y)$ instead of the standard metric $d(x,y) = |xy^{-1}| = d(0, xy^{-1})$, which is right invariant. The balls and cubes are then defined using d_L. Thus $B(a,r) = \{x \in \mathbb{H}^n : |a^{-1}x| < r\}$. With this definition we note that $B(a,r) = a \cdot B(0,r)$, a fact which is crucial. This allows us to conclude that when f is supported in $B(a,r)$ then $f * \sigma_s$ is supported in $B(a, r+s)$. Indeed, as the support of σ_s is contained in $\overline{B(0,s)}$ we see that $f * \sigma_s$ is supported in $B(a,r) \cdot \overline{B(0,s)} \subset a \cdot B(0,r) \cdot \overline{B(0,s)} \subset B(a, r+s)$. We have the following result by Hytonen [12].

Theorem 5.1. *Let $\delta \in (0,1)$ with $\delta \leq 1/96$. Then there exists a countable set of points $\{z_\nu^{k,\alpha} : \nu \in \mathcal{A}_k\}$, $k \in \mathbb{Z}$, $\alpha = 1, 2, \ldots, N$ and a finite number of dyadic systems $\mathcal{D}^\alpha := \cup_{k \in \mathbb{Z}} \mathcal{D}_k^\alpha$, $\mathcal{D}_k^\alpha := \{Q_\nu^{k,\alpha} : \nu \in \mathcal{A}_k\}$ such that*

(1) *For every $\alpha \in \{1, 2, \ldots, N\}$ and $k \in \mathbb{Z}$ we have*
   i) *$\mathbb{H}^n = \cup_{Q \in \mathcal{D}_k^\alpha} Q$ (disjoint union).*
   ii) *$Q, P \in \mathcal{D}^\alpha \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$.*
   iii) *$Q_\nu^{k,\alpha} \in \mathcal{D}^\alpha \Rightarrow B\big(z_\nu^{k,\alpha}, \frac{1}{12}\delta^k\big) \subseteq Q_\nu^{k,\alpha} \subseteq B\big(z_\nu^{k,\alpha}, 4\delta^k\big)$. In this situation $z_\nu^{k,\alpha}$ is called the centre of the cube and the side length $\ell(Q_\nu^{k,\alpha})$ is defined to be δ^k.*
(2) *For every ball $B = B(x,r)$, there exists a cube $Q_B \in \cup_\alpha \mathcal{D}^\alpha$ such that $B \subseteq Q_B$ and $\ell(Q_B) = \delta^{k-1}$, where k is the unique integer such that $\delta^{k+1} < r \leq \delta^k$.*

Once we have the above theorem we can proceed as in Bagchi et al. [1] to establish the sparse bounds. The main ingredients in the rather long proof are the L^p improving property of the spherical means A_r and their continuity property which we have established already. The proof presented in [1] can be repeated verbatim to get the sparse bounds. We refer the reader to [1] for all the details.

Theorem 5.2. *Assume $n \geq 2$ and fix $0 < \delta < \frac{1}{96}$. Let $1 < p, q < \infty$ be such that $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the triangle joining the points $(0,1), (1,0)$ and $(\frac{2n}{2n+1}, \frac{2n}{2n+1})$. Then for any pair of compactly supported bounded functions (f, g) there exists a (p,q)-sparse form such that $\langle M_{\mathbb{H}^n}^{lac} f, g \rangle \leq C \Lambda_{\mathcal{S}, p, q}(f, g)$.*

Sketch of proof: First we will reduce the case of getting sparse bounds for $M_{\mathbb{H}^n}^{\mathrm{lac}}$ to a simpler case. For this we need to set up some notations. For each $k \in \mathbb{Z}$ and a cube Q in $\mathbb{H}^n$ with $l(Q) = \delta^k$ writing $V_Q = \cup_{P \in \mathbb{V}_Q} P$ where $\mathbb{V}_Q = \{P \in \mathcal{D}_{k+3}^1 : B(z_P, \delta^{k+1}) \subseteq Q\}$ we define

$$A_Q f := A_{\delta^{k+2}}(f \mathbf{1}_{V_Q}).$$

From this definition it is not hard to see that

$$A_{\delta^{k+2}} f \leq \sum_{\alpha=1}^{N} \sum_{Q \in \mathcal{D}_k^\alpha} A_Q f$$

and also whenever $supp(f) \subseteq Q$, $supp(A_Q f) \subseteq Q$. Hence it is enough to prove sparse bound for each of the maximal function defined by

$$M_{\mathcal{D}^\alpha} f = \sup_{Q \in \mathcal{D}^\alpha} A_Q f \quad \text{for each} \quad \alpha = 1, 2, \ldots, N.$$

Now we use standard trick to linearize the supremum. Suppose Q_0 is a cube in $\mathcal{D}$ and $\mathcal{Q}$ be the collection of all dyadic subcubes of Q_0. We define

$$E_Q := \left\{ x \in Q : A_Q f(x) \geq \frac{1}{2} \sup_{P \in \mathcal{Q}} A_P f(x) \right\}$$

for $Q \in \mathcal{Q}$. Note that for any $x \in \mathbb{H}^n$ there exists a $Q \in \mathcal{Q}$ such that

$$A_Q f(x) \geq \frac{1}{2} \sup_{P \in \mathcal{Q}} A_P f(x)$$

and hence $x \in E_Q$. If we define $B_Q = E_Q \setminus \cup_{Q' \supsetneq Q} E_{Q'}$, then $\{B_Q : Q \in \mathcal{Q}\}$ are disjoint and also, $\cup_{Q \in \mathcal{Q}} B_Q = \cup_{Q \in \mathcal{Q}} E_Q$. Observing that $\{B_Q : Q \in \mathcal{Q}\}$, for any $f, g > 0$ an easy calculation yields

$$\langle \sup_{Q \in \mathcal{Q}} A_Q f, g \rangle \leq 2 \sum_{Q \in \mathcal{Q}} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle.$$

Now we will make use of the following lemma which is the main ingredient in proving the sparse bound.

Lemma 5.3. *Let $1 < p, q < \infty$ be such that $\left(\frac{1}{p}, \frac{1}{q} \right)$ in the interior of the triangle joining the points $(0, 1), (1, 0)$ and $\left(\frac{2n}{2n+1}, \frac{2n}{2n+1} \right)$. Let $f = \mathbf{1}_F$ and let g be any bounded function supported in Q_0. Let $C_0 > 1$ be a constant and let $\mathcal{Q}$ be a collection of dyadic sub-cubes of $Q_0 \in \mathcal{D}$ for which the following holds*

$$\sup_{Q' \in \mathcal{Q}} \sup_{Q: Q' \subset Q \subset Q_0} \frac{\langle f \rangle_{Q,p}}{\langle f \rangle_{Q_0, p}} < C_0. \tag{5.1}$$

Then there holds

$$\sum_{Q \in \mathcal{Q}} \langle A_Q f, g_Q \rangle \lesssim |Q_0| \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q}.$$

Using the continuity property, we have established in the previous section, this lemma can be proved using exactly the same argument as in Bagchi et al. [1]. We refer the reader to [1] for the proof of this lemma.

Now once we have this lemma, the sparse domination result is immediate. Since f is compactly supported, we may assume that the support of f is contained in some cube Q_0. Also we can take f to be non-negative. Then notice that for a fixed dyadic grid $\mathcal{D}$ we have

$$M_{\mathcal{D}} f = M_{\mathcal{D} \cap Q_0} f = \sup_{Q \in \mathcal{D} \cap Q_0} |A_Q f|.$$

Hence we need to prove a sparse bound for

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle.$$

Using the previous lemma we first prove for a special case when $f = \mathbf{1}_F$ where $F \subseteq Q_0$. We consider the following set

$$\mathcal{E}_{Q_0} := \{P \text{ maximal subcube of } Q_0 : \langle f \rangle_{P,p} > 2 \langle f \rangle_{Q_0,p} \}$$

Let $E_{Q_0} = \cup_{P \in \mathcal{E}_{Q_0}}$. For a suitable choice of $c_n > 1$ we can arrange $|E_{Q_0}| < \frac{1}{2}|Q_0|$. We let $F_{Q_0} = Q_0 \setminus E_{Q_0}$ so that $|F_{Q_0}| \geq \frac{1}{2}|Q_0|$. We define

$$\mathcal{Q}_0 = \{Q \in \mathcal{D} \cap Q_0 : Q \cap E_{Q_0} = \emptyset\}. \tag{5.2}$$

Note that when $Q \in \mathcal{Q}_0$ then $\langle f \rangle_{Q,p} \leq 2 \langle f \rangle_{Q_0,p}$. For otherwise, if $\langle f \rangle_{Q,p} > 2 \langle f \rangle_{Q_0,p}$ then there exists $P \in \mathcal{E}_{Q_0}$ such that $P \supset Q$, which is a contradiction. For the same reason, if $Q' \in \mathcal{Q}_0$ and $Q' \subset Q \subset Q_0$ then $\langle f \rangle_{Q,p} \leq 2 \langle f \rangle_{Q_0,p}$. Thus

$$\sup_{Q' \in \mathcal{Q}_0} \sup_{Q: Q' \subset Q \subset Q_0} \langle f \rangle_{Q,p} \leq 2 \langle f \rangle_{Q_0,p}.$$

Note that for any $Q \in \mathcal{D} \cap Q_0$, either $Q \in \mathcal{Q}_0$ or $Q \subset P$ for some $P \in \mathcal{E}_{Q_0}$. Thus

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle = \sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle + \sum_{P \in \mathcal{E}_{Q_0}} \sum_{Q \subset P} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle$$

for any $Q \in \mathcal{Q}_0$, $Q \subset F_{Q_0}$ and hence

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle = \sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g \mathbf{1}_{F_{Q_0}} \mathbf{1}_{B_Q} \rangle.$$

Applying Lemma 5.3 we obtain

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle \le C |Q_0| \langle f \rangle_{Q_0,p} \langle g \mathbf{1}_{F_{Q_0}} \rangle_{Q_0,q}.$$

Let $\{P_j\}$ be an enumeration of the cubes in $\mathcal{E}_{Q_0}$. Then the second sum above is given by

$$\sum_{j=1}^{\infty} \sum_{Q \in P_j \cap \mathcal{D}} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle.$$

For each j we can repeat the above argument recursively. Putting everything together we get a sparse collection $\mathcal{S}$ for which

$$\sum_{Q \in \mathcal{D} \cap Q_0} \langle A_Q f, g \mathbf{1}_{B_Q} \rangle \le C \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \mathbf{1}_{F_S} \rangle_{S,q}. \tag{5.3}$$

This proves the result when $f = \mathbf{1}_F$.

Now we will prove the above result (5.3) for any non-negative, compactly supported bounded function f. Here we modify the definition of $\mathcal{E}_{Q_0}$ as follows

$$\mathcal{E}_{Q_0} := \{ P \text{ maximal subcube of } Q_0 : \langle f \rangle_{P,p} > 2 \langle f \rangle_{Q_0,p} \text{ or } \langle g \rangle_{P,p} > 2 \langle g \rangle_{Q_0,p} \}$$

Now defining E_{Q_0}, F_{Q_0} and $\mathcal{Q}_0$ same as before, we have

$$\sup_{Q' \in \mathcal{Q}_0} \sup_{Q : Q' \subset Q \subset Q_0} \langle f \rangle_{Q,p} \le 2 \langle f \rangle_{Q_0,p} \text{ and } \sup_{Q' \in \mathcal{Q}_0} \sup_{Q : Q' \subset Q \subset Q_0} \langle g \rangle_{Q,q} \le 2 \langle g \rangle_{Q_0,q}.$$

Now we will decompose f in the following way so that we can use the sparse domination already proved for characteristic functions. We write $f = \sum_m f_m$ where $f_m := f \mathbf{1}_{E_m}$ and $E_m := \{ x \in Q_0 : 2^m \le f(x) < 2^{m+1} \}$. For each m applying the sparse domination to $\mathbf{1}_{E_m}$, we get a sparse family $\mathcal{S}_m$ such that

$$\sum_{Q \in \mathcal{Q}_0 \cap \mathcal{D}} \langle A_Q \mathbf{1}_{E_m}, g \mathbf{1}_{F_{Q_0}} \mathbf{1}_{B_Q} \rangle \le C \sum_{S \in \mathcal{S}_m} |S| \langle \mathbf{1}_{E_m} \rangle_{S,p} \langle g \mathbf{1}_{F_{Q_0}} \rangle_{S,q}$$

Hence using the fact that for any $Q \in \mathcal{Q}_0, Q \subset F_{Q_0}$ we have

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f_m, g \mathbf{1}_{B_Q} \rangle \le C 2^{m+1} \sum_{S \in \mathcal{S}_m} |S| \langle \mathbf{1}_{E_m} \rangle_{S,p} \langle g \mathbf{1}_{F_{Q_0}} \rangle_{S,q}.$$

Now it is clear that if $S \cap F_{Q_0} = \phi$ then $\langle g \mathbf{1}_{F_{Q_0}} \rangle_{S,q} = 0$. Also when $S \cap F_{Q_0} \ne \phi$, using the definition of $\mathcal{E}_{Q_0}$ and $\mathcal{Q}_0$ one can easily show that $\langle g \mathbf{1}_{F_{Q_0}} \rangle_{S,q} \le C \langle g \rangle_{Q_0,q}$. Hence we make use of the following lemma proved in [13] (see also [1]):

Lemma 5.4. *Let $\mathcal{S}$ be a collection of sparse sub-cubes of a fixed dyadic cube Q_0 and let $1 \leq s < t < \infty$. Then, for a bounded function ϕ,*

$$\sum_{Q \in \mathcal{S}} \langle\phi\rangle_{Q,s}|Q| \lesssim \langle\phi\rangle_{Q_0,t}|Q_0|.$$

For some $\rho_1 > p$ we have

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f_m, g\mathbf{1}_{B_Q}\rangle \leq C 2^{m+1} \langle g\rangle_{Q_0,q} \langle\mathbf{1}_{E_m}\rangle_{Q_0,\rho_1}|Q_0|.$$

But we know that $f = \sum_m f_m$. So, finally we obtain

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g\mathbf{1}_{B_Q}\rangle \leq C \langle g\rangle_{Q_0,q}|Q_0| \sum_m 2^m \langle\mathbf{1}_{E_m}\rangle_{Q_0,\rho_1}.$$

Note that a simple calculation yields $\sum_m 2^m \langle\mathbf{1}_{E_m}\rangle_{Q_0,\rho_1} \leq C\|f\|_{L^{\rho_1,1}(Q_0,\frac{1}{|Q_0|}dx)}$ where $\|.\|_{L^{\rho_1,1}}$ denotes the Lorentz space norm. Also it is a well-known fact that for any $\rho > \rho_1$, $L^{\rho_1,1}(Q_0, \frac{1}{|Q_0|}dx)$-norm is dominated by the $L^\rho(Q_0, \frac{1}{|Q_0|}dx)$-norm. Hence we have

$$\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g\mathbf{1}_{B_Q}\rangle \leq C \langle g\rangle_{Q_0,q}|Q_0|\langle f\rangle_{Q_0,\rho}.$$

Now proceeding same as in case $f = \mathbf{1}_F$, we get the following sparse domination

$$\langle M_{\mathcal{D}} f, g\rangle \leq C \sum_{S \in \mathcal{S}} |S|\langle f\rangle_{S,\rho}\langle g\rangle_{S,q},$$

which proves the theorem. $\square$

As consequences of the sparse bound we get some new weighted and unweighted inequalities for the lacunary maximal function under consideration. We now use the above sparse domination along with the following well-known boundedness property of the sparse forms proved in [8]:

Proposition 5.5. *Let $1 \leq r < s' \leq \infty$. Then,*

$$\Lambda_{r,s}(f,g) \lesssim \|f\|_{L^p}\|g\|_{L^{p'}}, \quad r < p < s'.$$

We have thus proved the following result which is the main theorem in this article.

Theorem 5.6. *For $n \geq 2$, the lacunary spherical maximal function $M_{\mathbb{H}^n}^{lac}$ is bounded on $L^p(\mathbb{H}^n)$ for all $1 < p < \infty$.*

Weighted norm inequalities are also well studied in the literature. To state the weighted boundedness properties of the sparse form we need to mention the following terminologies about classes of weights. A weight w is a non-negative locally integrable function defined on $\mathbb{H}^n$. Given $1 < p < \infty$, the Muckhenhoupt class of weights A_p consists of all w satisfying

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty$$

where the supremum is taken over all cubes Q in $\mathbb{H}^n$. On the other hand, a weight w is in the reverse Hölder class RH_p, $1 \le p < \infty$, if

$$[w]_{\mathrm{RH}_p} = \sup_Q \langle w \rangle_Q^{-1} \langle w \rangle_{Q,p} < \infty,$$

again the supremum taken over all cubes in $\mathbb{H}^n$. The following weighted inequality for sparse form has been proved in [2].

Proposition 5.7. *Let $1 \le p_0 < q_0' \le \infty$. Then,*

$$\Lambda_{p_0,q_0}(f,g) \le \{[w]_{A_{p/p_0}} \cdot [w]_{\mathrm{RH}_{(q_0'/p)'}}\}^\alpha \|f\|_{L^p(w)} \|g\|_{L^{p'}(\sigma)}, \quad p_0 < p < q_0',$$

with $\alpha = \max \left\{ \frac{1}{p-1}, \frac{q_0'-1}{q_0'-p} \right\}$.

Using this result we have the following weighted boundedness property for the lacunary maximal function:

Theorem 5.8. *Let $n \ge 2$ and define*

$$\frac{1}{\phi(1/p_0)} = \begin{cases} 1 - \frac{1}{2np_0}, & 0 < \frac{1}{p_0} \le \frac{2n}{2n+1}, \\ 2n\left(1 - \frac{1}{p_0}\right), & \frac{2n}{2n+1} < \frac{1}{p_0} < 1. \end{cases}$$

Then $M_{\mathbb{H}^n}^{lac}$ is bounded on $L^p(w)$ for $w \in A_{p/p_0} \cap \mathrm{RH}_{(\phi(1/p_0)'/p)'}$ and all $1 < p_0 < p < (\phi(1/p_0))'$.

Acknowledgments

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