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For the first interval  $[0, x]$  (with  $0 < x \leq (1 - \alpha)$ ), we have  $v_1(0, x) = x(1 + \alpha)$  and  $v_2(0, x) = x(1 - \alpha)$ . However, for any  $\alpha \in (0, 1)$  and  $x > 0$ , the following strict inequality holds:  $v_1(0, x) = x(1 + \alpha) > x(1 - \alpha) = v_2(0, x)$ . This leads to a contradiction and proves that  $C$  does not admit a perfect division with connected pieces.

Even though we might not have a perfect allocation, the work of Alon [2] proves that a perfect division with  $n(n - 1)$  cuts always exists. Hence, in the above-mentioned instance  $C$  with 2 agents, 2 cuts should suffice to form a perfect division. In particular, we note that the following division  $\mathcal{D}^* = \{D_1^*, D_2^*\}$  is perfect in  $C$ ; here  $D_1^* = [\frac{1}{2} - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}]$  and  $D_2^* = [0, \frac{1}{2} - \frac{\alpha}{2}] \cup [1 - \frac{\alpha}{2}, 1]$ . Here,

$$v_1(D_2^*) = (1 + \alpha) \left( \frac{1}{2} - \frac{\alpha}{2} \right) + \alpha \left( \frac{\alpha}{2} \right) = \frac{1}{2}$$

and

$$v_2(D_2^*) = (1 - \alpha) \left( \frac{1}{2} - \frac{\alpha}{2} \right) + (2 - \alpha) \left( \frac{\alpha}{2} \right) = \frac{1}{2}$$

That is, both the agents value the piece  $D_2^*$  at  $1/2$ . Since the valuations are normalized, we additionally have  $v_1(D_1^*) = v_2(D_1^*) = 1/2$ . This shows that  $\mathcal{D}^*$  is a perfect division (with disconnected pieces) in  $C$ .