Analysis of High-Speed Rotating Flow in 2D Polar $(r - \theta)$ Coordinate

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The generalized analytical model for the radial boundary layer in a high-speed rotating cylinder is formulated for studying the secondary radial flow due to insertion of mass, momentum and energy into the rotating cylinder in the polar $(r - \theta)$ plane. The analytical solution includes the sixth order differential equation for the radial boundary layer at the cylindrical surface in terms of master potential (χ), which is derived from the equations of motion in a polar $(r - \theta)$ plane. The linearization approximation is used, where the equations of motion are truncated at linear order in the velocity and pressure disturbances to the base flow, which is a solid-body rotation. Additional assumptions in the analytical model include constant temperature in the base state (isothermal condition), and high Reynolds number, but there is no limitation on the stratification parameter. In this limit, the gas flow is restricted to a boundary layer of thickness $(\text{Re}^{-1/3} R)$ at the wall of the cylinder. Here, the stratification parameter $\mathcal{A} = \sqrt{m\Omega^2 R^2 / (2k_B T)}$. This quantity ($\mathcal{A}$) is the ratio of the peripheral speed (ΩR) to the most probable molecular speed $\sqrt{2k_B T / m}$, the Reynolds number $\text{Re} = (\rho_w \Omega R^2) / \mu$, where m is the molecular mass, Ω and R are the rotational speed and radius of the cylinder, k_B is the Boltzmann constant, T is the gas temperature, ρ_w is the gas density at wall, and μ is the gas viscosity. For the case of mass/momentum/energy insertion into the flow, the separation-of-variables procedure is used, and the appropriate homogeneous boundary conditions are specified so that the linear operators in the meridional and radial directions are self-adjoint. The discrete eigenvalues and eigenfunctions of the linear operators (sixth-order and second-order in the radial and meridional directions for the generalized analytical equation) are obtained. The solutions for the secondary radial flows are determined in terms of these eigenvalues and eigenfunctions. The major advantage of the present formulation is that it is not restricted to the asymptotic limit of high stratification parameter, and since we have used the conservative form of the compressible mass, momentum and energy conservation equations in deriving the generalized analytical model, obviously, the shock type solutions (Rankine-Hugoniot relations) are automatically satisfied. These solutions are compared with direct simulation Monte Carlo (DSMC) simulations. The comparison shows that it is necessary to take certain precautions when making a quantitative comparison between analysis and simulations. The commonly used 'diffuse reflection' boundary conditions at solid walls in DSMC simulations result in a non-zero slip ((Pradhan & Kumaran, J. Fluid Mech., vol. 686, 2011, pp. 109-159); (Kumaran & Pradhan, J. Fluid Mech., vol. 753, 2014, pp. 307-359)). These have to be incorporated in the analysis in order to make quantitative predictions. In the case of mass/momentum/energy sources within the flow, it is necessary to ensure that the homogeneous boundary conditions (zero temperature perturbation) are accurately satisfied in the simulations. When these precautions are taken, there is quantitative agreement (with a difference of less than 10%) between the predictions of the analytical model and the DSMC simulations, even when stratification parameter is as low as 0.707, and the Reynolds number is as low as 100. The predictions of the generalized analytical model is also significantly better than that of the analytical model, which is restricted to thin boundary layers in the limit of high stratification

parameter. In a high speed rotating field we examine the mass flow rate through the stationary intake tube. The simulations show that the scaled mass flow rate $m/(H\Omega)$ increases progressively from zero, at the stagnation condition, to 0.024 as the equilibrium back pressure is reduced. Here, H is the total gas holdup. The slow down of the circumferential velocity of the bulk of the rotating fluid due to the presence of stationary intake tube is studied for stratification parameter in the range 0.707 – 3.535, and found significant slow down (between 8 to 28%), which induces the secondary radial flow towards the axis, and it further excites the secondary axial flow, which could be very important for the centrifugal gas separation processes. An important finding is that the stagnation pressure (no mass flow through the intake tube) is strongly affected by the wall gap, as well as with stratification parameter indicating a strong coupling between the local temperature, density, pressure and velocity fields. We have further carried out the DSMC simulations for a binary gas mixture with large mass difference ($\epsilon_a = (2\Delta m/(m_1 + m_2)) = 0.5$), and compare the simulation results with those of single-component gas incorporating the homogeneous boundary conditions. Here, $\Delta m = (m_1 - m_2)$ is the difference in the molecular masses of the two species. An important point to note is that in all cases, internal momentum source, mass source/sink, and stationary intake tube, the peak value of the secondary radial flow for a binary gas mixture with $\epsilon_a = 0.5$ increases by (19 - 30)% compared to the single-component gas, as stratification parameter $\mathcal{A}$ is increased from 0.707 to 3.535, for $Re = 10^2$.

Key words: High speed rotating flow, DSMC Simulation, rarefied gas flow.

1. Introduction

Centrifugal gas separation processes effect separation by utilizing the difference in the mole fraction in a high speed rotating cylinder caused by the difference in molecular mass, and consequently the centrifugal force density. These have been widely used in isotope separation because chemical separation methods cannot be used to separate isotopes of the same chemical species (Cohen (1951), Avery & Davies (1973), Von Halle, Wood & Lowry (1978), Brouwers (1978), Soubbaramayer (1979), Hoglund, Shacter & Von Halle (1979), Wood & Morton (1980), Sakurai (1981), Olander (1981), Wood & Sanders (1983), Wood, Jordan & Gunzburger (1984), Copper & Morton (1988), Wood & Babarsky (1992), Wood (1995), Wood, Mason & Soubbaramayer (1996), Omnes (2007)). Radioactive isotopes, including zinc, cadmium, iridium, selenium, tellurium, germanium and molybdenum, are used in medical and industrial applications. In medical applications, radio-isotopes are used both for diagnostics such as gallium scans for locations of inflammation and infection, indium for leukocyte imaging and molybdenum/technetium scans for high bone growth, as well as for therapy including cancer therapy using isotopes of iodine and iridium. An important industrial application is the separation of uranium isotopes for nuclear power generation. Other important applications include non-destructive testing using gamma cameras, and use of zinc isotopes to reduce corrosion in reactor cooling systems.

The principle of isotope separation in high speed centrifugal field in long rotating cylinders is based on the difference in the centrifugal force on different isotopes due to their difference in mass. Due to this, the concentration of the heavier isotope is higher than the mean concentration at the wall, while the lighter isotope is more concentrated near the center. In a closed rotating cylinder with no inflow/outflow, the gas rotated

like a solid body, and the concentration separation is easily predicted on the basis of hydrostatics alone. However, in an operating centrifuge, there is an inflow and outflow of gases, and these require stationary and rotating inserts. Secondary flows could also be generated due to temperature gradients along the side wall of the rotating cylinder as well as temperature difference between the end caps. The achievable separation is primarily determined by the secondary axial and radial flows. Therefore, the accurate modeling of these secondary flows and its numerical analysis based on molecular simulations is of critical importance in predicting the centrifuge performance.

Centrifugal separation in a high-speed rotating field is a widely used method for isotope separation (Fraenkel (1959); Barcilon & Pedlosky (1967); Hunter (1967); Salant (1968); Homsy & Hudson (1969); Greenspan (1968); Bark & Bark (1976); Hittinger, Holt & Soubbaramayer (1981); Hittinger, Holt, Soubbaramayer & Cortet (1983); Gunzburger & Wood (1982); Jones (1982); Jung (1983); Kai (1983); Gunzburger, Wood & Jordan (1984); Heuser, Ribando & Wood (1986); Copper & Morton (1988); Babarsky & Wood (1990, 1992); Bourn, Peterson & Wood (1999); Doneddu, Robin & Wood (2000); Babarsky, Herbst & Wood (2002); Maslen & Moore (1956); Stewartson (1957); Sozou (1959); Thompson & Chen (1980); Vieceili (1983)). The small difference in the molecular masses of the isotopes provides the only economically feasible method for effecting separation. The centrifugal force results in a small difference in concentration of the two isotopes, with a higher concentration of the heavier isotope near the wall of the cylinder, and a higher concentration of the lighter isotope near the axis. In order to separate the isotopes, stationary tubes and baffles are inserted into the rotating cylinder. The different types of external forcing (wall thermal drive, end-cap thermal drive, feed drive, mechanical drive) cause a axial and radial secondary flow in the cylinder. This secondary flow has a strong effect on the efficiency of separation.

Here, we have studied the secondary radial flow due to insertion of mass, momentum and energy into the rotating cylinder in the polar ($r - \theta$) plane. The presence of a stationary insert (intake tube) within the flow domain causes a disturbance to the base flow (solid body rotation). A detached bow shock is formed in front of the stationary insert (intake tube), and behind the detached bow shock, secondary radial flow is induced towards the axis, and it excites the secondary axial flow, which could be very important for the centrifugal gas separation processes. The inward radial flow also produces a vortex column near the stationary insert. A mechanically driven gas centrifuge utilizes stationary intake tube (which is used to collect the separated gases), to generate the secondary radial flow, and it further excites the secondary axial flow. For this reason, it is important to analyze the secondary radial flow due to insertion of mass, momentum and energy into the rotating cylinder in the polar ($r - \theta$) plane to assess the performance of a mechanically driven gas centrifuge.

There are two objectives of the present analysis of the rarefied gas flow in a rotating cylinder in the polar ($r - \theta$) plane. The first is to remove the restriction of high stratification parameter, and to generalize the solutions to low rotation speeds where the stratification parameter may be $O(1)$. Secondly, we would like to compare the predictions with molecular simulations based on the direct simulation Monte Carlo (DSMC) method for rarefied gas flows in a polar ($r - \theta$) plane, in order to quantify the errors resulting from the approximations made in the analytical solution. Here, we obtained the generalized analytical solution in a polar ($r - \theta$) plane and compared with molecular simulations based on direct simulation Monte Carlo (DSMC) method (Bird (1963, 1994)) due to insertion of mass, momentum and energy into the rotating cylinder in the polar ($r - \theta$) plane.

A complete flow simulation involves the solution of the compressible mass, momen-

tum and energy equations for the gas velocity in the cylinder, along with the ideal gas equation of state. There have been previous numerical simulations of axisymmetric flows in high speed rotating cylinder (Lahargue & Soubbaramayer (1978); Soubbaramayer (1979); Harada (1980); Hittinger, Holt & Soubbaramayer (1981); Hittinger, Holt, Soubbaramayer & Cortet (1983); Homsy & Hudson (1969); Hunter (1967); Kai (1983); Matsuda, Tamura & Sawada (1989); Mikami (1981); Sakurai (1981); Omnes (2007); Olander (1981); Brouwers (1978); Hoglund, Shacter & Von Halle (1979); Ribando (1984); Soubbaramayer & Billet (1980)). Apart from the complicated formulation of the differential equations for five variables (density, three components of velocity and temperature), there are two important conceptual difficulties in the numerical solution of the continuum equations. The flow in the cylinder is usually at high Mach number, where the gas is concentrated at the wall of the cylinder in a 'radial boundary layer' of thickness $\text{Re}^{-1/3}R$ much smaller than the cylinder diameter. While the continuum approximation is valid in the radial boundary layer, the density in the central region is sufficiently small that the flow is in the free-molecular regime. There is no simple way to determine the boundary between the continuum and free-molecular regimes, and to prescribe boundary conditions at this boundary. Another important issue is the modeling of the wall slip in the simulations. While the no-slip conditions are applicable to low Mach number flows, there is usually substantial slip at the wall which needs to be modeled in continuum simulations.

The analysis of flow field and concentration field in a high speed rotating cylinder is a challenging job which faces the following difficulties. In a high speed rotating cylinder the flow velocities are very large due to the high rotation rates, required to generate the necessary concentration separation between the species with nearly equal molecular masses. The stratification parameter, $\mathcal{A} = \sqrt{m\Omega^2 R^2 / (2k_B T)}$, which is the ratio of the peripheral speed (ΩR) to the most probable molecular speed $\sqrt{2k_B T / m}$, is typically much larger than 1. This generates two different complications in the simulation procedure. In contrast to most low stratification parameter flows commonly encountered, the high stratification parameter flows are 'compressible'. The simulation of high-speed rotating compressible flows is challenging due to the presence of curved detached bow shock waves involving sharp and nearly discontinuous density variations. There is a large variation in the total gas density across the high speed rotating cylinder, and the density at the wall is typically (5 – 6) orders of magnitude larger than that at the center. Associated with the large variation in the density is a similar variation in the 'mean free path', which is the average distance travel by a molecule between two successive collisions. The mean free path also varies by (5 – 6) orders of magnitude over the radius of the cylinder. The mean free path is typically small at the wall of the cylinder, but could be quite large at the center. The Knudsen number, which is the ratio of the mean free path and the macroscopic scale (cylinder radius), is small at the wall, but is greater than 1 at the center. Continuum Navier-Stokes equations cannot be applied when the Knudsen number is larger than 1, because these equations are derived assuming that the mean free path is small compared to the macroscopic scales, thus facilitating the use of a gradient expansion.

There are uncertainties in the specification of boundary conditions at the bounding surfaces. Usually, in continuum simulations, the velocity at a solid surface is equal to the velocity of the surface (no-slip condition), and the temperature of the fluid at the surface is equal to the surface temperature. However, these boundary conditions cannot be used in rarefied gas dynamics, where the mean free path is comparable to the system size, and

there could be slip at the bounding surfaces. General expressions are not available for the slip boundary condition, since the slip depends on the type of gas and solid surface.

In addition to the outer wall of the cylinder, it is also necessary to specify boundary conditions at the axis. For incompressible flows, symmetry boundary conditions are specified at the axis of the cylinder. However, in high speed rotating flows, the density at the center is very small and the mean free path is comparable to the macroscopic scale (cylinder radius). Due to this, it is not possible to extend the continuum description to the center of the cylinder. Boundary conditions have to be applied at an artificial 'inner boundary' at which the density is close to zero. There is no clearly defined way to specify the location of this boundary, but one has to show that the results of the simulation are insensitive to the inner boundary location. In addition, procedures have to be devised to specify velocity and temperature at this boundary.

In practical applications, there are rotating and stationary inserts in the cylinder for the inflow/outflow of gases. These inserts have to be incorporated as solid boundaries in a simulation, resulting in a complex simulation domain. This increases the complexity of the simulation grid required, and the number of grid points at which the hydrodynamic fields are evaluated.

All of the above difficulties in a continuum simulation can be overcome if we do a simulation at the molecular level, where the motion of the molecules and their interactions with the bounding surfaces are directly incorporated. For dilute gases, the most efficient simulation technique is the direct simulation Monte Carlo (DSMC) technique (Bird (1963, 1994); Vincenti (1967); Cercignani (2000); Chapman & Cowling (1970); Garcia (2000); Nanbu (1980); Quanhua & Boyd (2002)). A discrete simulation technique does not become invalid when the mean free path is larger than the system size, because the simulation does not use averaged equations. So even very low density regions can be simulated. In a discrete simulation technique, the interaction of molecules with each other and with the solid surfaces are specified, and there is no need to specify boundary conditions for the hydrodynamic fields. So complications with artificial boundaries do not arise. Though DSMC technique overcomes several of the difficulties of a continuum simulation, the computational power and time required for the simulation is much higher than that for a continuum simulation. This is due to the requirement for simulating, directly, a simplified version of the molecular motion. This requires the evaluation of the complete velocity distribution function of the molecule, and temperature fields.

In direct simulation Monte Carlo (DSMC) simulation procedure, the Boltzmann equation is simulated using a probabilistic Monte Carlo procedure to obtain the velocity distribution function, and the density, velocity and temperature are obtained by taking moments of the distribution function. Here, one 'simulated' molecule represents a large number of real molecules, and the collision probability is chosen in such a manner that the collision frequency, mean free path and the temperature of the simulated molecules is equal to those for the real molecules. This procedure can be used for the simulation of both the rarefied gas regime and the continuum regime, and this has been used previously for centrifugal flows (Roberts (1985); Roberts & Hausman (1988); Roblin & Doneddu (2000); Pradhan & Kumaran (2011); Kumaran & Pradhan (2014)). However, DSMC simulations are computationally intensive, due to the requirement of having a large number of simulated molecules in each differential volume in order to reduce statistical fluctuations.

Two simulation techniques are discussed here in the order of increasing computational complexity and decreasing analytical complexity. While the analysis of the analytical model is complicated, this method can be used for obtaining quick and approximate computational results. In contrast, the direct simulation Monte Carlo (DSMC) method

is conceptually simple, since it simulates the motion of molecules with specified inter-molecular interactions. However, since the number of molecules to be simulated is very large, this is computationally very expensive. This can be deployed to obtain accurate computational results in narrow regions of parameter space identified by more approximate method. Thus, the simulation methods in this hierarchy complement each other and provide a comprehensive framework for simulating strongly rotating flows.

In our previous study for the axial flow at the radial boundary layer and radial flow near the end caps (Pradhan & Kumaran (2011), Kumaran & Pradhan (2014)) in an axisymmetric ($r - z$) plane has revealed several novel features of high speed rotating flows. The generalized Onsager and Carrier-Maslen models (Pradhan & Kumaran (2011)) were formulated in the axisymmetric ($r - z$) plane where the approximation of high stratification parameter was relaxed. In this, a generalized sixth order operator was derived, which is applicable even when the stratification parameter is $O(1)$, though the assumption of high Reynolds number and high aspect ratio were made. This operator is also self-adjoint with real eigenvalues and orthogonal eigenfunctions, though these have to be determined numerically. For the secondary flow, it was found that the predictions of the generalized Onsager model were in very good agreement with DSMC results even when the secondary flow is as large as the 20% of the base solid-body rotation, the stratification parameter is as low as 0.707, the Reynolds number is as low as 100 and the cylinder length is only two times the diameter. However, these models still assume that the molecular masses of the two species are equal while calculating the secondary flows, and that the mole fractions are a constant. Next, the mass difference between the two species in a binary gas mixture was incorporated into the generalized Onsager and Carrier-Maslen models (Kumaran & Pradhan (2014)) to make it applicable to systems where the difference in molecular masses is not much smaller than the average molecular mass. In the base state, the flow is considered to be a solid body rotation with constant temperature, but the variation in the mole fraction is taken into account. Due to this, there is a variation in the viscosity and thermal conductivity in the radial direction. In addition, the total pressure is written as the sum of partial pressures of the two species, and is not approximated as an exponential function of the distance from the wall. The dilute gas equation of state is used for the pressure, while the Chapman-Enskog expressions (Chapman & Cowling (1970)) for hard sphere gases are used for the thermal conductivity and viscosity. In the analysis of the secondary flow, the equation for the master potential is expanded in an asymptotic series in the parameter $\epsilon = (\Delta m/m_{av})$, where Δm is the difference in the molecular masses of the two species, and m_{av} is a suitably defined average molecular mass. In the leading approximation where terms of $O(\epsilon)$ or smaller are neglected. The leading order equation and boundary conditions for the master potential are nearly identical to the sixth order differential equation of the generalized Onsager model (Pradhan & Kumaran (2011)); the important difference being that the density is not assumed to decrease exponentially from the wall, but the actual species densities are used. The mean velocity and temperature obtained by solving the leading order equations are used in the species balance equations for the mole fractions of the two species, and the corrections for the number density and mass density of the two species due to the secondary flow are obtained. These appear as inhomogeneous terms in the equation for the $O(\epsilon)$ correction for the master potential; this equation is solved with appropriate boundary conditions to obtain the $O(\epsilon)$ correction to the master potential and flow velocity. This is then used in the $O(\epsilon)$ contribution to the species balance equation, and the hierarchical procedure is repeated until the $O(\epsilon^2)$ correction to the secondary flow velocity and mole fraction are calculated. A similar hierarchical procedure is used for the Carrier-Maslen model for the end-cap boundary layers. The results of the Onsager hierarchy, up to

$O(\epsilon^2)$, are compared with the results of DSMC simulations for the axisymmetric flow of a two-species hard sphere molecular gas. The constitutive relations for mass diffusion, viscosity and heat conduction predicted by the DSMC simulations are compared with the Chapman-Enskog expressions (Chapman & Cowling (1970)) in order to validate the constitutive relations used in the analysis. After this, the results of the $O(\epsilon^2)$ Onsager hierarchy for the secondary flow are compared with DSMC simulations for three types of forcing, a linear wall temperature profile, inflow/outflow of gas along the axis, and mass/momentum input due to inserts within the flow (Pradhan & Kumaran (2011), Kumaran & Pradhan (2014)).

The condition which has not yet been relaxed in the analytical model is the condition on the linearization of the equations. It will be too complicated to include the non-linear terms in the conservation equations in the analytical model, and it will likely not be possible to obtain analytically tractable equations.

There have been previous studies on non-axisymmetric rotating compressible flows in a cylindrical geometry (Matsuda & Nakagawa (1983), Matsuda, & Hashimoto (1976), Wood & Babarsky (1992), Babarsky & Wood (1990)). Matsuda & Nakagawa (1983) studied the flow in a pie-shaped cylinder of infinite length with walls at fixed angles in the meridional direction, and reported the presence of 'buoyancy' boundary layer of thickness $Re^{-1/2}R$ at the walls perpendicular to the meridional direction. Wood & Babarsky (1992) and Babarsky & Wood (1990) extended the work of Matsuda & Nakagawa (1983) to a finite-length, pie-shaped cylinder about the vertex. They presented an eigenfunction expansion based on the spectral decomposition of the resulting differential operator. The present work provides an opportunity for validating both the analytical model as well as the DSMC method for high-speed rotating flows in the polar $(r - \theta)$ plane. The basic difference of flow dynamics between axisymmetric $(r - z)$ and a polar $(r - \theta)$ configuration is that, in $(r - z)$ plane secondary flow is subsonic, whereas in $(r - \theta)$ plane the azimuthal flow is supersonic and encounter a curved detached bow shock wave in front of the stationary inserts and generates a sub-sonic inward radial flow.

Babarsky, Herbst & Wood (2002); Bourn, Peterson & Wood (1999) introduced the properties of the temperature potential and its relationship to the original problem of axisymmetric flow in a rapidly rotating centrifuge based on Onsager master potential and provide a firm mathematical foundation for the solutions of these two problems. They have shown that the temperature potential plays an analogous role as Onsager master potential, and the Onsager minimum entropy principle for axisymmetric flow have been derived in order to place the discussion of temperature potential in context. The governing equations and accompanying boundary conditions have been derived, and the relationship of these new eigensolutions to the original Onsager eigensolution is explained in terms of supersymmetry. The authors state that one of the advantages of the temperature-potential method is that it is easier to study the non-axisymmetric flows, since the nonaxisymmetric temperature operator can be constructed by considering it to be a small perturbation of the self-adjoint operator associated with the axisymmetric temperature potential.

Two-dimensional continuum simulations have also been carried out in the $(r - \theta)$ plane for the rotating flow past a fixed object. Here, one of the objectives is to analyse the high-speed shock wave that forms when a rotating flow encounters a stationary object. Hittinger, Holt & Soubbaramayer (1981) and Hittinger, Holt, Soubbaramayer & Cortet (1983) obtained a bow shock in front of a stationary object in their simulations. Two-dimensional numerical simulation of flow past a stationary object have been done by Mikami (1981) and Mikami (1985), and three dimensional inlet-flows has been simulated by Aoki & Suzuki (1985). In the simulations, they assumed a uniform upstream flow,

and were able to resolve the shock waves that result. The centrifugally driven thermal convection near the cooled rotor cover of a centrifuge studied by Mikami (1973). In this analysis, the boundary layer equations are reduced to ordinary differential equations by the Lees-Levy transformation, and solved numerically by applying the quasi-linearization method. Sample solutions for radial and azimuthal velocity profiles are presented. Kai (1983) studied the drag on the gas due to a stationary cylindrical object, and then equated it with a viscous drag due to an axisymmetric equivalent disk. A modified version of the Newton method was adopted to get the full solution in static state for nonlinear equations.

The secondary flows generated inside a high speed rotating cylinder have been studied by numerical solutions of the compressible Navier-Stokes equations by many authors in two-dimensional axisymmetric ($r - z$) configuration, in the polar ($r - \theta$) configuration, and in three dimensional ($r - \theta - z$) configuration. These equations are solved, subject to boundary conditions, to determine the density, velocity and temperature fields. Nakayama & Torii (1974) analysed the Stewartson layers by assuming that the height is larger than the thickness of the $\text{Re}^{-\frac{1}{4}}R$ and $\text{Re}^{-\frac{1}{3}}R$ layers, by expanding the equations in terms of the ratio of thickness to the height. Bark & Bark (1976) determined the exact solution for the $\text{Re}^{-\frac{1}{4}}R$ layer and asymptotic solutions for the $\text{Re}^{-\frac{1}{3}}R$ layer, as well as numerical solutions, assuming that the cylinder walls are thermally conducting, and the wall temperature is a constant. The work of Matsuda (1976a) was one of the first to stress the importance of the thermal boundary condition on the flow field, particularly if the walls are thermally insulated. The combined effect of the compressibility of the fluid and the thermal insulation of the walls produces a flow distinctly different from that of Boussinesq fluids, and from that of compressible fluids confined in a cylinder with thermally conducting walls. Matsuda & Hashimoto (1978) divided the flow field in r - z plane into several regions, the inner inviscid core, the Stewartson $\text{Re}^{-\frac{1}{4}}R$ layer along the sidewall, its extension to the top and bottom Ekman layers with thickness $\text{Re}^{-\frac{1}{2}}R$, the Stewartson $\text{Re}^{-\frac{1}{3}}R$ layer and its extension to the Ekman layers. In a frame rotating with the angular velocity Ω there is no flow in the inner core on account of the Taylor-Proudman theorem. Along the sidewall, however, there must be an azimuthal flow in order to satisfy the side-wall conditions. This azimuthal flow does not satisfy the boundary conditions at the top and bottom end plates, leading to the formation of the top and bottom Ekman layers with thickness $\text{Re}^{-\frac{1}{2}}R$, which in turn induce a meridional flow of order $\text{Re}^{-\frac{1}{2}}R$ in the $\text{Re}^{-\frac{1}{4}}R$ layer owing to Ekman suction. This flow then circulates through the $\text{Re}^{-\frac{1}{3}}R$ layer and in the Ekman extensions.

Three dimensional Navier-Stokes simulations of flows affected by an inserted stationary object, modeled as a stationary disk extended in the meridional direction, has been performed by Volosciuk (1981) and Walz, Volosciuk & Schutz (1983). Numerical investigations have also been made of thermally, mechanically and thermo-mechanically driven flows of a weakly compressible gas in a cylinder assuming an axisymmetric continuum flow (Harada (1980)). The finite difference method was developed for these computations based on the Euler upwind scheme. It was found that an almost cylindrical Couette flow was established in an annulus, despite the end effects of the plates which generate mechanically driven circulations. The two mechanically driven circulations interact with the thermally driven circulations. The authors also investigated the axisymmetric continuum flow of a strongly compressible rotating gas driven thermally and mechanically in an annulus using finite difference method. The DuFort-Frankel upwind scheme was used for efficient computations of the strongly compressible rotating flows, and it was found that the thermally driven flow becomes weaker as the compressibility is increased. The

viscous dissipation increases the gas temperature of the mechanically driven flow despite cooling by the expansion work.

Sakurai (1981) analyzed the axisymmetric steady centrifugally driven thermal convection of a compressible fluid in a rapidly rotating circular. They clarified the importance of the effect of the flow-induced volume change of a fluid particle, and propose a method of solution. The effect of inserted objects was analysed in terms of linearized thin-wing theory by Sakurai (1981). Their analysis showed that the drag force on one inserted object was not affected by other upstream objects, or by the wake of the same object due to fluid circulation, though the objects do interact with respect to their effect on the flow.

Many previous studies have obtained velocity profiles in rotating flows using the compressible Navier-Stokes equations. There have been studies of high-speed stratified flows past objects (Hausman & Roberts (1984), Roberts (1985), Roberts & Hausman (1988), Roberts & Hausman (1988)). The present work provides an opportunity for validating the analytical model as well as the DSMC method for the high-speed rotating flows in a polar ($r-\theta$) plane. The molecular simulation procedure is more computationally intensive than the continuum Navier-Stokes equations, but it also has two important advantages. The Navier-Stokes equations are written for the macroscopic density and velocity fields, and these are applicable only when the mean free path in the gas is small compared to the macroscopic scale (radius of the cylinder or boundary layer thickness). There are many situations in high-speed rotating flows where the gas density in the central region of the cylinder is sufficiently small that the mean free path is comparable to the cylinder radius. In these cases, the Navier-Stokes equations are not valid near the center, and it is necessary to specify artificial 'boundary conditions' at a surface within the flow. In this context, Roblin & Doneddu (2000) has been performed DSMC simulations to study the feed gas expansion from the central pipe, and the residual gas contained in the casing in which the cylinder rotates. These simulations were used to provide more realistic boundary conditions for continuum fluid flow calculations. The possible dependence of the solutions on these artificial boundary conditions poses difficulties in the interpretation of results. No such difficulties exist in the molecular simulations, since it is only necessary to specify particle-wall interaction parameters (Pradhan & Kumaran (2011); Kumaran & Pradhan (2014)).

The another advantage of the molecular simulations is that the velocity slip at the wall resulting from the particle-wall interaction law can be obtained directly from the simulations (Pradhan & Kumaran (2011), Kumaran & Pradhan (2014)), whereas the slip velocity has to be modeled in the continuum numerical solutions. In the present simulations, we also find that there is substantial 'temperature slip' in the DSMC simulations, that is, the gas temperature at the wall is not equal to the wall temperature. Both velocity and temperature slip at the walls are measured in the DSMC simulations, and these are then incorporated into the analytical solutions, in order to make a quantitative comparison of the results of the analysis and simulations. The comparisons are carried out for stratification parameter in the range 0.707 – 3.535 and for Reynolds numbers as low as 100, in order to quantitatively estimate the accuracy of the analytical model.

In molecular simulations, the direct simulation Monte Carlo (DSMC) technique is most suited for the simulation of rarefied gases. In this technique, the Boltzmann equation is solved using a probabilistic solution procedure. The positions and velocities of simulated molecules in the flow domain are evolved in time. The macroscopic properties (density, velocity, temperature) are determined by averaging over the molecular properties in cells. The positions and velocities between collisions are evolved deterministically using Newton's laws, while a stochastic Monte Carlo technique is used for collisions between the

nearby molecules. Collisions are only permitted between molecules in the same sub-cell to ensure spatial localization. The number of simulated molecules is small compared to the number of real molecules in the system - typically, one simulated molecule represents $(10^{14} - 10^{18})$ real molecules. The advantage of the procedure is that the collision frequency and molecular diameter of the simulated molecules can be adjusted to obtain the same collision frequency and mean free path as the real molecules. Despite this reduction, the number of molecules per cell has to be large for accurate statistical averaging. Typically, $(50 - 100)$ simulated molecules are required per cell for reliable averages to be obtained. Due to this, the total number of molecules becomes very large (about 100 times the number of cells), and the procedure is computationally very intensive. The advantages of this procedure is that properties have to be specified only at the molecular level. It is not necessary to specify continuum parameters such as the equation of state, the viscosity, thermal conductivity and diffusivity. The properties specified are the inter-molecular interaction parameters such as the molecular diameter, molecular mass, and collision cross section, and the molecule-wall collision parameters. Molecular models range from the simplest hard-sphere model, where the molecules are treated as rigid hard spheres, to the soft-sphere model and to more sophisticated models which incorporate the internal energy and the anisotropy in molecular interactions. Similar parameters need to be specified for particle-wall interactions.

The DSMC simulations are computationally very intensive, both with regard to the computational speed and memory requirements. In two dimensions, a proper discretization requires of the order of $(10^4 - 10^5)$ cells, and consequently the number of molecules is $(10^6 - 10^7)$. In the computation, if we assign 100 bytes for the position and velocity of each molecule, the access memory required is $(10^8 - 10^9)$. Thus, 1 GB is the minimum memory required for carrying out simulations even in two dimensions; this is certainly feasible. In three dimensions, the number of cells increases by a further factor of $(10^2 - 10^3)$ in the third dimension, and a consequent increase in the memory requirement of the order of 100 GB to 1 TB. With regard to the speed of computation, it is necessary to generate a large number of random numbers at each collision in the stochastic collision model. This increases significantly the number of computations required at each time step. As computational power increases, discrete simulations are likely to become more and more attractive in comparison to other techniques.

Apart from the large computational requirements, this procedure does not have most of the disadvantages present in continuum models. It is not necessary to specify the equation of state and transport properties, since these emerge naturally by statistical averaging over the molecular properties. There is no difficulty in simulating high Mach number or high Knudsen number flows because a molecular description is being used. Velocity and temperature slip at the walls are accurately captured once the particle-wall interaction parameters are specified. There is no need to make a transition from a continuum to free-molecule regime because the description is at the molecular level. Consequently, there is also no need to specify inner boundary conditions.

The implementation of sources of mass, momentum and energy within the flow is technically non-trivial in the DSMC simulation. In order to mimic a Dirac delta function in the simulations, mass, momentum and energy are inserted/removed over a small source region of radius 10^{-3} times the cylinder radius at the designated location. When quantitative comparison has to be made with analytical solutions for specified rates of input/output of mass, momentum and energy, care has to be taken in the simulations due to the coupling between mass, momentum and energy sources. When molecules are added in simulations, there is the simultaneous addition of the momentum and energy of these molecules as well. In order to add mass alone, it is necessary to reduce the

momentum and energy of all molecules in the source region so that total momentum and energy are conserved. In a similar manner, momentum is added by increasing the mean velocity of the molecules in the source region, while addition of energy requires increasing the fluctuating velocity of the molecules while keeping the total mean velocity a constant. Another complication is that addition of energy at a location near the wall in the simulation causes an increase in the local fluctuating velocities of molecules, which heats up the wall. This violates the constant temperature boundary condition that is used in the analytical solution, and so it is necessary to extract heat locally at the wall in the simulation to ensure that the wall temperature is maintained a constant (Pradhan & Kumaran (2011), Kumaran & Pradhan (2014)).

Here, we formulate a generalized analytical model in the polar $(r - \theta)$ plane in a high speed rotating cylinder, where the restriction on high stratification parameter is removed. In order to set the context for the numerical comparison, we revisit the approximations in the analytical model presented here in the polar $(r - \theta)$ plane in a high speed rotating cylinder of radius R . The starting point of the analytical model is the axisymmetric Navier-Stokes, mass, momentum and energy conservation equations in the polar $(r - \theta)$ coordinate, where r and θ are the radial and azimuthal coordinates. The base state is a solid-body rotation, in which there is an exponential decay of the density from the wall of the cylinder proportional to $\rho_w \exp(-\mathcal{A}^2(1 - (r^2/R^2)))$ due to the centrifugal force. Here $\mathcal{A} = \sqrt{m\Omega^2 R^2/(2k_B T)}$ is the stratification parameter, m is the molecular mass, Ω and R are the angular velocity and radius of the cylinder, k_B is the Boltzmann constant, T is the absolute temperature, and ρ_w is the gas density at the wall. The equations are linearized in the amplitudes of the secondary radial flow velocity, density and temperature about a base state which is a solid body rotation. A scaled radial coordinate is defined by $\xi = \mathcal{A}^2 (1 - (r^2/R^2))$, and it is assumed that the Reynolds number $(\rho_w \Omega R^2/\mu)$ is large. However, there is no restriction on the stratification parameter, and since we have used the conservative form of the compressible mass, momentum and energy conservation equations, in deriving the generalized analytical model, obviously, the shock type solutions (Rankine-Hugoniot relations) are automatically satisfied. In this limit, the equations are reduced to a sixth-order differential equation in ξ for the 'master potential' χ , which is related to the stream function ψ by the relation $\psi = (1/r)(\partial\chi/\partial r)$. After the approximations, the linear operator in the generalized analytical model is dependent explicitly on the stratification parameter, though it is still a self-adjoint operator with discrete eigenvalues and orthogonal eigenfunctions if the boundary conditions are properly specified. In addition, the equations are solved on a finite domain, $0 \leq \xi \leq \mathcal{A}^2$.

The effect of a stationary insert in the flow domain is of three types. The first is the mass source or sink due to the addition or removal of gas in a particular region in the flow domain. The second effect is due to the force exerted on the gas by a stationary inserts, which is modelled locally as a momentum source or sink. The time rate of addition/removal of momentum is equal to the applied force. The third effect is a localised energy source, which increases or decreases the temperature of the gas. All the three effects are incorporated in the present molecular simulation (direct simulation Monte Carlo (DSMC) method) as follows. The addition or removal of mass is accomplished by either adding or removing molecules at specified locations in the simulation cell at specified rates. The pressure is specified at inflow and outflow boundaries by connecting a pressure bath, which is a large volume of gas maintained at a constant pressure. Forces are exerted by imposing impulses on the molecules at specified locations, in such a manner that the rate of change of impulse is equal to the applied force. Energy is added by

increasing the mean square velocities of the molecules at locations where heating takes place.

In §2, the mass, momentum and energy conservation equations in the polar ($r-\theta$) plane in a high speed rotating cylinder are presented and different dimensionless numbers are defined. The equations for perturbations to the base state (solid body rotation) are then used to get the equation for the master potential in §3. In §4, the radial boundary layer equations are solved to obtain the generalized analytical equation in the polar ($r-\theta$) plane. In §5, a brief summary of the molecular simulation based on direct simulation Monte Carlo (DSMC) method is presented. In §6, flow fields due to the insertion of mass, momentum and energy into the rotating cylinder are studied for a single-component gas for stratification parameter ($\mathcal{A}$) in the range (0.707 – 3.535), and analytical results are compared with direct simulation Monte Carlo (DSMC) simulations. In §7, DSMC simulations are carried out for a binary gas mixture and results are compared with single-component gas. The important conclusions of the analysis are summarized in §8.

2. Formulation

The configuration consists of a right circular cylinder of radius R and angle ($\theta = 2\pi$) rotating at a constant angular velocity Ω . A cylindrical coordinate system with origin at the centre of the cylinder ($r = 0$), with radial, and meridional coordinates r , and θ respectively, is used. Throughout this analysis, we assume that the flow is in the ($r-\theta$) plane, so there is no dependence on the axial coordinate z . Gravitational acceleration is neglected, since it has a negligible effect on high-speed gas flows. The compressible mass, momentum and energy conservation equations in cylindrical polar ($r-\theta$) coordinate system are provided in the appendix for reference. There are six variables in these equations, the density ρ , the three components of the velocity v_r, v_θ, v_z , the pressure p and the temperature T . In equations A 1 to A 5, S_m is the mass source density (mass input per unit volume per unit time), f_r, f_θ and f_z are the force densities in r, θ and z directions respectively, and S_e is the energy source density (energy input per unit volume per unit time).

The pressure is related to the density by the ideal gas equation of state,

$$\rho = \frac{pm}{k_B T}, \quad (2.1)$$

where m is the mass of a molecule and k_B is the Boltzmann constant.

If the temperature on the walls of the cylinder is uniform, and there is no inflow or outflow of gas, a steady solution for the equations of motion is one where the gas undergoes solid-body rotation with angular velocity Ω at constant temperature. The temperature $\bar{T}$ and the components of the velocity $\bar{\mathbf{v}}$ in the base state in this steady state are,

$$\bar{v}_\theta = \Omega r, \quad \bar{v}_z = 0, \quad \bar{v}_r = 0, \quad \bar{T} = T_0, \quad (2.2)$$

where T_0 is the constant temperature in the steady state. The pressure is obtained by solving the radial momentum equation,

$$\frac{d\bar{p}}{dr} = \bar{\rho}\Omega^2 r = \frac{\bar{p}m\Omega^2 r}{k_B T_0}, \quad (2.3)$$

where $\bar{\rho}$ is the density at steady state, together with the ideal gas equation of state,

$$\bar{\rho} = \frac{\bar{p}m}{k_B T_0}, \quad (2.4)$$

where m is the mass of a molecule and k_B is the Boltzmann constant. The above equation is easily solved to obtain

$$\frac{\bar{p}}{\bar{p}_w} = \frac{\bar{\rho}}{\bar{\rho}_w} = \exp\left(-\mathcal{A}^2\left(1 - \frac{r^2}{R^2}\right)\right), \quad (2.5)$$

where $\bar{p}_w$ and $\bar{\rho}_w$ are the pressure and density at the wall, and $\mathcal{A}$ is the stratification parameter. The total mass of gas in the cylinder is related to ρ_w by,

$$\text{Mass of gas} = \frac{\pi R^2 Z \bar{\rho}_w (1 - \exp(-\mathcal{A}^2))}{\mathcal{A}^2}, \quad (2.6)$$

where Z is the total height of the cylinder.

It is useful to discuss the different dimensionless numbers used in the analysis. The stratification parameter obtained from the equation for the density is,

$$\mathcal{A} = \sqrt{\frac{m\Omega^2 R^2}{2k_B T_0}} \propto \frac{R\Omega}{c_s}, \quad (2.7)$$

where $(R\Omega)$ is the characteristic flow velocity, and $c_s = \sqrt{\gamma k_B T_0 / m}$ is the speed of sound. The Reynolds number, used a little later, is defined by

$$\text{Re} = \frac{\rho_w \Omega R^2}{\bar{\mu}}, \quad (2.8)$$

where $\bar{\mu}$ is the viscosity of the fluid in the unperturbed state. Note that the viscosity of a dilute gas depends only on the temperature, and not on the density. Therefore, the viscosity is a constant in the base state where the temperature is uniform. The viscosity of a dilute gas scales as,

$$\bar{\mu} \sim \frac{\sqrt{mk_B T_0}}{d^2}, \quad (2.9)$$

where m is the molecular mass and d is the molecular diameter. Using the above expression for the viscosity, it can easily be verified that the Reynolds number can be rewritten as,

$$\text{Re} \sim \frac{\mathcal{A}}{\text{Kn}}, \quad (2.10)$$

where the Knudsen number is defined by

$$\text{Kn} = (m/\sqrt{2\pi}\rho_w d^2 R) = (\lambda/R), \quad (2.11)$$

where $\lambda = (m/\sqrt{2\pi}\rho_w d^2)$ is the mean free path at the wall. The other dimensionless groups used are the Prandtl number and the ratio of specific heats,

$$\text{Pr} = \frac{C_p \bar{\mu}}{\bar{K}}, \quad (2.12)$$

$$\gamma = \frac{C_p}{C_v}, \quad (2.13)$$

which are both $O(1)$ in the ideal gas limit.

3. Equations of motion in polar $(r - \theta)$ coordinate

The disturbance to the base state due to insertion of small mass, momentum and energy into the rotating cylinder, is incorporated by perturbing the base velocity and

pressure profile as follows:

$$\rho = \bar{\rho} + \hat{\rho}, \quad (3.1)$$

$$p = \bar{p} + \hat{p},$$

$$T = \bar{T} + \hat{T},$$

$$v_r = \hat{v}_r,$$

$$v_\theta = r\Omega + \hat{v}_\theta,$$

$$v_z = \hat{v}_z. \quad (3.2)$$

The above forms for the dynamical variables are inserted into the mass, momentum and energy conservation equations, and linearized in the perturbation (hatted) quantities. Steady state is assumed, so that all the time derivatives are zero. After subtracting out the corresponding base state equations, we obtain the following equations for the perturbations:

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{\rho} \hat{v}_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\bar{\rho} \hat{v}_\theta + \hat{\rho} \Omega r) = S_m, \quad (3.3)$$

$$-\hat{\rho} \Omega^2 r - 2\bar{\rho} \Omega \hat{v}_\theta + \bar{\rho} \Omega \frac{\partial \hat{v}_r}{\partial \theta} = -\frac{\partial \hat{p}}{\partial r} + \bar{\mu} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial \hat{v}_\theta}{\partial \theta} \right) + \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right), \quad (3.4)$$

$$2\bar{\rho} \Omega \hat{v}_r + \bar{\rho} \Omega \frac{\partial \hat{v}_\theta}{\partial \theta} = -\frac{1}{r} \frac{\partial \hat{p}}{\partial \theta} + \bar{\mu} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial \hat{v}_r}{\partial \theta} \right) + \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial \theta} \right), \quad (3.5)$$

$$\bar{\rho} \Omega \frac{\partial \hat{v}_z}{\partial \theta} = \bar{\mu} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{v}_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_z}{\partial \theta^2} \right) + f_z, \quad (3.6)$$

$$-\bar{\rho} \Omega^2 r \hat{v}_r + \bar{\rho} \Omega C_p \frac{\partial \hat{T}}{\partial \theta} = -\bar{p} S_m + \bar{K} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{T}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \hat{T}}{\partial \theta^2} \right) + S_e. \quad (3.7)$$

Here, $\bar{\mu}$ and $\bar{K}$ are the viscosity and thermal conductivity of the gas in the base state, where the temperature is T_0 . Note that in the ideal gas limit, the viscosity and thermal conductivity are functions of the temperature alone, and they do not depend on the density. In equation 3.4, the term on the left is the contribution to the r momentum equation due to the centrifugal force, while in equation 3.5, the left side represents the contribution to the θ momentum equation due to the Coriolis force. In the energy equation 3.7, the term on the left is the pressure work done due to the compression or expansion of the gas, and we have used the equation of state 2.4 and the mass conservation equation 3.3 to simplify this term. The perturbation to the equation of state is

$$\hat{\rho} = \left(\frac{m}{k_B T_0} \right) \hat{p} - \bar{\rho} \left(\frac{\hat{T}}{T_0} \right). \quad (3.8)$$

First, we eliminate the density in the r momentum equation, 3.4, using the equation of state, equation 3.8, to obtain

$$\begin{aligned} & \bar{\rho} \Omega \frac{\partial \hat{v}_r}{\partial \theta} - \frac{\Omega^2 r m \hat{p}}{k_B T_0} + \frac{\Omega^2 r \bar{\rho} \hat{T}}{T_0} - 2\bar{\rho} \Omega \hat{v}_\theta + \frac{\partial \hat{p}}{\partial r} \\ &= \bar{\mu} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial \hat{v}_\theta}{\partial \theta} \right) + \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right). \end{aligned} \quad (3.9)$$

The identity,

$$\begin{aligned}\bar{\rho} \frac{\partial}{\partial r} \left(\frac{\hat{p}}{\bar{\rho}} \right) &= \frac{\partial \hat{p}}{\partial r} - \frac{\hat{p}}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial r} \\ &= \frac{\partial \hat{p}}{\partial r} - \frac{\hat{p} m \Omega^2 r}{k_B T_0},\end{aligned}\quad (3.10)$$

can be inserted into the momentum conservation equation, 3.9, to obtain

$$\begin{aligned}&\frac{\Omega}{r} \frac{\partial \hat{v}_r}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\hat{p}}{\bar{\rho}} \right) - \frac{2\Omega \hat{v}_\theta}{r} - \frac{\Omega^2 \hat{T}}{T_0} \\ &= \frac{\bar{\mu}}{\bar{\rho} r} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial \hat{v}_\theta}{\partial \theta} \right) + \frac{1}{\bar{\rho} r} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right).\end{aligned}\quad (3.11)$$

If we take the derivative of the above equation with respect to the θ coordinate, we obtain

$$\begin{aligned}&\frac{\Omega}{r} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}} \frac{\partial \hat{p}}{\partial \theta} \right) - \Omega \frac{d}{d\theta} \left(\frac{2\hat{v}_\theta}{r} - \frac{\Omega \hat{T}}{T_0} \right) \\ &= \frac{\bar{\mu}}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial \hat{v}_\theta}{\partial \theta} \right) + \frac{1}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right).\end{aligned}\quad (3.12)$$

Equation 3.5 is used to substitute for $(\partial \hat{p} / \partial \theta)$ in the above equation, to obtain

$$\begin{aligned}&\frac{\Omega}{r} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(-2\bar{\rho} \Omega \hat{v}_r - \bar{\rho} \Omega \frac{\partial \hat{v}_\theta}{\partial \theta} + \bar{\mu} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_\theta) \right) \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{r^2} \frac{\partial^2 \hat{v}_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial \hat{v}_r}{\partial \theta} + \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial \theta} \right) - \Omega \frac{d}{d\theta} \left(\frac{2\hat{v}_\theta}{r} - \frac{\Omega \hat{T}}{T_0} \right) \right) \right. \\ &\quad \left. \left. \frac{\bar{\mu}}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) + \frac{1}{r^2} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial \hat{v}_\theta}{\partial \theta} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right) \right) \right).\end{aligned}\quad (3.13)$$

Divided by $\bar{\mu}$ and after simplification it gives,

$$\begin{aligned}&\frac{\Omega}{\bar{\mu} r} \frac{\partial^2 \hat{v}_r}{\partial \theta^2} + \frac{1}{\bar{\mu} r} \frac{\partial}{\partial r} \left(-2r\Omega \hat{v}_r - r\Omega \frac{\partial \hat{v}_\theta}{\partial \theta} \right) \\ &\quad + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_\theta) \right) \right) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho} r} \frac{\partial^2 \hat{v}_\theta}{\partial \theta^2} \right) \\ &\quad + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2}{\bar{\rho} r} \frac{\partial \hat{v}_r}{\partial \theta} \right) - \frac{1}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \hat{v}_r) \right) \right) \\ &\quad + \frac{1}{\bar{\rho} r^3} \left(-\frac{\partial^3 \hat{v}_r}{\partial \theta^3} + 2 \frac{\partial^2 \hat{v}_\theta}{\partial \theta^2} \right) - \frac{\Omega}{\bar{\mu}} \frac{d}{d\theta} \left(\frac{2\hat{v}_\theta}{r} - \frac{\Omega \hat{T}}{T_0} \right) = \\ &\quad \frac{1}{\bar{\mu}} \left(\frac{1}{\bar{\rho} r} \frac{\partial}{\partial \theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial \theta} \right) \right) \right).\end{aligned}\quad (3.14)$$

Here, the viscosity $\bar{\mu}$ in the base state is independent of position, because the temperature is uniform and the viscosity depends only on temperature.

The velocity components $\hat{v}_r$ and $\hat{v}_\theta$ can be expressed in terms of a stream function

using the mass conservation equation 3.3,

$$r\bar{\rho}\hat{v}_r = -\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr, \quad (3.15)$$

$$\bar{\rho}\hat{v}_\theta = \frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r, \quad (3.16)$$

where the difference in the stream function between two radial locations gives the angular mass flux in the region between these two locations,

$$\psi(a, \theta) - \psi(r, \theta) = \int_r^a (\bar{\rho}\hat{v}_\theta + \hat{\rho}\Omega r) dr. \quad (3.17)$$

Expressing in terms of the stream function, equation 3.14 is

$$\begin{aligned} & \frac{\Omega}{\bar{\mu}r} \frac{\partial^2}{\partial\theta^2} \left(\frac{1}{\bar{\rho}r} \left(-\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr \right) \right) + \frac{1}{\bar{\mu}r} \frac{\partial}{\partial r} \left(\frac{-2\Omega}{\bar{\rho}} \left(-\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr \right) \right) \\ & - r\Omega \frac{\partial}{\partial\theta} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r \right) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r \right) \right) \right) \right) \right) \right) \\ & + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial^2}{\partial\theta^2} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r \right) \right) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2}{\bar{\rho}^2 r^2} \frac{\partial}{\partial\theta} \left(-\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr \right) \right) \\ & - \frac{1}{\bar{\rho}r} \frac{\partial}{\partial\theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}} \left(-\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr \right) \right) \right) \right) - \frac{1}{\bar{\rho}^2 r^4} \frac{\partial^3}{\partial\theta^3} \left(-\frac{\partial\psi}{\partial\theta} + \int_0^r S_m r dr \right) \\ & + \frac{2}{\bar{\rho}^2 r^3} \frac{\partial^2}{\partial\theta^2} \left(\frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r \right) - \frac{\Omega}{\bar{\mu}} \frac{d}{d\theta} \left(\frac{2}{\bar{\rho}r} \left(\frac{\partial\psi}{\partial r} - \hat{\rho}\Omega r \right) - \frac{\Omega\hat{T}}{T_0} \right) = \\ & \frac{1}{\bar{\mu}} \left(\frac{1}{\bar{\rho}r} \frac{\partial}{\partial\theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial\theta} \right) \right) \right). \quad (3.18) \end{aligned}$$

The stream function ψ is expressed in terms of the ‘master potential’ χ ,

$$\psi = \frac{1}{r} \frac{\partial\chi}{\partial r}. \quad (3.19)$$

This is inserted into the momentum conservation equation 3.13 to obtain

$$\begin{aligned} & \frac{\Omega}{\bar{\mu}r} \frac{\partial^2}{\partial\theta^2} \left(\frac{1}{\bar{\rho}r} \left(-\frac{\partial}{\partial\theta} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) + \frac{1}{\bar{\mu}r} \frac{\partial}{\partial r} \left(\frac{-2\Omega}{\bar{\rho}} \left(-\frac{\partial}{\partial\theta} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \\ & - r\Omega \frac{\partial}{\partial\theta} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \right) \right) \right) \right) \\ & + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial^2}{\partial\theta^2} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2}{\bar{\rho}^2 r^2} \frac{\partial}{\partial\theta} \left(-\frac{\partial}{\partial\theta} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \\ & - \frac{1}{\bar{\rho}r} \frac{\partial}{\partial\theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}} \left(-\frac{\partial}{\partial\theta} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \right) \right) - \frac{1}{\bar{\rho}^2 r^4} \frac{\partial^3}{\partial\theta^3} \left(-\frac{\partial}{\partial\theta} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) + \int_0^r S_m r dr \right) \\ & + \frac{2}{\bar{\rho}^2 r^3} \frac{\partial^2}{\partial\theta^2} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) - \hat{\rho}\Omega r \right) - \frac{\Omega}{\bar{\mu}} \frac{d}{d\theta} \left(\frac{2}{\bar{\rho}r} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) - \frac{\Omega^2}{\bar{\mu}} \frac{\partial}{\partial\theta} \left(\frac{\hat{T}}{T_0} \right) = \\ & \frac{1}{\bar{\mu}} \left(\frac{1}{\bar{\rho}r} \frac{\partial}{\partial\theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial\theta} \right) \right) \right). \quad (3.20) \end{aligned}$$

4. Radial boundary layer

When the boundary layer is in the radial coordinate, the radial derivatives of $\hat{v}_\theta$ and $\hat{T}$ on the right sides of equations 3.5 and 3.7 are balanced by the left sides. The temperature perturbation $\hat{T}$ in 3.20 can be expressed in terms of the master potential from the energy equation, 3.7. The energy equation is

$$-\bar{\rho}\Omega^2 r \hat{v}_r + \bar{\rho}\Omega C_p \frac{\partial \hat{T}}{\partial \theta} = -\bar{p}S_m + \bar{K} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{T}}{\partial r} \right) + \frac{\partial^2 \hat{T}}{\partial z^2} \right) + S_e. \quad (4.1)$$

Expressing the radial velocity $\hat{v}_r$ in terms of the master potential using 3.15 and 3.19, we obtain

$$\frac{\Omega^2}{r} \frac{\partial}{\partial r} \left(\frac{\partial \chi}{\partial \theta} \right) - \Omega^2 \int_{r_0}^r S_m r dr + \bar{\rho} C_p \Omega \frac{\partial \hat{T}}{\partial \theta} = \bar{K} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{T}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \hat{T}}{\partial \theta^2} \right) + (S_e - \bar{p}S_m). \quad (4.2)$$

The above equation can be multiplied by r , integrated once with respect to the radial coordinate with lower limit r_0 and upper limit r , to obtain

$$\begin{aligned} \Omega^2 \left(\frac{\partial \chi}{\partial \theta} \right) - \Omega^2 \int_{r_0}^r r \left(\int_{r_0}^r S_m r dr \right) dr + \int_{r_0}^r r dr \bar{\rho} C_p \Omega \left(\frac{\partial \hat{T}}{\partial \theta} \right) = \\ \bar{K} \left(r \frac{\partial \hat{T}}{\partial r} \right) \Big|_{r=r_0} - \bar{K} r \frac{\partial \hat{T}}{\partial r} \Big|_{r=r_0} + \int_{r_0}^r \frac{dr}{r} \left(\frac{\partial^2 \hat{T}}{\partial \theta^2} \right) \\ + \int_{r_0}^r (S_e - \bar{p}S_m) r dr. \end{aligned} \quad (4.3)$$

This is then multiplied by $(\Omega^2/\bar{\mu}T_0r^2)$, and differentiated once with respect to θ , to obtain

$$\begin{aligned} \frac{\Omega^4}{\bar{K}\bar{\mu}T_0r^2} \left(\frac{\partial^2 \chi}{\partial \theta^2} \right) - \frac{\Omega^4}{\bar{K}\bar{\mu}T_0r^2} \frac{\partial}{\partial \theta} \int_{r_0}^r r \left(\int_{r_0}^r \frac{\partial S_m}{\partial \theta} r dr \right) dr + R_T + C_T \\ = \frac{\Omega^2}{\bar{\mu}} \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \left(\frac{\hat{T}}{T_0} \right) + \frac{\Omega^2}{\bar{K}\bar{\mu}T_0r^2} \frac{\partial}{\partial \theta} \int_{r_0}^r (S_e - \bar{p}S_m) r dr, \end{aligned} \quad (4.4)$$

where

$$R_T = \frac{\Omega^2}{\bar{K}\bar{\mu}T_0r^2} \int_{r_0}^r \frac{dr}{r} \left(\frac{\partial^3 \hat{T}}{\partial \theta^3} \right) + \frac{\Omega^2}{\bar{K}\bar{\mu}T_0r^2} \int_{r_0}^r r dr \bar{\rho} C_p \Omega \left(\frac{\partial^2 \hat{T}}{\partial \theta^2} \right), \quad (4.5)$$

and the term appearing due to the constant of integration in equation 4.3 is,

$$C_T = \frac{\Omega^2}{\bar{\mu}T_0r^2} \frac{\partial}{\partial \theta} \left(r \frac{\partial \hat{T}}{\partial r} \right) \Big|_{r=r_0} - \frac{\Omega^4}{\bar{K}\bar{\mu}T_0r^2} \left(\frac{\partial^2 \chi}{\partial \theta^2} \right) \Big|_{r=r_0}, \quad (4.6)$$

where r_0 is a reference location. For the flow in a cylinder, $r_0 = 0$ (at the centre).

We take the derivative $(1/r)(\partial/\partial r)$ of the equation for the master potential, equation 3.20, and then insert the equation 4.4, to obtain

$$\begin{aligned} \frac{\Omega}{\bar{\mu}r} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial \theta^2} \left(\frac{1}{\bar{\rho}r} \left(-\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \right) \\ + \frac{1}{\bar{\mu}} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{-2\Omega}{\bar{\rho}} \left(-\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \right) \end{aligned}$$

$$\begin{aligned}
& -r\Omega \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial}{\partial \theta} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) \right) \right) \right) \\
& -\hat{\rho}\Omega r + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \right) \right) \right) \right) \\
& + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial^2}{\partial \theta^2} \left(\frac{1}{\bar{\rho}} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \right) \right) \\
& + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2}{\bar{\rho}^2 r^2} \frac{\partial}{\partial \theta} \left(-\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \right) \\
& - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}} \left(-\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \right) \right) \right) \\
& - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}^2 r^4} \frac{\partial^3}{\partial \theta^3} \left(-\frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) + \int_0^r S_m r dr \right) \right) \\
& + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{2}{\bar{\rho}^2 r^3} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \\
& - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\Omega}{\bar{\mu}} \frac{d}{d\theta} \left(\frac{2}{\bar{\rho}r} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) - \hat{\rho}\Omega r \right) \right) \right) \\
& - \frac{\Omega^4}{\bar{K}\bar{\mu}T_0 r^2} \left(\frac{\partial^2 \chi}{\partial \theta^2} \right) + R_T + C_T \\
& = \frac{\Omega^2}{\bar{K}\bar{\mu}T_0 r^2} \frac{\partial}{\partial \theta} \int_{r_0}^r (S_e - \bar{p}S_m) r dr \\
& + \frac{1}{\bar{\mu}} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial}{\partial \theta} \left(f_r + \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial r} \right) \right) \\
& - \frac{1}{\bar{\mu}} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{\bar{\rho}r} \frac{\partial}{\partial r} \left(\frac{r}{\bar{\rho}} \left(f_\theta + \frac{1}{r} \frac{\bar{\mu}}{3} \frac{\partial S_m}{\partial \theta} \right) \right) \right). \quad (4.7)
\end{aligned}$$

The radial and azimuthal coordinates are expressed in terms of the scaled radial ξ , and azimuthal coordinates θ^* , defined by

$$\xi = \mathcal{A}^2 \left(1 - \frac{r^2}{R^2} \right), \quad (4.8)$$

$$\theta^* = \frac{\theta}{2\pi}. \quad (4.9)$$

With these transformation, it can easily be verified that

$$\frac{1}{r} \frac{\partial}{\partial r} = -\frac{2\mathcal{A}^2}{R^2} \frac{\partial}{\partial \xi}, \quad (4.10)$$

$$\frac{\partial}{\partial \theta} = \frac{1}{2\pi} \frac{\partial}{\partial \theta^*}, \quad (4.11)$$

where $\delta = (R/\sqrt{2}\mathcal{A})$ is the boundary layer thickness in the radial coordinate. When the equation for the master potential is expressed in terms of these scaled co-ordinates, and the base density profile

$$\bar{\rho} = \rho_w \exp(-\xi) \quad (4.12)$$

is used, we obtain

$$\frac{64\mathcal{A}^{12}}{R^8 \rho_w^2} \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 \chi}{\partial \xi^2} \right) \right)$$

$$\begin{aligned}
& + \frac{\Omega \mathcal{A}^4}{2\pi^3 \bar{\mu} \rho_w R^5} \frac{1}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \frac{\partial}{\partial \xi} \left(\frac{\partial^2}{\partial \theta^{*2}} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \frac{\partial}{\partial \theta^*} \frac{\partial \chi}{\partial \xi} \right) \right) \\
& \quad - \frac{8\Omega \mathcal{A}^6}{\pi \bar{\mu} R^6 \rho_w} \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(\frac{\partial}{\partial \theta^*} \frac{\partial \chi}{\partial \xi} \right) \right) \\
& \quad + \frac{4\mathcal{A}^6 \Omega}{\pi \rho_w R^4} \frac{\partial}{\partial \xi} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial}{\partial \theta^*} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2 \chi}{\partial \xi^2} \right) \\
& \quad + \frac{4\mathcal{A}^8}{\pi^2 \rho_w^2 R^8} \frac{\partial^2}{\partial \xi^2} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \left(\frac{\partial^2}{\partial \theta^{*2}} e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2 \chi}{\partial \xi^2} \right) \right) \\
& \quad + \frac{4\mathcal{A}^6}{\pi^2 \rho_w^2 R^8} \frac{\partial^2}{\partial \xi^2} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)} \left(\frac{\partial}{\partial \theta^*} \frac{\partial^2 \chi}{\partial \theta^* \partial \xi} \right) \right) \\
& \quad + \frac{4\mathcal{A}^8}{\pi^2 \rho_w^2 R^8} \frac{\partial}{\partial \xi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \left(\frac{\partial}{\partial \theta^*} \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \frac{\partial^2 \chi}{\partial \theta^* \partial \xi} \right) \right) \right) \\
& \quad + \frac{\mathcal{A}^4}{2\pi^3 \rho_w^2 R^8} \frac{\partial}{\partial \xi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^2} \left(\frac{\partial^3}{\partial \theta^{*3}} \left(\frac{\partial \chi}{\partial \xi} \right) \right) \right) \\
& \quad + \frac{4\mathcal{A}^6}{\pi^2 \rho_w^2 R^8} \frac{\partial}{\partial \xi} \left(\frac{e^{2\xi}}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{3/2}} \left(\frac{\partial^2}{\partial \theta^{*2}} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2 \chi}{\partial \xi^2} \right) \right) \\
& \quad + \frac{8\mathcal{A}^6 \Omega}{\pi \rho_w \bar{\mu} R^6} \frac{\partial}{\partial \xi} \left(\frac{\partial}{\partial \theta^*} \left(e^\xi \frac{\partial^2 \chi}{\partial \xi^2} \right) \right) \\
& \quad + \frac{\Omega^4}{2\pi \bar{K} T_0 \left(1 - \frac{\xi}{\mathcal{A}^2}\right)} \left(\frac{\partial^2 \chi}{\partial \theta^{*2}} \right) \\
& \quad + R_T + C_T = S, \tag{4.13}
\end{aligned}$$

where S represents the source term in the equation for the master potential obtained from the mass, momentum and energy sources.

The above equation can be reduced to dimensionless form by dividing by the prefactor of the first term on the left, and defining a scaled master potential. The scaling for master potential (χ) is best inferred by balancing the first term on the left and the term on the right in equation 4.2 which provides $\chi^* = (\chi / \bar{\mu} R^2 2\pi)$. Using this scaled master potential, we obtain

$$\begin{aligned}
& \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 \chi^*}{\partial \xi^2} \right) \right) \\
& + \frac{1}{128\pi^3 \mathcal{A}^8 \text{Re}} \frac{1}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \frac{\partial}{\partial \xi} \left(\frac{\partial^2}{\partial \theta^{*2}} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{1/2}} \frac{\partial}{\partial \theta^*} \frac{\partial \chi^*}{\partial \xi} \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8\pi\mathcal{A}^6\text{Re}} \frac{\partial^2}{\partial\xi^2} \left(e^\xi \left(\frac{\partial}{\partial\theta^*} \frac{\partial\chi^*}{\partial\xi} \right) \right) \\
& + \frac{1}{16\pi\mathcal{A}^6\text{Re}} \frac{\partial}{\partial\xi} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial}{\partial\theta^*} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2\chi^*}{\partial\xi^2} \right) \\
& + \frac{1}{16\pi\mathcal{A}^4\text{Re}} \frac{\partial^2}{\partial\xi^2} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2}} \left(\frac{\partial^2}{\partial\theta^{*2}} e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2\chi^*}{\partial\xi^2} \right) \right) \\
& + \frac{1}{16\pi^2\mathcal{A}^6\text{Re}} \frac{\partial^2}{\partial\xi^2} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)} \left(\frac{\partial}{\partial\theta^*} \frac{\partial^2\chi^*}{\partial\theta^*\partial\xi} \right) \right) \\
& + \frac{1}{16\pi^2\mathcal{A}^4\text{Re}} \frac{\partial}{\partial\xi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2}} \left(\frac{\partial}{\partial\theta^*} \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial\xi^2} \left(e^\xi \frac{\partial^2\chi^*}{\partial\theta^*\partial\xi} \right) \right) \right) \\
& + \frac{1}{128\pi^3\mathcal{A}^8\text{Re}} \frac{\partial}{\partial\xi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)^2} \left(\frac{\partial^3}{\partial\theta^{*3}} \left(\frac{\partial\chi^*}{\partial\xi} \right) \right) \right) \\
& + \frac{1}{16\pi^2\mathcal{A}^6\text{Re}} \frac{\partial}{\partial\xi} \left(\frac{e^{2\xi}}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{3/2}} \left(\frac{\partial^2}{\partial\theta^{*2}} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} \frac{\partial^2\chi^*}{\partial\xi^2} \right) \right) \\
& + \frac{1}{8\pi\mathcal{A}^6\text{Re}} \frac{\partial}{\partial\xi} \left(\frac{\partial}{\partial\theta^*} \left(e^\xi \frac{\partial^2\chi^*}{\partial\xi^2} \right) \right) \\
& + \frac{2\mathcal{A}^2\text{Pr}(1-\gamma^{-1})}{\left(1 - \frac{\xi}{\mathcal{A}^2} \right)} \frac{\text{Re}^2}{64(2\pi)^2\mathcal{A}^{12}} \left(\frac{\partial^2\chi^*}{\partial\theta^{*2}} \right) \\
& + R_T^* + C_T^* = S^*, \quad (4.14)
\end{aligned}$$

where the Reynolds number, stratification parameter and the Prandtl number are given in equations 2.8, 2.7 and 2.12 respectively, and

$$\begin{aligned}
R_{\hat{T}}^* &= - \left(\frac{1}{2\pi} \right)^4 \frac{1}{\mathcal{A}^{14}} \left(\left(1 - \xi/\mathcal{A}^2 \right)^{-1} \int_{\xi_0}^{\xi} \frac{d\xi}{\left(1 - \xi/\mathcal{A}^2 \right)} \left(\frac{\partial^3 \hat{T}^*}{\partial\theta^{*3}} \right) \right) - \\
& \quad \left(\frac{1}{8\pi^2} \right) \frac{1}{\mathcal{A}^{14}} \left(\left(1 - \xi/\mathcal{A}^2 \right)^{-1} \int_{\xi_0}^{\xi} e^{-\xi} d\xi \left(\frac{\partial^2 \hat{T}^*}{\partial\theta^{*2}} \right) \right), \quad (4.15)
\end{aligned}$$

$$\begin{aligned}
C_{\hat{T}}^* &= - \left(\frac{1}{2\pi} \right) \frac{1}{\mathcal{A}^{10}} \left(\left(1 - \xi/\mathcal{A}^2 \right)^{-1} \frac{\partial}{\partial\theta^*} \left(\left(1 - \xi/\mathcal{A}^2 \right) \frac{\partial \hat{T}^*}{\partial\xi} \right) \Big|_{\xi=\xi_0} \right) + \\
& \quad \left(\frac{1}{2\pi} \right) \frac{1}{\mathcal{A}^{12}} \left(\left(1 - \xi/\mathcal{A}^2 \right)^{-1} \frac{\partial^2 \chi^*}{\partial\theta^{*2}} \Big|_{\xi=\xi_0} \right), \quad (4.16)
\end{aligned}$$

where $\hat{v}_\theta^* = \hat{v}_\theta/(\Omega R)$ and $\hat{T}^* = \hat{T}/T_0$ and S^* is the scaled source term, where

$$\begin{aligned}
S^*(\xi, \theta^*) = & \frac{\Omega}{\bar{\mu} R \rho_w} \frac{1}{4\pi^2} \left(\frac{1}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left(\frac{\partial^2}{\partial \theta^{*2}} \left(e^\xi \int_{\xi_0}^\xi S_m d\xi^* \right) \right) \right) \\
& + \frac{4\mathcal{A}^2 \Omega}{\bar{\mu} R^2 \rho_w} \left(\frac{\partial^2}{\partial \xi^2} \left(e^\xi \int_{\xi_0}^\xi S_m d\xi^* \right) \right) \\
& - \frac{4\mathcal{A}^2}{R^4 \rho_w^2} \frac{1}{2\pi} \left(\frac{\partial^2}{\partial \xi^2} \left(\frac{e^{2\xi}}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)} \int_{\xi_0}^\xi \frac{\partial S_m(\xi^{**}, \theta^*)}{\partial \theta^*} d\xi^* \right) \right) \\
& + \frac{4\mathcal{A}^4}{R^6 \rho_w^2} \frac{1}{2\pi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{\frac{1}{2}}} \left(\frac{\partial}{\partial \theta^*} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{\frac{1}{2}} \frac{\partial^2}{\partial \xi^2} \left(e^\xi \int_{\xi_0}^\xi S_m d\xi^* \right) \right) \right) \\
& - \frac{1}{R^4 \rho_w^2} \frac{1}{8\pi^3} \left(\frac{\partial}{\partial \xi} \left(\frac{e^{2\xi}}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^2} \left(\frac{\partial^3}{\partial \theta^{*3}} \int_{\xi_0}^\xi S_m d\xi^* \right) \right) \right) \\
& - \left(\frac{2\text{Re}^2 \mathcal{A}^2 \text{Pr} (1 - \gamma^{-1})}{(1 - \xi/\mathcal{A}^2)} \right) \frac{1}{256\pi^2 \mathcal{A}^{16}} \frac{R\Omega}{\rho_w} \times \\
& \quad \int_{\xi_0}^\xi \left[\int_{\xi_0}^{\xi^*} \frac{\partial S_m(\xi^{**}, \theta^*)}{\partial \theta^*} d\xi^{**} \right] d\xi^* \\
& + \frac{\Omega^2}{2\mathcal{A}^2 \bar{K} \bar{\mu} T_0} \frac{1}{2\pi} \frac{1}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)} \int_{\xi_0}^\xi \frac{\partial (S_e - p_w e^{-\xi} S_m)}{\partial \theta^*} d\xi^* \\
& - \frac{2\mathcal{A}^2}{\bar{\mu} R^3 \rho_w} \frac{1}{2\pi} \left(\frac{\partial}{\partial \xi} \left(\frac{e^\xi}{\left(1 - \frac{\xi}{\mathcal{A}^2}\right)^{\frac{1}{2}}} \frac{\partial}{\partial \theta^*} \left(f_r - \frac{2\bar{\mu}}{3} \frac{\mathcal{A}^2}{R} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{\frac{1}{2}} \frac{\partial S_m}{\partial \xi} \right) \right) \right) \\
& - \frac{4\mathcal{A}^4}{\bar{\mu} R^3 \rho_w} \left(\frac{\partial^2}{\partial \xi^2} \left(\left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{\frac{1}{2}} e^\xi \left(f_\theta + \frac{1}{R \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{\frac{1}{2}}} \frac{\bar{\mu}}{3} \frac{1}{2\pi} \frac{\partial S_m}{\partial \theta^*} \right) \right) \right). \quad (4.17)
\end{aligned}$$

The equation 4.17 provides the conditions under which the source strengths are significant, that is, for $S_m \sim (R^4 \rho_w^2 / \mathcal{A}^2)$, $f_r \sim (\bar{\mu} R^3 \rho_w / \mathcal{A}^2)$, $f_\theta \sim (\bar{\mu} R^3 \rho_w / \mathcal{A}^4)$ and $S_e \sim (\bar{K} T_0 \bar{\mu} \mathcal{A}^2 / \Omega^2)$.

The following terms in equation 4.14 are neglected to formulate the 'analytical equation' in the polar $(r - \theta)$ plane.

(a) The second term on the left side of equation 4.13 is neglected, because the parameter $(1 / (128\pi^3 \mathcal{A}^8 \text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(b) The third term on the left side of equation 4.13 is neglected, because the parameter $(1 / (8\pi \mathcal{A}^6 \text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(c) The fourth term on the left side of equation 4.13 is neglected, because the parameter $(1 / (16\pi \mathcal{A}^6 \text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(d) The fifth term on the left side of equation 4.13 is neglected, because the parameter $(1/(16\pi\mathcal{A}^4\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(e) The sixth term on the left side of equation 4.13 is neglected, because the parameter $(1/(16\pi^2\mathcal{A}^6\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large. ,

(f) The seventh term on the left side of equation 4.13 is neglected, because the parameter $(1/(16\pi^2\mathcal{A}^4\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(g) The eighth term on the left side of equation 4.13 is neglected, because the parameter $(1/(128\pi^3\mathcal{A}^8\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(h) The ninth term on the left side of equation 4.13 is neglected, because the parameter $(1/(16\pi^2\mathcal{A}^6\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(i) The tenth term on the left side of equation 4.13 is neglected, because the parameter $(1/(8\pi\mathcal{A}^6\text{Re})) \ll 1$. Clearly, this is valid when the Reynolds number (Re) and stratification parameter ($\mathcal{A}$) is large.

(j) The terms R_T^* and C_T^* on the left side of equation 4.13 is also neglected. The value of these terms depends on the location of the boundary surface $\xi = \xi_0$. In the present case, this boundary is at the center of the cylinder. Symmetry requires that the $(\partial T/\partial\theta) = 0$, and $(\partial T/\partial r) = 0$. Therefore, R_T^* and C_T^* are identically zero in this case.

(k) Within the first term on the left of equation 4.13, a term proportional to $(\xi/\mathcal{A}^2)$ can be neglected by assuming that $(1 - \xi/\mathcal{A}^2) \approx 1$. Clearly, this is valid when the stratification parameter $\mathcal{A} \gg 1$. Similarly, within the eleventh term on the left, a term proportional to $(\xi/\mathcal{A}^2)$ can be neglected by assuming that $(1 - \xi/\mathcal{A}^2) \approx 1$. Clearly, this is valid when the stratification parameter $\mathcal{A} \gg 1$.

Therefore, in the limit of $\text{Re} \gg 1$ and $\mathcal{A} \gg 1$, the equations 4.14 can be reduced to

$$\frac{\partial^2}{\partial \xi^2} \left(e^\xi \frac{\partial^2}{\partial \xi^2} \left(e^\xi \frac{\partial^2 \chi^*}{\partial \xi^2} \right) \right) + (2\mathcal{A}^2 \text{Pr} (1 - \gamma^{-1})) \frac{\text{Re}^2}{64 (2\pi)^2 \mathcal{A}^{12}} \frac{\partial^2 \chi^*}{\partial \theta^{*2}} = S^*. \quad (4.18)$$

We refer the equation (4.18) as the 'analytical equation' in the polar $(r-\theta)$ plane. Clearly, this is valid when the stratification parameter $\mathcal{A} \gg 1$ and Reynolds number $\text{Re} \gg 1$.

The limitation $\mathcal{A} \gg 1$ imposed by condition 11 above can be relaxed if we formulate a 'generalized analytical equation' in the polar $(r-\theta)$ plane,

$$L_6^{\mathcal{A}}(\chi^*) + \frac{\text{Re}^2}{64 (2\pi)^2 \mathcal{A}^{12}} \frac{\partial^2 \chi^*}{\partial \theta^{*2}} = S^*, \quad (4.19)$$

where the linear differential operator $L_6^{\mathcal{A}}$ in the ξ co-ordinate is defined by

$$L_6^{\mathcal{A}} = \left(\frac{2\mathcal{A}^2 \text{Pr} (1 - \gamma^{-1})}{(1 - \xi/\mathcal{A}^2)} \right)^{-1} \times \left(\frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \right) \right) \right). \quad (4.20)$$

The operator has been written with the superscript $\mathcal{A}$ to indicate that there is an explicit dependence on the stratification parameter. This operator also depends on the Prandtl number Pr and the ratio of specific heats, but these are gas properties and not flow properties, and are both $O(1)$ for an ideal gas. Therefore, the the only flow

dependence in the operator L_6^A enters through the stratification parameter ($\mathcal{A}$), while the Prandtl number and ideal gas constant are determined by the nature of the gas. Here, we will assume an ideal gas of hard spheres, for which $\text{Pr} = 0.95$ and $\gamma = 1.066$.

It is easily verified that this operator is also self-adjoint, when the inner product is defined by

$$(A_1, B_1) = \int_0^{\xi_0} d\xi w(\xi) A_1(\xi) B_1(\xi), \quad (4.21)$$

where the weighting function $w(\xi)$ is given by

$$w(\xi) = \left(\frac{2\mathcal{A}^2 \text{Pr} (1 - \gamma^{-1})}{(1 - \xi/\mathcal{A}^2)} \right), \quad (4.22)$$

and the boundary conditions are appropriately specified, as discussed next. Therefore, the eigenvalues of this operator are real, and the eigenfunctions are orthogonal. We refer to equation 4.19 as the 'generalized analytical equation' in the polar ($r - \theta$) plane, where the scaled source term (S^*) is

Since the equation 4.19 is a sixth order differential equation, it is necessary to specify three boundary conditions at each boundary. Here we will investigate the secondary radial flow due to mass/momentum/energy sources within the domain, with homogeneous boundary conditions (constant temperature). As shown below, the solution for the master potential with internal sources can be obtained by the method of separation of variables, since the differential operators in both the axial and meridional directions are self-adjoint. The following expressions for the velocity components are useful for specifying the boundary conditions,

$$\hat{v}_r = \frac{2R\Omega\mathcal{A}^2}{\text{Re}} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{-1/2} e^\xi \frac{\partial^2 \chi^*}{\partial \xi \partial \theta^*}, \quad (4.23)$$

$$\hat{v}_\theta = \frac{8\pi R\Omega\mathcal{A}^4}{\text{Re}} \left(1 - \frac{\xi}{\mathcal{A}^2} \right)^{1/2} e^\xi \frac{\partial^2 \chi^*}{\partial \xi^2}. \quad (4.24)$$

The homogeneous boundary conditions for 4.19 are as follows. At the wall ($(r = R); (r^* = r/R = 1); (\xi = 0)$), the boundary conditions are the no-penetration condition in the radial direction, the no-slip in the meridional direction and the zero temperature perturbation condition. The homogeneous radial and meridional velocity boundary conditions, $\hat{v}_r = 0$ and $\hat{v}_\theta = 0$ at $((r = R); (r/R = 1); (\xi = 0))$, reduce to,

$$\frac{\partial \chi^*}{\partial \xi} = 0, \quad (4.25)$$

$$\frac{\partial^2 \chi^*}{\partial \xi^2} = 0, \quad (4.26)$$

while a combination of the zero meridional velocity and temperature conditions is formulated from equation 3.20 for the master potential,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\rho} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{\rho} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} \right) \right) \right) \right) \right) \right) = 0, \quad (4.27)$$

which is expressed in terms of scaled coordinates as

$$\frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 \chi^*}{\partial \xi^2} \right) \right) = 0 \quad (4.28)$$

or

$$L_5^A(\chi^*) = 0, \quad (4.29)$$

where the operator L_5^A is

$$L_5^A = \frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \right) \right). \quad (4.30)$$

At the center of the cylinder, $(\xi = \xi_0 = \mathcal{A}^2)$ (which corresponds to $((r = 0); (r^* = r/R = 0))$), the symmetry conditions are zero radial velocity $\hat{v}_r = 0$, zero azimuthal shear stress $\tau_{r\theta} = 0$. The zero radial velocity boundary condition at the centre requires that $(\partial\psi/\partial\theta) = 0$, or

$$\frac{\partial \chi^*}{\partial \xi} = 0. \quad (4.31)$$

Symmetry requires that the azimuthal shear stress $\tau_{r\theta}$ has to be zero at the centre of the cylinder. Therefore,

$$r \frac{\partial}{\partial r} \left(\frac{\hat{v}_\theta}{r} \right) = 0, \quad (4.32)$$

or

$$r \frac{\partial}{\partial r} \left(\frac{1}{\rho r} \frac{\partial \psi}{\partial r} \right) = 0, \quad (4.33)$$

which is expressed in terms of scaled coordinates as

$$\frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 \chi^*}{\partial \xi^2} \right) = 0, \quad (4.34)$$

or

$$L_3^A(\chi^*) = 0, \quad (4.35)$$

where

$$L_3^A = \frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \right). \quad (4.36)$$

In addition, since the velocity and temperature perturbations are related to the derivatives of the master potential, the master potential itself can be set equal to zero, without loss of generality, at the axis of the cylinder. We choose to set the master potential equal to zero at the axis. Then, from the boundary condition 4.31, the master potential is equal to zero at the axis $((r = 0); (r^* = r/R = 0), (\xi = \xi_0))$, that is,

$$\chi^* = 0. \quad (4.37)$$

In the θ direction, the boundary conditions are as follows,

$$\chi^*(\theta^* = 0) = \chi^*(\theta^* = 1), \quad (4.38)$$

To obtain the solution, we use the separation-of-variables procedure. First, we define the eigenvalues and eigenfunctions of the operators L_6^A and $(\partial^2/\partial\theta^{*2})$. For the L_6^A operator, the eigen-relation can be written as,

$$L_6^A F_n(\xi) = \alpha_n F_n(\xi), \quad (4.39)$$

where α_n are the discrete eigenvalues. The adjoint of the linear operator L_6^A , with bound-

Eigenvalue	$\mathcal{A} = 0.707$	$\mathcal{A} = 2.121$	$\mathcal{A} = 3.535$
α_1	1.163426781	1.018965341	0.988653417
α_2	3.894532109	2.101305067	1.990673421
α_3	5.384321785	3.102328976	2.979863209
α_4	6.550432198	3.757192418	3.590768210
α_5	7.522923167	4.314588795	4.100844231
α_6	8.408542189	4.748457210	4.539599803
α_7	9.269542087	5.106905623	4.978805681
α_8	9.980945328	5.411974328	5.267641095
α_9	10.521989056	5.608567231	5.428856018
α_{10}	10.798215649	5.700896743	5.560698274

TABLE 1. First 10 eigenvalues of $L_6^{\mathcal{A}}$ operator at stratification parameter = 0.707, 2.121, 3.535.

any conditions equations 4.25 to 4.37, can be calculated as,

$$\begin{aligned}
& \int_0^{\xi_0} d\xi A_1(\xi) L_6^{\mathcal{A}}(B_1(\xi)) \\
&= \left(A_1 L_5 B_1 - (d_\xi A_1) L_4 B_1 + (d_\xi^2 A_1) \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \right) L_3 B_1 \right. \\
&\quad \left. - (d_\xi^2 B_1) \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \right) L_3 A_1 + (d_\xi A_1) L_4 B_1 - B_1 L_5 A_1 \right) \Big|_{\xi=0}^{\xi=\xi_0} \\
&+ \int_0^{\xi_0} d\xi B_1(\xi) L_6^{\mathcal{A}}(A_1(\xi))
\end{aligned} \tag{4.40}$$

where

$$L_5^{\mathcal{A}} A_1 = \frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 A_1}{\partial \xi^2} \right) \right), \tag{4.41}$$

$$L_4^{\mathcal{A}} A_1 = e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2}{\partial \xi^2} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 A_1}{\partial \xi^2} \right), \tag{4.42}$$

$$L_3^{\mathcal{A}} A_1 = \frac{\partial}{\partial \xi} \left(e^\xi \left(1 - \frac{\xi}{\mathcal{A}^2} \right) \frac{\partial^2 A_1}{\partial \xi^2} \right). \tag{4.43}$$

It is clear, from equation 4.40, that the operator $L_6^{\mathcal{A}}$, with boundary conditions (equations 4.25 to 4.37) is a self-adjoint operator, since all the terms on the right side obtained by partial integration are zero. Therefore, the eigenvalues α_n in equation 4.39 are real, and the eigenfunctions $F_n(\xi)$ are orthogonal. We define the eigen-functions so that they are orthonormal:

$$\int_0^{\xi_0} d\xi F_n(\xi) F_m(\xi) = \delta_{mn}. \tag{4.44}$$

The eigenvalues and eigenfunctions are determined numerically using fourth-order Runge-Kutta integration and the shooting technique. The first ten eigenvalues (α_n) of the operator $L_6^{\mathcal{A}}$ at $\mathcal{A} = 0.707, 2.121, \text{ and } 3.535$ are given in table 1.

The eigen equation for the θ direction is,

$$\frac{\partial^2 G_n(\theta^*)}{\partial \theta^{*2}} = \beta_n G_n(\theta^*), \quad (4.45)$$

with boundary condition 4.38. For this case, the eigen-values are given by $\beta_n = (n\pi)^2$, and the orthonormal eigen-functions are,

$$G_n(\theta^*) = \sqrt{2} \sin(n\pi\theta^*). \quad (4.46)$$

The master potential can be expressed in terms of the eigenfunctions of the L_6^A and the $(\partial^2/\partial\theta^{*2})$ operators,

$$\chi^* = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} F_n(\xi) G_m(\theta^*). \quad (4.47)$$

The above expressions are substituted into the differential equation A 4, and the orthogonality condition is used, to obtain the coefficients A_{mn} ,

$$A_{nm} = \frac{1}{\alpha_n + B^* \beta_m} \int_0^{\xi_0} d\xi \int_0^1 d\theta^* S^*(\xi, \theta^*) F_n(\xi) G_m(\theta^*). \quad (4.48)$$

In the case of delta function sources,

$$S^*(\xi, \theta^*) = \sum_{p=1}^P S_p^* \delta(\xi - \xi_p, \theta^* - \theta_p^*), \quad (4.49)$$

the coefficients A_{nm} reduce to,

$$A_{nm} = \frac{1}{\alpha_n + B^* \beta_m} \sum_{p=1}^P S_p^* F_n(\xi_p) G_m(\theta_p^*). \quad (4.50)$$

Here, P is the total number of sources. In the case of a the complicated source expression 4.17, which involves derivatives and integrals, if the mass, momentum and energy sources are of the form,

$$\begin{pmatrix} S_m \\ f_r \\ f_\theta \\ f_z \\ S_e \end{pmatrix} = \sum_{p=1}^P \begin{pmatrix} S_{mp} \\ f_{rp} \\ f_{\theta p} \\ f_{zp} \\ S_{ep} \end{pmatrix} \delta(\xi - \xi_p) \delta(\theta^* - \theta_p^*), \quad (4.51)$$

the expression for the coefficients can be determined using integration by parts.

5. DSMC simulation

The direct simulation Monte Carlo method (DSMC) is a numerical method for modeling rarefied gas flows. Here, the Boltzmann equation for the probability distribution function for gas molecules is solved using a probabilistic (Monte Carlo) simulation technique. The Boltzmann equation is more fundamental than the Navier-Stokes equations, since it describes the evolution of the velocity distribution function of the gas molecules. The Navier-Stokes equations are obtained by taking the first, second and third moments of the Boltzmann equation with respect to the molecular velocities. While taking moments, the assumption of 'local equilibrium' (velocity distribution function close to the Maxwell-Boltzmann distribution) is made, thereby restricting the Navier-Stokes equations to near-equilibrium flows. Since there is no assumption of local equilibrium in the

Boltzmann equation, it is valid even for far-from equilibrium flows, provided the molecular chaos approximation is satisfied.

The DSMC method uses 'simulated molecules' which represent a large number of real gas molecules. A basic assumption is that the inter-molecular collisions and molecular motion between collisions can be decoupled over time scales comparable to the time between collisions. The translation of molecules between collisions is deterministically modeled, while the inter-molecular collisions and molecular-surfaces are incorporated using probabilistic models. The domain is divided into a large number of cells, and properties are evaluated in these cells. The cells are further divided into 'sub-cells' for collision localization.

In the DSMC simulations, inter-molecular collisions are executed at random from all molecules within 'sub-cells' within the simulation cell, whose linear dimension is much smaller than the mean free path. The probability of collisions between a pair of molecules within a sub-cell is selected equal to that for a dilute gas with the same temperature as that in the cell. In this manner, the temperature (mean square velocity), mean free path and collision frequency for the simulated molecules are equal to those for the real molecules, provided the sub-cell dimension is much smaller than the mean free path.

The most common molecular model is the hard-sphere model, but other models such as the variable hard sphere and variable soft sphere models are also used in order to accurately reproduce the density and temperature dependence of the viscosity. The hydrodynamic fields (density, velocity, temperature) are obtained by averaging over the molecular distribution functions in the DSMC simulations.

At the boundaries, it is necessary to prescribe interaction laws between the molecules and the solid surfaces. Accurate specification of the interaction laws requires a detailed knowledge of the molecule-surface interaction. However, there are standard interaction laws based on the surface velocity and temperature which are sufficient for predicting the hydrodynamic fields accurately. The most commonly used is the 'diffuse reflection' conditions, where a molecule incident on the surface gets reflected with a Gaussian velocity distribution with the same mean velocity and temperature as that of the surface. Another commonly used condition is the specular reflection condition, that the velocity normal to the surface is reversed, while that tangential to the surface is unchanged. It is important to note that these boundary conditions are different from the no-slip conditions in the continuum models. In high Knudsen number flows, the diffuse and specular reflection conditions do predict a slip velocity at the surface. Therefore, these simulations could be used even in situations where there is a slip at the surface.

From kinetic theory of dilute gases Chapman & Cowling (1970), the viscosity of a hard-sphere gas in the dilute limit is given by,

$$\mu = \frac{5\sqrt{mk_B T}}{16\sqrt{\pi}d^2}, \quad (5.1)$$

where k_B is the Boltzmann constant, while the speed of sound c_s is given by the relation

$$mnc_s^2 = \gamma p = \gamma nk_B T, \quad (5.2)$$

where m is the molecular mass, n is the number density, and p is the pressure. From this the stratification parameter $\mathcal{A}$, Reynolds number Re , cylinder radius R , molecular diameter d and volume fraction of the molecules (ϕ) are related by the following equation:

$$Re = \frac{96}{5} \times \frac{\sqrt{2}}{\sqrt{\pi}} \mathcal{A} \phi \frac{R}{d}, \quad (5.3)$$

where the volume fraction of the molecules is given by

$$\phi = \frac{\pi}{6} d^3 n, \quad (5.4)$$

where n is the number density of the molecules. The important point to note, from 5.3, is that the Reynolds number is proportional to the Mach number times $\phi(R/d)$, where R is the macroscopic scale. Therefore if we are keeping the volume fraction (density) a constant, increasing (decreasing) the Reynolds number at constant Mach number corresponds to increasing (decreasing) the system size R in relation to the molecular diameter.

The details of the DSMC simulation procedure are discussed in Bird (1963) and Bird (1994), and so we do not repeat the discussion here, but restrict attention to the parameters in the simulations. The number of cells along the radial direction is taken as 400 with clustering towards the rotating sidewall as well as near the stationary insert, while 3200 cells are taken along the azimuthal direction with clustering near the stationary insert. The cell width is about one-third of the local mean free path. The number of subcells per cell is taken as 4 for both radial and azimuthal directions. The number of simulated particles per cell is taken as 100. In our simulations, the number of real molecules per simulated particles varied between 10^{16} and 10^{18} so that the statistical fluctuations not becomes large.

The hard sphere (HS) molecular model is used in the present simulations, with the molecular mass $58.442636 \times 10^{-26}$ kg/molecule, molecular diameter $= 5.5 \times 10^{-10}$ m, viscosity index 0.933, specific heat ratio $\gamma = 1.066$, and Prandtl number $Pr = 0.95$. It is assumed that the gas molecules undergo diffuse reflection from all the solid surfaces with the thermal accommodation coefficient equal to 1. The time step in the simulation is chosen to be small compared to the time between collisions. The mean collision time is calculated as, $t = (\lambda / c_{av})$, where λ is the mean free path and c_{av} is the average molecular velocity. The minimum value of mean free path at the rotating sidewall, calculated from the gas density at the wall, is $\lambda = 2.316742 \times 10^{-7}$ m. The average molecular velocity is calculated as $\sqrt{k_B T / m} = 84.17717$ m/sec. This gives the mean collision time (t) $= 2.752221 \times 10^{-9}$ sec. The time step used in the simulations is 5.0×10^{-10} sec, which is less than mean collision time.

Two-dimensional polar ($r - \theta$) configuration is considered in the present simulations. The flow field is studied by placing the stationary insert within the flow domain which causes the disturbance to the base flow (solid body rotation). Due to the presence of stationary insert in the high speed rotating field, there is a shock wave formation and subsequently generation of secondary radial flow. The effect of a stationary insert in the flow domain is of three types. The first is the mass source or sink due to the addition or removal of gas in a particular region in the flow domain. The second effect is due to the force exerted on the gas by a stationary inserts, which is modelled locally as a momentum source or sink. The time rate of addition/removal of momentum is equal to the applied force. The third effect is a localized energy source, which increases or decreases the temperature of the gas. All the three effects are incorporated in the molecular simulation as follows. The addition or removal of mass is accomplished by either adding or removing molecules at specified locations in the simulation cell at specified rates. The pressure is specified at inflow and outflow boundaries by connecting a pressure bath, which is a large volume of gas maintained at a constant pressure. Forces are exerted by imposing impulses on the molecules at specified locations, in such a manner that the rate of change of impulse is equal to the applied force. Energy is added by increasing the mean square velocities of the molecules at locations where heating takes place.

In the present analysis, we have varied the stratification parameter in the range (0.707–

FIGURE 1. Computational domain and coordinate system in the two-dimensional polar ($r - \theta$) coordinate for momentum source (a); mass source/sink (b); stationary intake tube (c); and the cell arrangement in the flow domain (sectional zoomed view) (d). In sub-figure (a), the momentum source is placed in two different radial locations, one outside the radial boundary layer ($(r^* = 0.86)$; $(\xi_s^* = 0.25)$), and other inside the radial boundary layer ($(r^* = 0.99)$; $(\xi_s^* = 0.01)$). In sub-figure (b), the mass source is placed at the centre ($(r^* = 0)$), and mass sink is placed in two different radial locations, one outside the radial boundary layer ($(r^* = 0.86)$; $(\xi_s^* = 0.25)$), and other inside the radial boundary layer ($(r^* = 0.99)$; $(\xi_s^* = 0.01)$). In sub-figure (c), the stationary intake tube is placed inside the rotating field with inner radius r_i and outer radius r_o . The intake tube is extended in the θ direction from 60° to 75° . The back pressure is applied at the rear-end of the intake tube at $\theta = 60^\circ$. Here, $\xi = \mathcal{A}^2 (1 - (r^2/R^2)) = \mathcal{A}^2 (1 - r^{*2})$, and $r^* = \sqrt{(1 - (\xi/\mathcal{A}^2))} = \sqrt{(1 - \xi^*)}$, where $r^* = r/R$, $\xi^* = \xi/\mathcal{A}^2$, and the angle θ is expressed in degree.

3.535), Reynolds number in the range $(10^2 - 10^6)$, dimensionless scaled wall gap $d^* = (R - r)/R$ in the range $(0.01 - 0.15)$. We place the sources/sinks of mass/momentum/energy at two locations, one outside the radial boundary layer at $((r^* = 0.86)$; $(\xi_s^* = 0.25)$), and the second inside the radial boundary layer at $((r^* = 0.99)$; $(\xi_s^* = 0.01)$), where $\xi^* = \xi/\mathcal{A}^2$. The computational domain and coordinate system is shown in figure 1.

6. Results (single-component gas)

First, we consider the single-component gas of hard sphere (HS) molecular model, with the molecular mass $58.442636 \times 10^{-26}$ kg/molecule, molecular diameter $= 5.5 \times 10^{-10}$ m, viscosity index 0.933, specific heat ratio $\gamma = 1.066$, and Prandtl number $Pr = 0.95$. It

is assumed that the gas molecules undergo diffuse reflection from all the solid surfaces with the thermal accommodation coefficient equal to 1.

Before proceeding to simulate the flow field in the two-dimensional polar $(r - \theta)$ coordinate by placing a stationary intake tube within the flow domain, we first study the simplified flow field by placing a small circular source domain of radius 10^{-3} times the cylinder radius R for adding/removing mass, momentum and energy. In the case of mass source, a molecule is added at random from the source domain at fixed time intervals determined by the desired rate of mass input. The velocity of the added molecules is chosen from a random Gaussian distribution with the same mean velocity and temperature as that of the gas in the domain. When a molecule is added, the mean velocity and the fluctuating velocity of all other molecules in the domain are scaled by a constant factor so that the total momentum and energy of all the molecules in the source domain remain unchanged due to the addition of a molecule. In the case of a mass sink, molecules are removed at random from the domain at time intervals fixed by the desired rate of mass output. When a molecule is removed, the mean and the fluctuating velocity of all other molecules in the source domain are scaled by a constant factor so that the total momentum and energy in the domain remain unchanged.

Forces (momentum sources) exerted on the gas due to stationary inserts are modelled by imposing impulses (increasing the momentum) of all the molecules in the direction of a force at each collision step. The magnitude of the impulse is determined such that the ratio of the impulse and the time between collisions is equal to the applied force. A localised energy source, which increases or decreases the temperature of the gas, is imposed by scaling the fluctuating velocity of all the molecules in the source domain. The scaling of the velocity is determined such that the increase in energy divided by the time between collisions is equal to the desired energy input rate.

The flow geometry and coordinate system are shown in figure 1 (a), for the momentum source, and in figure 1 (b), for mass source/sink. In our calculation, we place the sources/sinks at two locations, one outside the radial boundary layer at $((r^* = 0.86); (\xi_s^* = 0.25))$, and the second inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$, where $r^* = r/R$; and $\xi_s^* = \xi_s/\mathcal{A}^2$. Here, we choose to express the results in terms of the scaled radial coordinate r^* , where $r^* = r/R$ for notational simplicity. Note that $\xi = \mathcal{A}^2 (1 - (r^2/R^2)) = \mathcal{A}^2 (1 - r^{*2})$. Therefore, $r^* = \sqrt{(1 - (\xi/\mathcal{A}^2))} = \sqrt{(1 - \xi^*)}$, where $r^* = r/R$, $\xi^* = \xi/\mathcal{A}^2$.

In order to check the convergence of the simulations, we have varied both the time step and the number of simulated molecules per cell, keeping the number of real molecules a constant. In all cases, the time step is decreased by a factor of 5 and the number of simulated molecules increased by a factor of 2, and verify that there is less than 1.5 % change in the resulting secondary radial flow profiles. The comparisons for a momentum source for $Re = 10^2$, $\mathcal{A} = 3.535$, and source strength $F^* = 0.2$ is shown in figure 2. Different time steps (5×10^{-10} , 1×10^{-9} , 1.5×10^{-9} sec) and different numbers of simulated molecules per cell (100, 80, 60) are used to test the convergence. It is found that the results are insensitive to the time step and the number of simulated molecules per cell. This clearly shows the convergence of the simulation results. Similar results are obtained for mass source/sink within the flow, but we do not provide the details here.

6.1. Momentum source (single-component gas)

The secondary radial flow generated by a momentum source, as shown in figure 1 (a), has been studied at different Reynolds numbers and stratification parameters. Results for the stratification parameters $\mathcal{A} = 3.535, 2.121, 0.707$, Reynolds number $Re = 10^2$ are shown in figure 5. This because the smallest Reynolds number provides the most stringent test.

FIGURE 2. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$; and $\text{Re} = 10^2$. Flow induced by the momentum source inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$. Sub-figure (a) for different time steps (t) in seconds. The symbols show the DSMC results for sub-figure (a) : ($t = 5 \times 10^{-10}$ (o), $t = 1 \times 10^{-9}$ ($\square$), $t = 1.5 \times 10^{-9}$ ($\triangle$)). Sub-figure (b) for different numbers of simulated molecules per cell (n_s). The symbols show the DSMC results for sub-figure (b) : ($n_s = 100$ ($\triangleright$), $n_s = 80$ (o), $n_s = 60$ (∇)). The lines show the generalized analytical model (4.19) with homogeneous (zero temperature perturbation) boundary condition (---). Here, $\xi = \mathcal{A}^2 (1 - (r^2/R^2)) = \mathcal{A}^2 (1 - r^{*2})$, and $r^* = \sqrt{(1 - (\xi/\mathcal{A}^2))} = \sqrt{(1 - \xi^*)}$, where $r^* = r/R$, $\xi^* = \xi/\mathcal{A}^2$.

The momentum source is placed in two different radial locations, one outside the radial boundary layer at $((r^* = 0.86); (\xi_s^* = 0.25))$, and other inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$.

The influx of momentum/energy near the wall is imposed by increasing the mean/fluctuating velocity of the molecules at the location. When the location is near the wall, the molecules with augmented momentum/energy from the source collide with the wall and locally increase the temperature. This results in a non-zero temperature perturbation at the wall. The dimensionless temperature slip (T_s^*) is defined by

$$T_s^* = \frac{T - T_a}{T}, \quad (6.1)$$

where T is the gas temperature at the wall obtained from DSMC simulation, and T_a is the constant gas temperature at the wall assumed in the analytical solution.

We have examined, in detail, the temperature slip at the rotating sidewall near the source in the DSMC simulations. The variation of temperature slip for momentum source and mass source/sink is shown in figure 3 (a, b), for the momentum source with configuration shown in figure 1 (a), and for the mass source/sink configuration shown in figure 1 (b). We find that the temperature slip (T_s^*) decreases as the Reynolds number is increased and temperature slip (T_s^*) increases as the stratification parameter is increased for these cases, and is at a maximum for $\text{Re} = 10^2$, and stratification parameter $\mathcal{A} = 3.535$, shown in figure 3 (a, b). As expected, the temperature slip is larger when the source is placed inside the boundary layer $((r^* = 0.99); (\xi_s^* = 0.01))$, owing to the larger number density and the higher frequency of molecular collisions. There is little variation in the temperature slip with Reynolds number, but there is a systematic increase in the temperature slip as the stratification parameter is increased. The largest scaled temperature slip (T_s^*) is close to 0.72 for the momentum source in figure 3 (a), and is close to 0.65 for the mass source/sink in figure 3 (b).

FIGURE 3. The variation of dimensionless scaled temperature slip (T_s^*) as a function of stratification parameter ($\mathcal{A}$) with source strength $F^* = 0.2$. Sub-figure (a) for momentum source; and sub-figure (b) for mass source/sink. The symbols show the DSMC results for the sources inside the radial boundary layer at ($r^* = 0.99$); ($\xi_s^* = 0.01$) for : ($\text{Re} = 10^6$ ($\square$), $\text{Re} = 10^4$ ($\triangle$), $\text{Re} = 10^2$ ($\circ$)); and for the sources outside the radial boundary layer at ($r^* = 0.86$); ($\xi_s^* = 0.25$) for : ($\text{Re} = 10^6$ ($\blacksquare$), $\text{Re} = 10^4$ ($\blacktriangle$), $\text{Re} = 10^2$ ($\bullet$)). Here, $r^* = r/R$, $\xi_s^* = \xi/\mathcal{A}^2$.

The temperature slip violates the homogeneous (zero temperature perturbation) boundary condition used in the analytical solution, and so results in a difference in the magnitudes of the secondary flow predicted. It is observed that there is a significant difference between the analytical and simulation results. In order to mimic the homogeneous boundary condition for the gas temperature assumed in the analytical solutions, it is necessary to remove energy from the wall. This is done by reducing the wall temperature as explained in the simulation section. Once the homogeneous boundary conditions are satisfied in the simulations, it is observed that there is quantitative agreement (with a difference of less than 10 %) between the analytical and simulation results as shown in figure 4.

The analytical solutions are compared with the results of DSMC simulations, where we have taken the precautions discussed above to ensure that homogeneous boundary conditions are satisfied. The analytical solutions are obtained by considering homogeneous boundary condition with internal source. The radial velocity, scaled by (ΩR) , is plotted as a function of normalized radial coordinate ($r^* = r/R$) for stratification parameter $\mathcal{A} = 3.535, 2.121, 0.707$, Reynolds number $\text{Re} = 10^2$, for isothermal sidewall (with energy removal from the wall) as shown in figure 4. We find that the secondary radial flow is weaker when stratification parameter is reduced from 3.535 to 0.707.

The results show quantitative agreement between the DSMC simulations and the analytical results of the generalized analytical model (equation 4.19), provided the care is taken to ensure that the temperature is a constant at the sidewall. There is agreement to within 10 % even when the Reynolds number is as low as 100, the stratification parameter is as low as 0.707. The results for higher Reynolds numbers and stratification parameters, not shown here, are in even better agreement with the simulation results. This shows that the 'generalized analytical model' (equation 4.19) captures the secondary radial flows quantitatively even at stratification parameter as low as 0.707, provided the boundary conditions in the analytical model are accurately specified. The predictions of the 'generalized analytical model' (equation 4.19) is also significantly better than that of the 'analytical model' (equation 4.18), which is restricted to thin boundary layers in the limit of high stratification parameter. We have also carried out further tests, whose

FIGURE 4. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$ (a); $\mathcal{A} = 2.121$ (b); $\mathcal{A} = 0.707$ (c), for $\text{Re} = 10^2$. Flow induced by momentum source inside the radial boundary layer at $(r^* = 0.99)$; $(\xi_s^* = 0.01)$ with source strength = 0.2. The symbols show the DSMC results for isothermal sidewall (with energy removal from wall) for : $\mathcal{A} = 3.535$ ($\square$), $\mathcal{A} = 2.121$ ($\triangle$), $\mathcal{A} = 0.707$ ($\circ$). The lines show the generalized analytical model (4.19) with homogeneous (zero temperature perturbation) boundary condition ($--$), and analytical model (4.18) with homogeneous (zero temperature perturbation) boundary condition ($\cdot - \cdot$). Here, $\xi = \mathcal{A}^2 (1 - (r^2/R^2)) = \mathcal{A}^2 (1 - r^{*2})$, and $r^* = \sqrt{(1 - (\xi/\mathcal{A}^2))} = \sqrt{(1 - \xi^*)}$, where $r^* = r/R$, $\xi^* = \xi/\mathcal{A}^2$.

results are not shown here for conciseness. When constant temperature boundary conditions are not enforced in the simulations, we find that the difference between analytical and simulation results is more than one and half times that when the constant temperature boundary conditions are enforced.

It should be noted that energy removal is necessary because the energy dissipation term (double dot product of the viscous stress and rate of deformation tensors) is second-order in the velocity perturbation. This is not included in the linearization approximation that we have used, as can be seen from the energy equation. Therefore, the viscous heating is not captured by the generalized analytical model (equation 4.19), and it is necessary to explicitly remove energy to maintain a constant temperature at the wall.

6.2. Mass source/sink (single-component gas)

The secondary radial flow generated by a mass source/sink, as shown in figure 1 (b), has also been studied at different Reynolds numbers and stratification parameters. Results for the stratification parameters $\mathcal{A} = 3.535, 2.121, 0.707$, $\text{Re} = 10^2$, with $F^* = 0.2$,

are shown in figure 5. This because the smallest Reynolds number provides the most stringent test. Here, the dimensionless scaled source strength is defined by

$$F^* = \frac{F}{H\Omega}, \quad (6.2)$$

where F is the mass flow rate, H is the total gas holdup, and Ω is the angular velocity. The scaled source strength (F^*) is a measure of the perturbation to the base flow due to the mass flow rate for mass source/sink type of forcing. Here, we have carried out simulations for $F^* = 0.2$. The actual mass flow rate (F) is varied in such a way that the desired scaled source strength is obtained at different stratification parameter, and Reynolds number. The mass sink is placed in two different radial locations, one inside ($(r^* = 0.99); (\xi_s^* = 0.01)$) and other outside ($(r^* = 0.86); (\xi_s^* = 0.25)$) the radial boundary layer. Mass is introduced in the flow domain by adding molecules to the system at $r^* = 0$ through the circle of radius $10^{-3}R$ which is connected to a pressure bath. Similarly, the mass is removed by removing molecules through a circle of the same radius $10^{-3}R$ at $(r^* = 0.86); (\xi_s^* = 0.25)$ (outside the radial boundary layer) and at $(r^* = 0.99); (\xi_s^* = 0.01)$ (inside the radial boundary layer) which is also connected to a pressure bath to get the same desired scaled mass flow rate $F^* = 0.2$.

The analytical solutions are compared with the results of DSMC simulations, where we have taken the precautions to ensure that the homogeneous boundary conditions are satisfied. The radial velocity, scaled by (ΩR) , is plotted as a function of normalized radial coordinate ($r^* = r/R$) for the stratification parameters $\mathcal{A} = 3.535, 2.121, 0.707$, $Re = 10^2$, with $F^* = 0.2$, for isothermal sidewall (with energy removal from the wall) as shown in figure 5. We find that the secondary radial flow is weaker when stratification parameter is reduced from 3.535 to 0.707 in this case as well.

The results show quantitative agreement between the DSMC simulations and the analytical results of the 'generalized analytical model' (equation 4.19), provided the care is taken to ensure that the temperature is a constant at the sidewall. There is agreement to within 10 % even when the Reynolds number is as low as 100, the stratification parameter is as low as 0.707. The results for higher Reynolds numbers and stratification parameters, not shown here, are in even better agreement with the simulation results. This shows that the 'generalized analytical model' (equation 4.19) captures the secondary radial flows quantitatively even at stratification parameter as low as 0.707, provided the boundary conditions in the analytical model are accurately specified. The predictions of the 'generalized analytical model' (equation 4.19) is also significantly better than that of the 'analytical model' (equation 4.18).

6.3. Stationary intake tube (single-component gas)

Next, we have studied the flow field in polar ($r - \theta$) plane by placing a stationary intake tube within the flow domain. The effect of an insert in the flow domain is of three types. The first is the mass source/sink due to the addition or removal of gas in a particular region in the flow domain. The second effect is due to the force exerted on the gas by a stationary inserts, which is modelled locally as a momentum source or sink. The time rate of addition/removal of momentum is equal to the applied force. The third effect is a localized energy source, which increases or decreases the temperature of the gas. All the three effects are incorporated in the molecular simulation as follows. The addition or removal of mass is accomplished by either adding or removing molecules at specified locations in the simulation cell at specified rates. The pressure is specified at inflow and outflow boundaries by connecting a pressure bath, which is a large volume of gas maintained at a constant pressure. Forces are exerted by imposing impulses on the

FIGURE 5. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$ (a); $\mathcal{A} = 2.121$ (b); $\mathcal{A} = 0.707$ (c), for $\text{Re} = 10^2$, with source strength $F^* = 0.2$. Mass is introduced in the flow domain by adding molecules to the system at $r^* = 0$ through the circle of radius $10^{-3}R$. The mass sink is placed inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$. The mass is removed by removing molecules through a circle of the same radius $10^{-3}R$ at the radial location $((r^* = 0.99); (\xi_s^* = 0.01))$. The symbols show the DSMC results for isothermal sidewall (with energy removal from wall) for : ($\mathcal{A} = 3.535$ ($\square$), $\mathcal{A} = 2.121$ ($\triangle$), $\mathcal{A} = 0.707$ ($\circ$)). The lines show the generalized analytical model (4.19) with homogeneous (zero temperature perturbation) boundary condition ($--$), and analytical model (4.18) with homogeneous (zero temperature perturbation) boundary condition ($\cdot - \cdot$). Here, $\xi = \mathcal{A}^2 (1 - (r^2/R^2)) = \mathcal{A}^2 (1 - r^{*2})$, and $r^* = \sqrt{(1 - (\xi/\mathcal{A}^2))} = \sqrt{(1 - \xi^*)}$, where $r^* = r/R$, $\xi^* = \xi/\mathcal{A}^2$.

molecules at specified locations, in such a manner that the rate of change of impulse is equal to the applied force. Energy is added by increasing the mean square velocities of the molecules at locations where heating takes place.

The computational domain and coordinate system is shown in figure 1 (c). The density and temperature contours close to the stationary intake tube (zoomed view) for $\mathcal{A} = 3.535$, $\text{Re} = 10^2$, for the forward flow (equilibrium back pressure is less than the stagnation pressure) is shown in figure 6 (a), and figure 7 (a); and for the reverse flow (equilibrium back pressure is more than the stagnation pressure) is shown in figure 6 (b), and figure 7 (b). A detached bow shock wave is clearly observed in the simulations.

We have also examined, in detail, the temperature slip at the rotating sidewall in the DSMC simulations. The stationary intake tube inside the rotating field acts as a source/sink of mass/momentum/energy. When the location is near the wall, the molecules

FIGURE 6. Density contours close to the stationary intake tube (zoomed view) for $\mathcal{A} = 3.535$, $\text{Re} = 10^2$, for the forward flow (equilibrium back pressure is less than the stagnation pressure) (a); and for the reverse flow (equilibrium back pressure is more than the stagnation pressure) (b). A detached bow shock is formed in front of the stationary insert (intake tube), and behind the detached bow shock, secondary radial flow is induced towards the axis, and it excites the secondary axial flow.

FIGURE 7. Temperature contours close to the stationary intake tube (zoomed view) for $\mathcal{A} = 3.535$, $\text{Re} = 10^2$, for the forward flow (when the equilibrium back pressure is less than the stagnation pressure) (a); and for the reverse flow (when the equilibrium back pressure is more than the stagnation pressure) (b). A detached bow shock wave is clearly observed.

with augmented momentum/energy from the source collide with the wall and locally increase the temperature. This results in a non-zero temperature perturbation at the wall, as shown by the symbols (o) in figure 8 (a), for $\mathcal{A} = 3.535$, $\text{Re} = 10^2$. At the wall nearest the source, there is an increase of close to 91 % in the temperature at the wall. This results in a substantial increase in the secondary radial flow as shown in figure 9. We find that the magnitude of the secondary radial flow velocity is about 1.3 – 1.75 times when constant temperature boundary conditions are not enforced. We find that the dimensionless temperature slip (T_s^*) increases as the stratification parameter is increased, as shown in figure 8 (b). It is also observed that the temperature slip is more in front of the intake tube ($\theta = 75^\circ$) compared to at the rear-end ($\theta = 60^\circ$.)

Azimuthal velocity in the flow domain, scaled by (ΩR) , is plotted along the radial coordinate (r^*) for the stratification parameters $\mathcal{A} = 0.707, 2.121, 3.535$, and $\text{Re} = 10^2$, at $\theta = 75^\circ$ (in front of the intake tube) and $\theta = 60^\circ$ (rear-end of the intake tube), as shown in figure 10. It is clearly observed that there is a substantial slow down of the circumferential velocity (or equivalent angular momentum loss) $v_{\theta+(slow\ down)}$ of the bulk of the rotating fluid due to the presence of stationary insert (intake tube) within the high speed rotating field.

FIGURE 8. Sub-figure (a) shows the variation of dimensionless scaled temperature slip (T_s^*) along θ at the rotating wall for $\mathcal{A} = 3.535$, and $\text{Re} = 10^2$, with energy removal ($\cdot - \cdot$), without energy removal ($\circ$); sub-figure (b) shows the variation of dimensionless scaled temperature slip (T_s^*) as a function of stratification parameter ($\mathcal{A}$) for $\text{Re} = 10^2$. The symbols show the DSMC results without energy removal in sub-figure (b) for : ($\theta = 75^\circ$ ($\circ$) : in front of the intake tube, $\theta = 60^\circ$ ($\square$) : rear-end of the intake tube).

FIGURE 9. Radial velocity, scaled by (ΩR) , as a function of r^* at $\theta = 75^\circ$ for $\mathcal{A} = 3.535$ (a); and $\mathcal{A} = 0.707$ (b), for $\text{Re} = 10^2$. The symbols show the DSMC results for sub-figure (a) : (with energy removal ($\square$), without energy removal ($\triangle$)). The symbols show the DSMC results for sub-figure (b) : (with energy removal ($\circ$), without energy removal (∇)).

The slow down of the circumferential velocity $v_{\theta(\text{slow down})}$ of the bulk of the rotating fluid in the polar ($r - \theta$) plane can be calculated as

$$v_{\theta(\text{slow down})} = \frac{1}{2\pi R} \int_0^{2\pi} \left(\int_0^R (v_{\theta(\text{sbr})} - v_\theta) dr \right) d\theta. \quad (6.3)$$

Using the scaled variables, $r^* = r/R$, $\theta^* = \theta/(2\pi)$, and $v_\theta^* = v_\theta/(R\Omega)$, we obtain

$$v_{\theta(\text{slow down})}^* = \frac{v_{\theta(\text{slow down})}}{R\Omega} = \int_0^1 \left(\int_0^1 (v_{\theta(\text{sbr})}^* - v_\theta^*) dr^* \right) d\theta^*. \quad (6.4)$$

Here, $v_{\theta(\text{sbr})}$ is the circumferential velocity in the base state (solid body rotation). The slow down of the circumferential velocity $v_{\theta(\text{slow down})}$ of the bulk of the rotating fluid for

FIGURE 10. Azimuthal velocity, scaled by $((\Omega R))$, as a function of r^* at two different locations of θ . Sub-figure (a), in front of the intake tube: $(\theta = 75^\circ)$; and sub-figure (b), at the rear-end of the intake tube: $(\theta = 60^\circ)$ for $\mathcal{A} = 0.707, 2.121, 3.535$; for $Re = 10^2$. The symbols show the DSMC results for : ($\mathcal{A} = 3.535$ ($\square$), $\mathcal{A} = 2.121$ ($\triangle$), $\mathcal{A} = 0.707$ ($\circ$)).

Stratification parameter	$v_{\theta(slow\ down)} (\%)$
$\mathcal{A} = 0.707$	8
$\mathcal{A} = 2.121$	17
$\mathcal{A} = 3.535$	28

TABLE 2. The slow down of the circumferential velocity $v_{\theta(slow\ down)}$ of the bulk of the rotating fluid for stratification parameters $\mathcal{A} = 0.707, 2.121, 3.535$, at $Re = 10^2$

stratification parameters $\mathcal{A} = 0.707, 2.121, 3.535$, at $Re = 10^2$, is given in table 2. This clearly indicates that the $v_{\theta(slow\ down)}$ increases from 8 % to 28 % as the stratification parameter $\mathcal{A}$ is increased from 0.707 to 3.535, at $Re = 10^2$, which induces the secondary radial flow towards the axis, and it further excites the secondary axial flow, which could be very important for the centrifugal gas separation processes.

The mass flow rate through the stationary intake tube, scaled by $(H\Omega)$, is studied for different back pressures, as shown in figure 11 for $\mathcal{A} = 2.121$; $Re = 10^2$, and $d^* = 0.125$. Here, H is the gas holdup in kg , Ω is the angular velocity sec^{-1} , d is the wall gap $d = (R - r)$, and $d^* = d/R$. The simulations show that the scaled mass flow rate $m/(H\Omega)$ increases progressively from zero, at the stagnation condition $P_S/P_W = 1.6489$, for $\mathcal{A} = 2.121$, $Re = 10^2$, $d^* = 0.125$ to 0.024 when the back pressure is $P_B/P_W = 0.4692$, for $\mathcal{A} = 2.121$, $Re = 10^2$, $d^* = 0.125$.

The stagnation pressure, scaled by the wall pressure (P_W) is plotted as a function of dimensionless wall gap (d^*) for the stratification parameters ($\mathcal{A}$) in the range (0.707 - 3.535), for $Re = 10^2$, as shown in figure 12. Here, $d^* = d/R$.

It is clearly observed that the stagnation pressure increases with stratification parameter ($\mathcal{A}$), and decreases as the wall gap is increased. In the base state (solid body rotation), the stagnation pressure is expressed as

$$P_S = \left(1 + \mathcal{A}^2 \frac{r^2}{R^2}\right) \times \exp\left(-\mathcal{A}^2 \left(1 - \frac{r^2}{R^2}\right)\right), \quad (6.5)$$

FIGURE 11. Mass flow rate through the intake tube, scaled by $(H\Omega)$, as a function of equilibrium back pressure, scaled by the wall pressure (P_W) , for $\mathcal{A} = 2.121$; $\text{Re} = 10^2$, $d^* = 0.125$. DSMC results are shown by symbols ($\circ$). The line $(\cdot - \cdot)$ corresponds to the zero mass flow rate to get the stagnation pressure. Here, d is the wall gap $d = (R - r)$, and $d^* = d/R$.

where the wall gap $d = (R - r)$.

7. Binary gas mixture

Next, we have carried out the DSMC simulations in two-dimensional polar $(r - \theta)$ plane for a binary gas mixture with large mass difference ($\epsilon_a = (2\Delta m / (m_1 + m_2)) = 0.5$), and we compare the simulation results with those of single component gas. Here, Δm is the difference in the molecular masses of the two species $\Delta m = (m_1 - m_2)$. The fluid is a hard sphere gas mixture in which both species have equal molecular diameter $d = 5.5 \times 10^{-10}$ m, average molecular mass $m_{av} = (m_1 + m_2)/2 = 58.4426 \times 10^{-26}$ kg/molecule, specific heat ratio $\gamma = 1.066$, and Prandtl number $\text{Pr} = 0.95$. It is assumed that the gas molecules undergo diffuse reflection from all the solid surfaces with the thermal accommodation coefficient equal to 1.

The stratification parameter $\mathcal{A}$ is varied in the range 0.707-3.535, and Reynolds number Re in the range $10^2 - 10^6$. The flow geometry and coordinate system are shown in figures 1.

7.1. Momentum source (binary gas mixture)

We find that there is substantial temperature slip at the rotating sidewall near the source, and slip increases as the mass difference between two species in a binary gas mixture increases. Here, we have presented the results for $\epsilon_a = (2\Delta m / (m_1 + m_2)) = 0.5$; for Reynolds number (Re) in the range $(10^2 - 10^6)$; and stratification parameters ($\mathcal{A}$) in the range (0.707 - 3.535) as shown in figure 13.

We have examined, in detail, the temperature slip at the rotating sidewall near the source in the DSMC simulations. The variation of temperature slip for momentum source and mass source/sink is shown in figure 13 (a, b), for the momentum source with configuration shown in figure 1 (a), and for the mass source/sink configuration shown in figure 1 (b). We find that the temperature slip (T_s^*) decreases as the Reynolds number is increased and temperature slip (T_s^*) increases as the stratification parameter is increased for these cases, and is at a maximum for $\text{Re} = 10^2$, and stratification parameter $\mathcal{A} = 3.535$, shown in figure 13 (a, b). We also find that temperature slip increases as the mass difference between two species increases in a binary gas mixture. Here, we have shown the

FIGURE 12. The stagnation pressure, scaled by wall pressure(P_W), as a function of scaled wall gap ($d^* = d/R$), for $\mathcal{A} = 3.535$ (a); $\mathcal{A} = 2.121$ (b); $\mathcal{A} = 0.707$ (c), for $Re = 10^2$. The symbols show the DSMC results for : ($\mathcal{A} = 3.535$ (Δ)) (a), ($\mathcal{A} = 2.121$ ($\square$)) (b), ($\mathcal{A} = 0.707$ ($\circ$)) (c). The lines (—) show the results for the base state (solid body rotation) (Equation 6.5).

results for $\epsilon_a = (2\Delta m/(m_1 + m_2)) = 0.5$; and Reynolds number Re in the range $10^2 - 10^6$; and stratification parameters $\mathcal{A}$ in the range $0.707 - 3.535$ as shown in figure 13.

Radial velocity in the flow domain, scaled by (ΩR) , is plotted along the radial coordinate (r^*) as shown in figure 14. Here, we have shown the results only for the case where energy is removed at the wall so that the homogeneous temperature boundary condition is maintained, and the momentum source is placed inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$. We find that the secondary radial flow is weaker when stratification parameter is reduced. The peak value of the secondary radial flow for a binary gas mixture with $\epsilon_a = 0.5$ increases by (19 - 30)% compared to the single-component gas, for the stratification parameter $\mathcal{A}$ in the range $0.707 - 3.535$ at $Re = 10^2$.

7.2. Mass source/sink (binary gas mixture)

Next, we study the secondary radial flow for the mass source/sink type of forcing. Substantial temperature slip at the rotating sidewall near the source is obtained in this case as well. Here, we have presented the results for $\epsilon_a = (2\Delta m/(m_1 + m_2)) = 0.5$; for Reynolds number (Re) in the range $(10^2 - 10^6)$; and stratification parameters ($\mathcal{A}$) in the range $(0.707 - 3.535)$ as shown in figure 13. We find that the temperature slip increases as the mass difference between two species in a binary gas mixture is increased.

The radial velocity in the flow domain, scaled by (ΩR) , is plotted along the scaled radial

FIGURE 13. The variation of dimensionless scaled temperature slip (T_s^*) as a function of stratification parameter (A). Sub-figure (a) for momentum source; and sub-figure (b) for mass source/sink. The symbols show the DSMC results for the sources inside the radial boundary layer at ($r^* = 0.99$); ($\xi_s^* = 0.01$) for single-component gas : ($\text{Re} = 10^6$ (□), $\text{Re} = 10^4$ (△), $\text{Re} = 10^2$ (○)); and the DSMC results for a binary gas mixture with $\epsilon_a = 0.5$ for : ($\text{Re} = 10^6$ (■), $\text{Re} = 10^4$ (▲), $\text{Re} = 10^2$ (●)). Here, $r^* = r/R$; $\xi_s^* = \xi_s/A^2$, and $\epsilon_a = (2\Delta m/(m_1 + m_2))$.

coordinate (r^*) for the isothermal sidewall as shown in figure 15. The mass source/sink is placed inside the radial boundary layer at ($r^* = 0.99$); ($\xi_s^* = 0.01$). We find that the secondary radial flow is weaker when stratification parameter is reduced in this case as well. The peak value of the secondary radial flow for a binary gas mixture with $\epsilon_a = 0.5$ increases by (20 - 25)% compared to the single-component gas, for the stratification parameter A in the range 0.707 – 3.535 at $\text{Re} = 10^2$.

7.3. Stationary intake tube (binary gas mixture)

We have further carried out the DSMC simulations for a binary gas mixture by placing a stationary intake tube within the flow domain. First, we calculate the temperature slip at the rotating sidewall for a binary gas mixture with $\epsilon_a = 0.5$ as shown in figure 16. We find that the temperature slip increases as the mass difference between two species in a binary gas mixture is increased.

The radial velocity in the flow domain, scaled by (ΩR), is plotted along the radial coordinate (r^*) for the isothermal sidewall as shown in figure 17. We find that the secondary radial flow is weaker when stratification parameter is reduced, and the peak value of the secondary radial flow for a binary gas mixture with $\epsilon_a = 0.5$ increases by (19 - 30)% compared to the single-component gas, for the stratification parameter A in the range 0.707 – 3.535 at $\text{Re} = 10^2$.

8. Conclusions

The objective of the present work is to carry out a comprehensive study of the comparison of the results obtained from analytical equation (4.18), and generalized analytical equation (4.19), derived in §4, in the polar ($r-\theta$) plane, with molecular simulations based on the direct simulation Monte Carlo (DSMC) method, for the secondary radial flow in a rotating cylinder. Several situations have been considered, including delta-function momentum source, and mass source/sink within the domain of the flow. Simulations have been carried out for Reynolds number as low as 100, stratification parameter as low as 0.707, and with source strength = 0.2.

FIGURE 14. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$ (a); $\mathcal{A} = 2.121$ (b); $\mathcal{A} = 0.707$ (c), for $\text{Re} = 10^2$. Flow induced by momentum source inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$. The symbols show the DSMC results for isothermal sidewall for single-component gas : ($\mathcal{A} = 3.535$ ($\square$), $\mathcal{A} = 2.121$ ($\triangle$), $\mathcal{A} = 0.707$ ($\circ$)); and the DSMC results for a binary gas mixture with $\epsilon_a = 0.5$ for : ($\mathcal{A} = 3.535$ ($\blacksquare$), $\mathcal{A} = 2.121$ ($\blacktriangle$), $\mathcal{A} = 0.707$ ($\bullet$)). Here, $r^* = r/R$; and $\xi_s^* = \xi_s/\mathcal{A}^2$.

Here, we have studied the secondary radial flow due to insertion of mass, momentum and energy into the rotating cylinder in the polar $(r - \theta)$ plane. The presence of a stationary insert (intake tube) within the flow domain causes a disturbance to the base flow (solid body rotation). A detached bow shock is formed in front of the stationary insert (intake tube), and behind the detached bow shock, secondary radial flow is induced towards the axis, and it excites the secondary axial flow, which could be very important for the centrifugal gas separation processes. The inward radial flow also produces a vortex column near the stationary insert. A mechanically driven gas centrifuge utilizes stationary intake tube (which is used to collect the separated gases), to generate the secondary radial flow, and it further excites the secondary axial flow.

The generalized analytical model for the radial boundary layer (4.19) in a high-speed rotating cylinder is formulated for studying the gas flow field due to insertion of mass, momentum and energy into the rotating cylinder in the polar $(r - \theta)$ plane. The analytical solution includes the sixth order differential equation for the radial boundary layer at the cylindrical surface in terms of master potential (χ), which is derived from the equations of motion in a polar $(r - \theta)$ plane. The linearization approximation is used, where the equations of motion are truncated at linear order in the velocity and pressure disturbances

FIGURE 15. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$ (a); $\mathcal{A} = 2.121$ (b); $\mathcal{A} = 0.707$ (c), for $\text{Re} = 10^2$, with source strength $F^* = 0.2$. Flow induced by a mass source at $(r^* = 0)$ and a mass sink inside the radial boundary layer at $((r^* = 0.99); (\xi_s^* = 0.01))$. The symbols show the DSMC results for isothermal sidewall for single-component gas : ($\mathcal{A} = 3.535$ ($\square$), $\mathcal{A} = 2.121$ ($\triangle$), $\mathcal{A} = 0.707$ ($\circ$)); and the DSMC results for a binary gas mixture with $\epsilon_a = 0.5$ for : ($\mathcal{A} = 3.535$ ($\blacksquare$), $\mathcal{A} = 2.121$ ($\blacktriangle$), $\mathcal{A} = 0.707$ ($\bullet$)). Here, $r^* = r/R$; and $\xi_s^* = \xi_s/\mathcal{A}^2$.

FIGURE 16. The variation of dimensionless scaled temperature slip (T_s^*) as a function of stratification parameter ($\mathcal{A}$) for $\text{Re} = 10^2$. The symbols show the DSMC results without energy removal for single-component gas : ($\theta = 75^\circ$ ($\circ$), $\theta = 60^\circ$ ($\square$)); and the DSMC results for a binary gas mixture with $\epsilon_a = 0.5$ for : ($\theta = 75^\circ$ ($\bullet$), $\theta = 60^\circ$ ($\blacksquare$)).

FIGURE 17. Radial velocity, scaled by (ΩR) , as a function of r^* for $\mathcal{A} = 3.535$ (a); and $\mathcal{A} = 0.707$ (b), for $\text{Re} = 10^2$. The symbols show the DSMC results for sub-figure (a) : (single-component gas ($\square$), binary gas mixture with $\epsilon_a = 0.5$ ($\blacksquare$); The symbols show the DSMC results for sub-figure (b) : (single-component gas ($\circ$), binary gas mixture with $\epsilon_a = 0.5$ ($\bullet$).

to the base flow, which is a solid-body rotation. Additional assumptions in the analytical model include constant temperature in the base state (isothermal condition), and high Reynolds number, but there is no limitation on the stratification parameter. In this limit, the gas flow is restricted to a boundary layer of thickness $(\text{Re}^{-1/3}R)$ at the wall of the cylinder.

For the case of mass/momentum/energy insertion into the flow, the separation-of-variables procedure is used, and the appropriate homogeneous boundary conditions are specified so that the linear operators in the meridional and radial directions are self-adjoint. The discrete eigenvalues and eigenfunctions of the linear operators (sixth-order and second-order in the radial and meridional directions for the generalized analytical equation) are obtained. The solutions for the secondary radial flows are determined in terms of these eigenvalues and eigenfunctions.

The major advantage of the present formulation is that it is not restricted to the asymptotic limit of high stratification parameter. Consequently, the model is not restricted to just boundary layer flows at the curved surface of the cylinder, but is even applicable to situations where the flow domain extends all the way to the centre of the cylinder. Therefore, the model can be used for much slower flows, and since we have used the conservative form of the compressible mass, momentum and energy conservation equations in deriving the generalized analytical model, obviously, the shock type solutions (Rankine-Hugoniot relations) are automatically satisfied.

The analytical solutions are compared with direct simulation Monte Carlo (DSMC) simulations for two types of forcing, the momentum source, and mass source/sink for stratification parameter in the range $0.707 - 3.535$, at Reynolds number as low as 100, and source strength as high as 0.2. Even for this strong forcing, we find that the generalized analytical solutions are in excellent agreement (error within 10%) with the results of DSMC simulations even when the stratification parameter is as low as 0.707, and the Reynolds number is as low as 100, provided precautions are taken to ensure that the homogeneous boundary conditions assumed in the generalized analytical equation (4.19) are accurately reproduced in the simulations. This shows that the generalized analytical model has excellent predictive capabilities even at relatively low Mach and Reynolds numbers, provided the care is taken in the specification of boundary conditions. The

generalized analytical equation (4.19) also represent a significant better agreement with the results of DSMC simulations, over the analytical equation (4.18), which is restricted to thin boundary layers in the limit of high stratification parameter.

In a high speed rotating field we examine the mass flow rate through the stationary intake tube. The simulations show that the scaled mass flow rate $m/(H\Omega)$ increases progressively from zero, at the stagnation condition $P_S/P_W = 1.6489$, $\mathcal{A} = 2.121$, $\text{Re} = 10^2$, $d^* = 0.125$, to, 0.024 when the back pressure is reduced to $P_B/P_W = 0.4692$. The slow down of the circumferential velocity of the bulk of the rotating fluid due to the presence of stationary intake tube within the rotating flow field is studied for stratification parameter in the range 0.707 – 3.535, and found significant slow down (between 8 to 28%), which induces the secondary radial flow towards the axis, and it further excites the secondary axial flow, which could be very important for the centrifugal gas separation processes. An important finding is that the stagnation pressure (no mass flow through the intake tube) is strongly affected by the wall gap, as well as with stratification parameter indicating a strong coupling between the local temperature, density, pressure and velocity fields.

We have also carried out the DSMC simulations for a binary gas mixture with large mass difference ($\epsilon_a = (2\Delta m/(m_1 + m_2)) = 0.5$), and compare the simulation results with those of single-component gas incorporating the homogeneous boundary conditions. An important point to note is that in all cases, internal momentum source, mass source/sink, and stationary intake tube, the peak value of the secondary radial flow for a binary gas mixture with $\epsilon_a = 0.5$ increases by (19 - 30)% compared to the single-component gas, as stratification parameter $\mathcal{A}$ is increased from 0.707 to 3.535, for $\text{Re} = 10^2$.

The present work provides an opportunity for validating the analytical model as well as the DSMC method for the high-speed rotating flow in the polar ($r - \theta$) coordinate. DSMC simulation technique does not become invalid when the mean free path is comparable to the system size (cylinder radius), because the simulation does not use averaged equations. There are many situations in high-speed rotating flows where the gas density in the central region of the cylinder is sufficiently small that the mean free path is comparable to the cylinder radius. In these cases, the Navier-Stokes equations are not valid near the centre, and it is necessary to specify 'artificial boundary conditions' at a fixed inner surface within the flow. The possible dependence of the solutions on these 'artificial boundary conditions' poses difficulties in the interpretation of results. No such difficulties exist in the molecular simulations, since it is only necessary to specify particle-wall interaction parameters. Molecular simulations also have the distinct advantage that the wall slip emerges naturally from the particle-wall collision laws, and it does not have to be modelled. Though DSMC simulation technique overcomes several of the difficulties of a continuum simulation, the computational power and time required for the simulation is much higher than that for a continuum simulation. This is due to the requirement for simulating, directly, a simplified version of the molecular motion. This requires the evaluation of the complete velocity distribution function of the molecule, and temperature fields.

Appendix A. Conservation equations in cylindrical coordinates

The compressible mass, momentum and energy conservation equations in cylindrical polar ($r - \theta$) coordinate system are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = S_m, \quad (\text{A } 1)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} \right) \\ &= -\frac{\partial p}{\partial r} + \left[\frac{1}{r} \frac{\partial}{\partial r} (\tau_{rr} r) - \frac{\tau_{\theta\theta}}{r} + \frac{1}{r} \frac{\partial (\tau_{\theta r})}{\partial \theta} \right] + f_r, \end{aligned} \quad (\text{A } 2)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} \right) \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial (\tau_{\theta\theta})}{\partial \theta} \right] + f_\theta, \end{aligned} \quad (\text{A } 3)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} \right) \\ &= \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial (\tau_{\theta z})}{\partial \theta} \right] + f_z, \end{aligned} \quad (\text{A } 4)$$

$$\begin{aligned} & \rho C_v \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} \right) \\ &= -p(\nabla \cdot \mathbf{v}) + \tau_{rr} \frac{\partial v_r}{\partial r} + \tau_{\theta\theta} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \tau_{r\theta} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) \right) + \tau_{\theta z} \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} \right) + \tau_{rz} \left(\frac{\partial v_z}{\partial r} \right) \\ &+ \frac{1}{r} \frac{\partial}{\partial r} \left(r K \frac{\partial T}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{K}{r} \frac{\partial T}{\partial \theta} \right) + S_e, \end{aligned} \quad (\text{A } 5)$$

where

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta}, \quad (\text{A } 6)$$

$$\tau_{rr} = \mu \left(2 \frac{\partial v_r}{\partial r} - \frac{2}{3} \nabla \cdot \mathbf{v} \right), \quad (\text{A } 7)$$

$$\tau_{\theta\theta} = \mu \left(2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) - \frac{2}{3} \nabla \cdot \mathbf{v} \right), \quad (\text{A } 8)$$

$$\tau_{zz} = \mu \left(-\frac{2}{3} \nabla \cdot \mathbf{v} \right), \quad (\text{A } 9)$$

$$\tau_{r\theta} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right), \quad (\text{A } 10)$$

$$\tau_{rz} = \mu \left(\frac{\partial v_z}{\partial r} \right), \quad (\text{A } 11)$$

$$\tau_{\theta z} = \frac{\mu}{r} \left(\frac{\partial v_z}{\partial \theta} \right). \quad (\text{A } 12)$$

Equation (A 1) is the mass conservation equation, equations (A 2), (A 3) and (A 4) are the momentum conservation equations in the r , θ and z directions, and (A 5) is the energy conservation equation.

Appendix B. Supersonic flow past a circular cylinder:

Time-marching solution and DSMC simulations

Before proceeding to simulate the flow field for the case of mass/momentum/energy insertion into the high speed rotating cylinder, we first compare the DSMC simulation

<i>Fluid</i>	<i>m</i>	<i>d</i>	γ
<i>Fluid</i> – 1	4.6480×10^{-26}	4.170×10^{-10}	1.4
<i>Fluid</i> – 2	58.442×10^{-26}	5.500×10^{-10}	1.066

TABLE 3. The molecular mass m (kg/molecule), molecular diameter d (m) and the ratio of specific heats γ of Fluid-1, and Fluid-2.

results for the supersonic flow past a circular cylinder with the time-marching solution (Anderson (2005)) in order to validate the analysis and simulation procedure. The flow field generated in a high speed rotating cylinder in the polar ($r-\theta$) plane due to stationary insert is represented as an unsymmetrical blunt body flow. A strong curved bow shock is created in front of the stationary body, as a detached shock. The shape and location of the detached shock wave as well as the flow field properties are calculated for a range of free-stream Mach number (M_∞) and angle of incidence (α). A strong curved bow shock is created in front of the stationary body, with the shock detached from the nose of the body by a distance δ . At point a in figure (18), the free-stream flow is normal to the wave; hence point a corresponds to a normal shock wave. Away from the centerline, the shock wave becomes curved and weaker, eventually evolving into a Mach wave at large distances from the body.

The computational domain consists of the flow over a stationary circular cylinder. The number of cells along the radial direction is taken as 200, while 1600 cells are taken along the azimuthal direction. The cell width is about one-third of the local mean free path. The number of subcells per cell is taken as 4 for both radial and azimuthal directions. The number of simulated particles per cell is taken as 100. The number of real molecules per simulated particles varied between 10^{16} and 10^{18} so that the statistical fluctuations not becomes large. The surface is maintained at a constant temperature of 300 K.

Here, we have analyzed the flow field using two fluids. The fluid-1, with the molecular diameter $d = 4.170 \times 10^{-10}$ m, molecular mass $m = 4.648 \times 10^{-26}$ kg/molecule, and specific heat ratio $\gamma = 1.4$; and fluid-2, with the molecular diameter $d = 5.5 \times 10^{-10}$ m, molecular mass $m = 58.442 \times 10^{-26}$ kg/molecule, and specific heat ratio $\gamma = 1.066$ as given in table 3. The hard sphere (HS) molecular model is used in the simulations. It is assumed that the gas molecules undergo diffuse reflection from all the solid surfaces with the thermal accommodation coefficient equal to 1.

Between point a and at large distances from the body, the curved shock goes through all possible conditions allowed for oblique shocks for an free-stream Mach number of (M_∞). Slightly above the centerline at point b , the shock is oblique but pertains to the strong-shock solution. Further along the shock, point c is the dividing point between strong and weak solutions; the streamline through point c experiences the maximum deflection. Slightly above point c , at point c' , the flow becomes sonic behind the shock. From points a to c' , the flow behind the shock is subsonic. Above point c' the flow is supersonic behind the shock. Hence, the flow field between the blunt body and its curved bow shock is mixed subsonic-supersonic flow, and the imaginary dividing curve between these two regions (where $M = 1$) is denoted as the sonic line. The shape of the detached shock wave, its detachment distance δ , and the complete flow field between the shock and the body depend on free-stream Mach number (M_∞) and the size and shape of the body. The steady supersonic flowfields are governed by the hyperbolic differential equations,

FIGURE 18. Computational domain and coordinate system for the supersonic flow past a circular cylinder. Flow condition and detached bow-shock wave in front of the stationary circular cylinder is shown.

FIGURE 19. Representative finite difference grid in physical space for the flow over a circular cylinder (a); and the sectional zoomed view of the simulation grid (b).

whereas the steady subsonic flowfields are described by elliptic differential equations. Here, the flow field generated between the blunt body (stationary circular cylinder) and its curved bow shock is mixed subsonic-supersonic flow, and hence the flowfields contain both subsonic and supersonic regions. The nature of the governing equations changes from elliptic to hyperbolic across the sonic line causes severe mathematical and numerical difficulties so much so that steady flow solutions of the subsonic and supersonic regions are treated separately and differently, and then patched in the transonic region near the sonic line. The powerful time-marching technique have the advantage for the solution of steady mixed subsonic-supersonic flow. For a given free-stream Mach number (M_∞), and the body shape, the shock wave shape and the flow field are calculated. In the time-marching approach, the initial shock wave shape is assumed at time $t = 0$. The abscissa of the shock and body are denoted by s , and b , respectively. The flow field between the assumed shock wave and the specified body is divided into a number of grid points, as

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FIGURE 20. Finite difference grid in transformed space for the flow over a circular cylinder.

FIGURE 21. Shock oriented coordinates for the characteristic treatment of the shock boundary conditions.

shown in figure 19. At each point, values of all the flow field variables are arbitrary set. Then, starting from these gussed values and using time-marching technique, new values of the flow field variables, shock detachment distance, and shock shape are calculated in steps of time. After a number of time steps, the flow field converges to the proper steady state value; this steady state is the desired result, and the time-marching technique is a means to that end. The governing equations for two-dimensional isentropic flow are,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad (\text{B } 1)$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \quad (\text{B } 2)$$

FIGURE 22. Construction of the unsteady characteristic at the shock wave.

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y}, \quad (\text{B } 3)$$

$$\frac{Ds}{Dt} = 0. \quad (\text{B } 4)$$

Here, the energy equation is stated in terms of the isentropic assumptions. For isentropic condition $p/\rho^\gamma = \text{constant}$ for a calorically perfect gas. Hence, the B 4 can be written alternatively as,

$$\frac{D(p/\rho^\gamma)}{Dt} = 0. \quad (\text{B } 5)$$

Expanding the derivative in equation B 5, and defining $\psi = \ln p - \gamma \ln \rho$, we obtain the energy equation in the form

$$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} = 0, \quad (\text{B } 6)$$

The system of equations to be solved are equation B 1 through B 3, and B 6, four equations for the four unknowns p , ρ , u , and v .

The grid shown in figure (19) is not rectangular; hence, it is inconvenient for the formation of finite difference. To transform the shock layer into a rectangular grid, a new independent variable ζ is defined such that

$$\zeta = (x - b) / \delta, \quad (\text{B } 7)$$

where $\delta = s - b$, i.e., δ is the local shock detachment distances. In this fashion, ζ always varies between 0 (at the body) and 1 (at the shock), and the grid now appears as sketched as figure (20). In addition, let $w = ds/dt$ be the x component of the shock wave velocity, and let θ represent the angle between the tangent to the shock and x axis. In addition, the following variables are defined,

$$C = (\zeta - 1) \frac{db}{dy} - \zeta \cot \theta,$$

$$P = \ln p,$$

$$Ro = \ln \rho,$$

$$B = \frac{(u - W\zeta + vC)}{\delta}, \quad (\text{B } 8)$$

where,

$$\begin{aligned} \psi &= \ln p - \gamma \ln \rho, \\ &= P - \gamma Ro. \end{aligned} \quad (\text{B } 9)$$

Finally, all the variables are nondimensionalized as follows : Divide p and ρ by their free stream values p_∞ , and ρ_∞ ; divide the velocities by $(p_\infty/\rho_\infty)^{1/2}$; divide the length by a characteristic length L , and nondimensional time is obtained by dividing the dimensional time by $L/(p_\infty/\rho_\infty)^{1/2}$. In terms of these nondimensional parameters the resulting equations are,

$$\frac{\partial Ro}{\partial t} = - \left[B \frac{\partial R}{\partial \zeta} + \frac{1}{\delta} \frac{\partial u}{\partial \zeta} + \frac{C}{\delta} \frac{\partial v}{\partial \zeta} + \frac{\partial v}{\partial y} + v \frac{\partial Ro}{\partial y} \right] = 0, \quad (\text{B } 10)$$

$$\frac{\partial u}{\partial t} = - \left[B \frac{\partial u}{\partial \zeta} + v \frac{\partial u}{\partial y} + \frac{P}{\rho \delta} \frac{\partial P}{\partial \zeta} \right], \quad (\text{B } 11)$$

$$\frac{\partial v}{\partial t} = - \left[B \frac{\partial v}{\partial \zeta} + v \frac{\partial v}{\partial y} + \frac{PC}{\rho \delta} \frac{\partial P}{\partial \zeta} + \frac{p}{\rho} \frac{\partial P}{\partial y} \right], \quad (\text{B } 12)$$

$$\frac{\partial \psi}{\partial t} = - \left[B \frac{\partial \psi}{\partial \zeta} + v \frac{\partial \psi}{\partial y} \right]. \quad (\text{B } 13)$$

To advance the flow field in steps of time, the grid points in figure (20) are treated as four distinct sets; interior points (all grid points except on the shock, body, downstream boundary); the shock points ($\zeta = 1$); the body points ($\zeta = 0$); and the downstream boundary points. The flow field at the interior points is advanced in time by means of MacCormack's predictor-corrector method. Note that equations B 10 through B 13 are written with the known spatial derivatives on the right hand side; these derivatives are replaced with forward differences on the predictor step and backward differences on the corrector step. This allows the calculation of the time derivatives that appear on the left hand side of equation B 10 through B 13. In turn, these time derivatives ultimately lead to the advancement of the flow field in steps of time.

For the shock points, the values of the flow variables behind the shock at time $t + \Delta t$ can be obtained from the Rankine-Hugoniot relations for a moving shock wave. However, this implies that a value of the shock wave velocity, $W(t + \Delta t)$, must first be assumed, since it is not known at the beginning of each time step in the computations. Therefore, an iterative process must be established wherein the values of the flow field variables at the shock grid points must be obtained from some independent calculation, and then compared with those obtained from the shock relations for the assumed W . The independent calculation is made via characteristic technique utilizing information from the interior points. Specifically, the characteristic equations are obtained from the two dimensional unsteady governing equations written for a (ξ, η, t) coordinate frame, where ξ and η are cartesian coordinates locally normal and tangential, respectively, to the shock wave as illustrated in figure (21). The assumption is made in obtaining the compatibility equations that the governing equations can be written as quasi-one-dimensional in the ξ direction, modified by forcing terms containing derivatives in the tangential direction. That is, the characteristic directions are drawn in the (ξ, η) plane only. as shown in figure (22). This characteristic line, along with the compatibility equations, allows the

flow field to be calculated at $\xi = 0$ (the shock point) at time $t + \Delta t$, i.e., point Q in figure (22), from the known flow field at point A at time t . Finally, the information at point Q obtained from this characteristics approach is compared with the information calculated from the shock relations for the assumed W , and if the agreement is not obtained, new values of W are assumed until the iteration converges.

A similar characteristics calculation is performed for the body points. Here, the analysis is simpler because the body is stationary, and also the entropy is known at the body. Note that, $Ds/Dt = 0$, i.e., the entropy of a given fluid element is constant, even when the flow is unsteady. The entropy at the stagnation point must be chosen equal to its proper steady state value for a normal shock, because this entropy wets the entire body surface and is constant throughout the time marching calculation. Finally, the flow field values at the downstream boundary are obtained by linear extrapolation from the upstream grid points.

The time-marching technique with the use of MacCormack's approach is explicit. For such an explicit solution, the value of Δt cannot be any arbitrary value; indeed; it must be less than or equal to some maximum value. This maximum value is usually estimated from a stability analysis performed on a set of approximate, linear equations after Courant, Friedrichs, and Lewy (Anderson (2005)), the so called CFL criteria. The physical significance of CFL criteria is that Δt must be less than or at most equal to the time required for a sound wave to propagate between two adjacent grid points. If we consider a two-dimensional rectangular grid as shown in figure 20, where at any grid point the flow field velocity is V , with x and y components u and v , respectively. The velocity of propagation of a sound wave in x direction is $u + a$, and the time of propagation is

$$\Delta t_x = \frac{\Delta x}{u + a}. \quad (\text{B } 14)$$

Similarly, in the y direction,

$$\Delta t_y = \frac{\Delta y}{v + a}. \quad (\text{B } 15)$$

The CFL criteria can then be expressed as

$$\Delta t \leq \min(\Delta t_x, \Delta t_y). \quad (\text{B } 16)$$

In practice, Δt is chosen such that

$$\Delta t = K \min(\Delta t_x, \Delta t_y), \quad (\text{B } 17)$$

where K is less than unity, typically on the order of 0.5 to 0.8. In the present analysis we have taken $K = 0.5$.

Now we examine the results obtained with the time-marching solution for the supersonic flow over a stationary circular cylinder. These solutions are then compared with direct simulation Monte Carlo (DSMC) simulations. The steady state shock shapes and the sonic lines are shown in figure 23 for the circular cylinder at free-stream Mach number (M_∞) = 3, 4, 6, with zero angle of attack ($\alpha = 0$). Note that, as the free-stream Mach number increases, the shock detachment distance decreases and the sonic line shifted to the downward direction and the subsonic region become smaller. Next, we have compared the results for fluid-1 and fluid-2. If we compare the mechanical properties of fluid-2 to those of fluid-1, we can see that the ratio of heat capacity ($C_P/C_v = \gamma$) is very low, $\gamma = 1.066$ for fluid-2, whereas for fluid-1 $\gamma = 1.4$, and that the density of fluid-2 is about 10 times that of fluid-1. From the results as shown in figure 24, we see that the shock wave is closer to the body for fluid-2 compare to fluid-1.

Next, we have performed the analysis to study the effects of free-stream Mach number

FIGURE 23. Flow with various free-stream Mach number (M_∞) with zero angle of attack ($\alpha = 0$). The lines show the results obtained with the time marching solution for : ($M_\infty = 6$ ($\cdot\cdot-\cdot\cdot$), $M_\infty = 4$ ($--$), $M_\infty = 3$ ($\cdot-\cdot$)). The sybmols show the DSMC results for : ($M_\infty = 6$ ($\square$), $M_\infty = 4$ ($\circ$), $M_\infty = 3$ ($\triangle$)). The detached shock wave and the corresponding sonic lines are shown for fluid-2.

FIGURE 24. Comparison of flow field for fluid-1 and fluid-2 with free-stream Mach number $M_\infty = 4$ with zero angle of attack ($\alpha = 0$). The lines show the results obtained with the time marching solution for : (fluid-1 ($--$), fluid-2 ($\cdot-\cdot$)). The sybmols show the DSMC results for : (fluid-1 ($\circ$), fluid-2 ($\triangle$)). The detached shock wave and the corresponding sonic lines are shown.

(M_∞) on the shape of the shock wave for an incident angle of 5 as shown in figure 25. We observed that, as the free-stream Mach number (M_∞) increases, the shock wave gets closer to the body, the sonic line shifted to the downward direction, and subsonic region gets smaller.

We find that there is excellent agreement between the time-marching solutions and the DSMC simulations, to within 5-10 %, for the ranges of values of parameters studied here.

FIGURE 25. Flow with free-stream Mach number (M_∞) in the range (3–6) with angle of incident $\alpha = 5$. The lines show the results obtained with the time marching solution for : ($M_\infty = 6$ ($\cdots$)), $M_\infty = 4$ ($--$), $M_\infty = 3$ ($\cdot - \cdot$)). The symbols show the DSMC results for : ($M_\infty = 6$ ($\square$), $M_\infty = 4$ ($\circ$), $M_\infty = 3$ ($\triangle$)). The detached shock wave and the corresponding sonic lines are shown for fluid-2.

REFERENCES

- ANDERSON, J. D. 2005 Fundamentals of Aerodynamics. *McGraw-Hill*.
- AOKI, E. & SUZUKI, M. 1985 Numerical studies on gas flow around scoop of a gas centrifuge. *In Proc. Sixth Workshop on Gases in Strong Rotation, Tokyo* (ed. Y. Takashima), 11.
- EVERY, D. G. & DAVIES, E. 1973 Uranium enrichment by gas centrifuge *Mills and Boon Ltd. London* **1**, 1.
- BABARSKY, R. J. & WOOD, H. G. 1990 Approximate eigensolutions for non-axisymmetric rotating compressible flows. *Computer Meth. Appl. Mech. and Engg.* **81**, 317.
- BABARSKY, R. J. & WOOD, H. G. 1992 Analysis of the flow field in a gas centrifuge by means of a temperature potential. *In Proc. of the Third International Workshop on Separation Phenomena in Liquids and Gases* (ed. Wood, H. G.), 27.
- BABARSKY, R. J. & HERBST, W. I. & WOOD, H. G. 2002 A new variational approach to gas flow in a rotating system. *Physics of fluids*. **14**(10), 3624.
- BARCILON, V. & PEDLOSKY, J. 1967 On the steady motions produced by a stable stratification in a rapidly rotating fluid. *J. Fluid Mech.* **29**, 673.
- BARK, F. H. & BARK, T. H. 1976 On vertical boundary layers in a rapidly rotating gas. *J. Fluid Mech.* **78**, 749.
- BIRD, G. A. 1963 Approach to translational equilibrium in a rigid sphere gas. *Physics of fluids* **6**, 1518.
- BIRD, G. A. 1994 Molecular gas dynamics and the direct simulation of gas flows. *Clarendon Press, Oxford*.
- BOURN, R. & PETERSON, T. D. & WOOD, H. G. 1999 Solution of the pancake model for flow in a gas centrifuge by means of a temperature potential. *Comput. Meth. Appl. Mech. Engng.* **178**, 183.
- BROUWERS, J. J. H. 1978 On compressible flow in a gas centrifuge and its effect on maximum separative power. *Nucl. Technology*. **39**, 311.
- CERCIGNANI, C. 2000 Rarefied gas dynamics. From basic concepts to actual calculations. *Cambridge University Press*.
- CHAPMAN, S. & COWLING, T. G. 1970 The Mathematical Theory of Non-Uniform Gases. *Cambridge University Press*.
- COHEN, K. 1951 The theory of isotope separation as applied to the large scale production of U235. *McGraw-Hill*.
- COPPER, G. R. & MORTON, J. B. 1988 Driven Wave Motions in a Rotating Gas. *J. Fluid Struct* **2**, 453.
- DONEDDU, F. & ROBIN, P. & WOOD, H. G. 2000 Optimization Studies for Gas Centrifuges. *Separation Science and Technology*. **35**, 1207.
- FRAENKAL, L. E. 1959 A Cylindrical sound pulse in a rotating gas. *J. Fluid Mech.* **36**, 637.
- GARCIA. 2000 Time step truncation error in direct simulation Monte Carlo. *Physics of fluids* **12**, 2621.
- GREENSPAN, H. P. 1968 The theory of Rotating Fluids. *Cambridge University Press* **1**, 1.
- GUNZBURGER, M. D. & WOOD, H. G. 1982 A finite element method for the Onsager pancake equation. *Comp. Meth. Appl. Mech. Engg.* **31**, 43.
- GUNZBURGER, M. D. & WOOD, H. G. & JORDAN, J. A. 1984 A finite element method for gas centrifuge problem. *SIAM J. Sci. Stat. Comp.* **5**.
- HARADA, I. 1980 A Numerical Study of Weakly Compressible Rotating Flows in a Gas Centrifuge. *Nucleas Sc. and Engg.* **73**, 225.
- HARADA, I. 1980 Computation of Strongly Compressible Rotating Flows. *J. Computational Physics*. **38**, 335.
- HAUSMAN, M. A. & ROBERTS, W. W. 1984 Three dimensional stratified gas flows past an obstacle. *J. Comp. Phys.* **55**, 347.
- HEUSER, G. E. & RIBANDO, R. J. & WOOD, H. G. 1986 A numerical simulation of internal waves in a rotating fluid. *Comput. Methods Appl. Mech. Engng.* **57**, 207.
- HITTINGER, M. & HOLT, M. & SOUBBARAMAYER, B. 1981 Numerical solution of the flow field near a gas centrifuge scoop. *In Proc. Fourth Workshop on Gases in Strong Rotation, Oxford* (ed. E. Ratz), 22.
- HITTINGER, M. & HOLT, M. & SOUBBARAMAYER, B. & CORTET, C. 1983 Simulation of gas

- flow in front of the scoop of a gas centrifuge. *In Proc. Fifth Workshop on Gases in Strong Rotation, Charlottesville* (ed. H. G. Wood) , 515.
- HOGLUND, R. L. & SHACTER, J. & VON HALLE, E. 1979 Diffusion separation methods. *Encyclopedia of chemical technology*. **7**, 3rd ed.
- HOMSY, G. M. & HUDSON, J. L. 1967 Centrifugally driven thermal convection in a rotating cylinder. *J. Fluid Mech.* **35**, 33.
- HUNTER, C. 1967 The axisymmetric flow in a rotating annulus due to horizontally applied temperature gradient. *J. Fluid Mech.* **27**, 753.
- JONES, C. A. 1982 On flow between counter-rotating cylinders. *J. Fluid Mech.* **120**, 433.
- JUNG, E. 1983 An analytic solution of the linearized flow equations using the method of eigenvalues. *In Proc. Fifth Workshop on Gases in Strong Rotation, Charlottesville* (ed. H. G. Wood) , 247.
- KAI, T. 1983 Analysis of Separation Performance of Gas Centrifuge. *In Proc. Fifth Workshop on Gases in Strong Rotation, Charlottesville* (ed. H. G. Wood) , 1.
- KUMARAN, V. & PRADHAN, S. 2014 The generalized Onsager model for a binary gas mixture. *J. Fluid Mech.* **753**, 307.
- LAHARGUE, J. P. & SOUBBARAMAYER 1978 A numerical model for the investigation of the flow and isotope concentration field in an ultracentrifuge. *Comp. Meth. Appl. Mech. Engg.* **15**, 259.
- MASLEN, S. H. & MOORE, F. K. 1956 On strong transverse waves without shocks in a circular cylinder. *J. Aeronautical Sciences* **23**, 583.
- MATSUDA, T. & HASHIMOTKO, K. 1976 Thermally, mechanically or externally driven flow in a gas centrifuge with insulated horizontal end plates. *J. Fluid Mech.* **78**, 337.
- MATSUDA, T. 1976 A New proposal of gas centrifuge with desirable counter-current. *J. Nucl.Sci. Tech.* **13**, 98.
- MATSUDA, T. & HASHIMOTO, K. 1978 The Structure of the Stewartson layers in a gas centrifuge. Part 1: Insulated end plates. *J. Fluid Mech.* **85**, 433.
- MATSUDA, T. & NAKAGAWA, K. 1983 A new type of boundary layer in a rapidly rotating gas. *J. Fluid Mech.* **126**, 431.
- MATSUDA, T. & TAMURA, N. & SAWADA, K. 1989 Three-dimensional numerical simulation of flows past scoops in a gas centrifuge. *J. Fluid Mech.* **201**, 203.
- MIKAMI, H. 1973 Thermally induced flow in a gas centrifuge. *J.Nucl.Sci.Tech.* **10**, 580.
- MIKAMI, H. 1981 Rotating supersonic flow about scoop inlet using an unsteady implicit technique. *In Proc. Fourth Workshop on Gases in Strong Rotation, Oxford* (ed. E. Ratz), 94.
- MIKAMI, H. 1985 Two dimensional numerical calculation of flow fields about scoop inlet by the piecewise linear method. *In Proc. Sixth Workshop on Gases in Strong Rotation, Tokyo* (ed. Y. Takashima), 67.
- NAKAYAMA, W. & TORII, T. 1974 Numerical analysis of separative power of isotope centrifuges. *J. Nucl. Sci. Technology*. **11**, 495.
- NANBU. 1980 Direct simulation scheme derived from the Boltzmann equation I. Multicomponent gases. *J. Phys. Soc. of Japan*. **45**, 2042.
- OLANDER, D. R. 1981 The theory of uranium enrichment by the gas centrifuge. *Progress in Nuclear Energy*. **8**, 1.
- OMNES, P. 2007 Numerical and physical comparisons of two models of a gas centrifuge. *Computers Fluids* **36**, 1028.
- PRADHAN, S. & KUMARAN, V. 2011 The generalized Onsager model for the secondary flow in a high-speed rotating cylinder. *J. Fluid Mech.* **686**, 109.
- QUANHUA, S. & BOYD, I. D. 2002 A Direct Simulation Method for Subsonic Microscale Gas Flows . *J. Computational Physics*. **179**, 400.
- RIBANDO, R. J. 1984 A Finite difference solution of Onsager's model for flow in a gas centrifuge. *Computers Fluids* **12 (3)**, 235.
- ROBERTS, W. W. 1985 Three dimensional stratified gas flows past impact probes and scoops : N-body Monte Carlo calculations. *In Proc. Sixth Workshop on Gases in Strong Rotation, Tokyo* (ed. Y. Takashima.), 115.
- ROBERTS, W. W. & HAUSMAN, M. A. 1988 Hypersonic, stratified gas flows past an obstacle : Direct Simulation Monte Carlo calculations. *J. Comp. Phys.* **77**, 283.
- ROBLIN, P. & DONEDDU, F. 2000 Direct Monte-Carlo Simulations In a Gas Centrifuge. *In*

- Proc. 22nd International Symposium on Rarefied Gas Dynamics.* (ed. T. J. Bartel & M. C. Gallis.), 169.
- SAKURAI, T. 1981 Linearized thin-wing theory of gas-centrifuge scoops. *J. Fluid Mech.* **103**, 257.
- SALANT, R. F. 1968 Symmetric normal modes in a uniformly rotating fluid. *J. Acoustical Society of America.* **43**, 1302.
- STEWARTSON, K. 1957 On almost rigid rotations. *J. Fluid Mech.* **3**, 17.
- SOUBBARAMAYER. 1979 Centrifugation. In *Uranium Enrichment* (Ed. S. Villani, Springer) **1**, 1.
- SOUBBARAMAYER & BILLET, J. 1980 A numerical method for optimizing the gas flow field in a centrifuge. *Comp. Meth. Appl. Mech. Engg.* **24**, 165.
- SOZOU, C. 1959 Adiabatic transverse modes in a uniformly rotating fluid. *J. Fluid Mech.* **36**, 605.
- THOMPSON, J. J. & CHEN, Y. P. 1980 Eigen vector solution of the linearized equations of thermal convection in a closed gas centrifuge. Part-1 : Problem formulation and solution by collocation. *Nuclear Engineering and Design* **58**, 97.
- VIECELLI, J. A. 1983 Exponential difference operator approximation for the sixth order Onsager equation. *J. Comp. Phys.* **50**, 162.
- VINCENTI. 1967 Introduction to Physical Gas Dynamics. Wiley, New York.
- VOLOSCIUK, K. 1981 Application of the vortex transport equations to the calculation of 2 and 3 dimensional extraction chamber flows. In *Proc. Fourth Workshop on Gases in Strong Rotation, Oxford* (ed. E. Ratz), 604.
- VON HALLE E. & WOOD, H. G., LOWRY, R. A. 1978 The effect of vacuum core boundary conditions on separation in the gas centrifuge. *Nucl. Technology.* **62**, 325.
- WALZ, A. & VOLOSCIUK, K. & SCHUTZ, H. 1983 Numerical investigations of the flow field near a model of a scoop using the vortex transport equations. In *Proc. Fifth Workshop on Gases in Strong Rotation, Charlottesville* (ed. H. G. Wood), 425.
- WOOD, H. G. & MORTON, J. B. 1980 Onsager's pancake approximation for the fluid dynamics of a gas centrifuge. *J. Fluid Mech.* **101**, 1.
- WOOD, H. G. & SANDERS, G. 1983 Rotating compressible flows with internal sources and sinks. *J. Fluid Mech.* **127**, 299.
- WOOD, H. G. & JORDAN, J. A. & GUNZBURGER, M. D. 1984 The effect of curvature on the flow field in rapidly rotating gas centrifuges. *J. Fluid Mech.* **140**, 373.
- WOOD, H. G. & BABARSKY, R. J. 1992 Analysis of a rapidly rotating gas in a pie-shaped cylinder. *J. Fluid Mech.* **239**, 249.
- WOOD, H. G. 1995 Analysis of feed effects on a single-stage gas centrifuge cascade. *Separation Science and Technology.* **30(13)**, 2631.
- WOOD, H. G. & MASON, T. C. & SOUBBARAMAYER. 1996 Multi-Isotope separation in a gas centrifuge using Onsager's Pancake model. *Separation Science and Technology.* **31(9)**, 1185.