On Permutation Weights and q -Eulerian Polynomials

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Abstract. Weights of permutations were originally introduced by Dugan et al. (*Journal of Combinatorial Theory, Series A* 164:24–49, 2019) in their study of the combinatorics of tiered trees. Given a permutation σ viewed as a sequence of integers, computing the weight of σ involves recursively counting descents of certain subpermutations of σ . Using this weight function, one can define a q -analog $E_n(x, q)$ of the Eulerian polynomials. We prove two main results regarding weights of permutations and the polynomials $E_n(x, q)$. First, we show that the coefficients of $E_n(x, q)$ stabilize as n goes to infinity, which was conjectured by Dugan et al. (*Journal of Combinatorial Theory, Series A* 164:24–49, 2019), and enables the definition of the formal power series $W_d(t)$, which has interesting combinatorial properties. Second, we derive a recurrence relation for $E_n(x, q)$, similar to the known recurrence for the classical Eulerian polynomials $A_n(x)$. Finally, we give a recursive formula for the numbers of certain integer partitions and, from this, conjecture a recursive formula for the stabilized coefficients mentioned above.

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1. Introduction

Dugan et al. defined certain weights of permutations in their work on the combinatorics of tiered trees [2]. Tiered trees are a generalization of maxmin trees, which were originally introduced by Postnikov [8] and appear in the study of local binary search trees [4], hypergeometric functions [5], and hyperplane arrangements [9, 13]. In [2], Dugan et al. defined weight for tiered trees, motivated by their role in the enumerations of absolutely irreducible representations of certain quivers and torus orbits on certain homogeneous varieties [6]. They showed that weight 0 maxmin trees are in natural bijection with permutations. They defined the weight of a permutation σ as the largest weight

of a maxmin tree that can be constructed from σ . Computing the weight of σ involves recursively counting descents of certain subpermutations of σ .

Using the weight of a permutation, a new q -analog of the Eulerian polynomials $E_n(x, q)$ was defined in [2]. The q -Eulerian polynomial $E_n(x, q)$ differs from other q -Eulerian polynomials by Carlitz [1], Foata-Schützenberger [3], Stanley [12], and Shareshian-Wachs [10], and presents interesting combinatorial properties. It was observed in [2] that the coefficients of $E_n(x, q)$ seemed to exhibit a certain *stabilization phenomenon*. From this conjecture, the formal power series $W_d(t)$ was extracted from $E_n(x, q)$ and observed to have a connection with enumerations of a certain type of partition $T(n, k)$, see sequence A256193 in [11].

We present two main results regarding permutation weights and the q -Eulerian polynomials $E_n(x, q)$. We prove the stabilization phenomenon conjectured in [2], which gives an explicit formula for the formal power series $W_d(t)$ defined in [2]. We derive a recurrence relation for the q -Eulerian polynomials $E_n(x, q)$, similar to the known recurrence for the classical Eulerian polynomials $A_n(x)$.

This paper is organized as follows. In Sect. 2, we introduce preliminary definitions and notation. In Sect. 3, we define the weight disparity of a permutation and find a lower bound for weight disparity. We then use this result to show that the coefficients of x^d in the formal power series $W_d(t)$ do indeed stabilize. In Sect. 4, we derive a recurrence relation for $E_n(x, q)$. In Sect. 5, we determine the conditions for which coefficients of $E_n(x, q)$ are stabilized and give a second proof of the stabilization phenomenon. We conclude with a conjecture regarding the coefficients of $W_d(t)$ in Sect. 6.

2. Preliminaries

Let S_n denote the set of permutations of $[n] = \{1, 2, \dots, n\}$. We consider the permutation $\sigma \in S_n$ as an ordered sequence of integers $a_1 a_2 \dots a_n$. We say that $\sigma \in S_n$ is a permutation of length n . We say that $i \in [n-1]$ is a descent of σ if $a_i > a_{i+1}$, and we write $\text{des}(\sigma)$ to denote the number of descents in σ . The definitions in this section follow from [2].

Definition 2.1. Given a permutation $\sigma = a_1 a_2 \dots a_n$, we define a method for splitting σ into certain subpermutations $\sigma_1, \sigma_2, \dots, \sigma_j$.

- (i) Find the minimum element of σ , call it a_m . Divide σ into subpermutations σ_ℓ , a_m , and σ_r , where $\sigma_\ell = a_1 a_2 \dots a_{m-1}$ and $\sigma_r = a_{m+1} a_{m+2} \dots a_n$. We have $\sigma = \sigma_\ell \cdot a_m \cdot \sigma_r$.
- (ii) Find the largest element of σ_ℓ , call it a_k . Let $\sigma_{\ell_1} = a_1 a_2 \dots a_k$ and let $\sigma_{\ell_2} = a_{k+1} a_{k+2} \dots a_{m-1}$. We have $\sigma = \sigma_{\ell_1} \cdot \sigma_{\ell_2} \cdot a_m \cdot \sigma_r$.
- (iii) Repeat step (ii) for σ_{ℓ_2} until σ_ℓ cannot be divided further. This process results in a collection of subpermutations $\sigma_{\ell_1}, \sigma_{\ell_2}, \dots, \sigma_{\ell_s}, \sigma_r$.

Example 2.1. We split the permutation $\sigma = 839562147$.

By step (i), we have $839562 \cdot 1 \cdot 47$.

By step (ii), we have $839 \cdot 562 \cdot 1 \cdot 47$.

By step (iii), we have $839 \cdot 56 \cdot 2 \cdot 1 \cdot 47$.

Since σ cannot be split further, we are done.

Definition 2.2. Let $\sigma \in S_n$. Then the *weight* of σ is the function $w : S_n \rightarrow \mathbb{Z}^{\geq 0}$ defined recursively as follows:

- (i) If σ is the identity permutation or $n = 1$, then $w(\sigma) = 0$.
- (ii) Otherwise, consider σ as a permutation of S_{n+1} by appending $n + 1$ to the right of σ . Let $\sigma_1, \sigma_2, \dots, \sigma_j$ be the subpermutations of this new permutation as in Definition 2.1. Then the weight of σ is defined as

$$w(\sigma) = \sum_{i=1}^j (\text{des}(\sigma_i) + w(\sigma_i)).$$

Example 2.2. Consider the permutation $\sigma = 781659243 \in S_9$. We complete σ to $781659243A$ by adding a maximal element A to the end. After splitting, we have $78 \cdot 1 \cdot 659243A$. The weight of σ is given by

$$w(\sigma) = (w(78) + \text{des}(78)) + (w(1) + \text{des}(1)) + (w(659243A) + \text{des}(659243A)).$$

We compute $w(\sigma)$ as follows:

- Since the permutation 1 is of length 1 , we have $w(1) = 0$ and $\text{des}(1) = 0$.
- It is clear that $w(78) = 0$ and $\text{des}(78) = 0$.
- If we consider the numbers in the permutation $659243A$ relative to each other, we can see that $659243A$ flattens to the permutation 5461327 . We complete the permutation to 54613278 and split to get $546 \cdot 1 \cdot 3278$. So the weight of $659243A$ is given by $w(659243A) = (w(546) + \text{des}(546)) + (w(1) + \text{des}(1)) + (w(3278) + \text{des}(3278))$. A quick computation shows that $w(546) = 0$ and $w(3278) = 0$, so we have $w(65392) = (0 + 1) + (0 + 0) + (0 + 1) = 2$.

Thus $w(\sigma) = (0 + 0) + (0 + 0) + (2 + 3) = 5$.

Definition 2.3. The *Eulerian polynomial* $A_n(x)$ is defined as

$$A_n(x) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)}.$$

The coefficient of x^m in $A_n(x)$ count the permutations in S_n with m descents and are the *Eulerian numbers*, denoted $A(n, m)$.

Using the permutation statistic of weight, a new generalization of the Eulerian polynomials $E_n(x, q)$ was defined in [2].

Definition 2.4. The *q -Eulerian polynomial* $E_n(x, q)$ is defined as

$$E_n(x, q) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} q^{w(\sigma)}.$$

Example 2.3. We list the q -Eulerian polynomials $E_n(x, q)$ up to $n = 7$:

$$E_0(x, q) = 1$$

$$E_1(x, q) = 1$$

$$E_2(x, q) = 1 + x$$

$$E_3(x, q) = 1 + x(q + 3) + x^2$$

$$E_4(x, q) = 1 + x(q^2 + 3q + 7) + x^2(q^2 + 4q + 6) + x^3$$

$$E_5(x, q) = 1 + x(q^3 + 3q^2 + 7q + 15) + x^2(q^4 + 4q^3 + 11q^2 + 25q + 25) \\ + x^3(q^3 + 5q^2 + 10q + 10) + x^4$$

$$E_6(x, q) = 1 + x(q^4 + 3q^3 + 7q^2 + 15q + 31) \\ + x^2(q^6 + 4q^5 + 11q^4 + 31q^3 + 58q^2 + 107q + 90) \\ + x^3(q^6 + 5q^5 + 16q^4 + 34q^3 + 76q^2 + 105q + 65) \\ + x^4(q^4 + 6q^3 + 15q^2 + 20q + 15) + x^5$$

$$E_7(x, q) = 1 + x(q^5 + 3q^4 + 7q^3 + 15q^2 + 31q + 63) \\ + x^2(q^8 + 4q^7 + 11q^6 + 31q^5 + 65q^4 \\ + 149q^3 + 237q^2 + 392q + 301) + x^3(q^9 \\ + 5q^8 + 16q^7 + 41q^6 + 104q^5 + 203q^4 + 380q^3 \\ + 609q^2 + 707q + 350) \\ + x^4(q^8 + 6q^7 + 22q^6 + 55q^5 + 106q^4 + 210q^3 + 336q^2 + 315q + 140) \\ + x^5(q^5 + 7q^4 + 21q^3 + 35q^2 + 35q + 21) + x^6$$

3. Proof of Stabilization Phenomenon

Let us fix k and consider the coefficients of x^k in $E_n(x, q)$. It was conjectured in [2] that as n goes to infinity, these coefficients converge to a fixed sequence and thus display a *stabilization phenomenon*. Observe that for $k = 2$ in Example 2.3, the coefficients of x^2 seem to stabilize to

$$q^N + 4q^{N-1} + 11q^{N-2} + 31q^{N-3} + 65q^{N-4} + \dots$$

where N is the maximum weight of a permutation of length n with 2 descents. Using this conjectural observation, the formal power series $W_d(t) \in \mathbb{Z}[[t]]$ was extracted in [2] and observed to have interesting combinatorial properties.

In this section, we prove the stabilization phenomenon conjectured in [2]. We first define the weight disparity of a permutation and find a lower bound for weight disparity in Theorem 3.4. We then use this bound to prove Theorem 3.6. We thus give an explicit formula for the formal power series $W_d(t)$, enabling the study of certain combinatorial properties in the coefficients of $W_d(t)$.

Definition 3.1. We denote the *maximum weight* of a permutation of length n with d descents by $\text{maxwt}(n, d)$.

It was shown in [2] that the maximum weight of such a permutation is $d(n - d - 1)$.

The following lemma follows directly from the proof of Theorem 6.10 in [2]. A permutation that ends in 1 is of maximum weight when its subpermutation π_L is a single component, in which case $w(\sigma) = \text{des}(\pi_L) + w(\pi_L) \leq (d-1)(n-d-1)$.

Lemma 3.2. *Let $\sigma \in S_n$ be a permutation with d descents. If we fix the last number in σ to be 1, then $\text{maxwt}(\sigma) = (d-1)(n-d-1)$.*

Definition 3.3. Let $\sigma \in S_n$ be a permutation with d descents. We define the *weight disparity* $\Delta(\sigma)$ of σ as $\Delta(\sigma) = \text{maxwt}(n, d) - w(\sigma)$.

We now prove a lower bound for weight disparity, $\Delta(\sigma)$.

Theorem 3.4. *Let $\sigma \in S_n$ be a permutation with d descents. If σ does not start with 1, then $\Delta(\sigma) \geq n-d-1$.*

Let $\sigma \in S_n$ be a permutation with d descents such that $\sigma = \pi_L \cdot 1 \cdot \pi_R$, where the subpermutation π_L is nonempty. We have two cases:

- (i) The subpermutation π_R is empty. Then $\sigma = \pi_L \cdot 1$, and by Lemma 3.1, we have $w(\sigma) \leq (d-1)(n-d-1)$ so $\Delta(\sigma) \geq n-d-1$.
- (ii) The subpermutation π_R is nonempty. Then $\sigma = \pi_L \cdot 1 \cdot \pi_R$. Let π_L be a permutation on ℓ numbers with $\text{des}(\pi_L) = q$. We have $\pi_R \in S_{n-\ell-1}$ and $\text{des}(\pi_R) = d-q-1$. We proceed to bound $w(\sigma)$. Computing the weight of σ , we have

$$w(\sigma) = w(\pi_L \cdot 1) + w(\pi_R) + \text{des}(\pi_R),$$

which implies

$$w(\sigma) \leq q(\ell - q - 1) + (d - q - 1)(n - \ell - d + q).$$

Since $\text{des}(\pi_R) = d - q - 1 \geq 0$ and $\text{des}(\pi_L) = q < \ell$, we have $(q - d + 1)(\ell - q - 1) \leq 0$. Similarly, we have $\text{des}(\pi_R) = d - q - 1 < n - \ell - 1$ and so $q(-n + \ell + d - q) \leq 0$. Combining these two inequalities, we get

$$(q - d + 1)(\ell - q - 1) + q(-n + \ell + d - q) \leq 0. \quad (3.1)$$

Adding $(d-1)(n-d-1)$ to (3.1) and substituting for $w(\sigma)$, we obtain

$$w(\sigma) \leq (d-1)(n-d-1),$$

or equivalently,

$$\Delta(\sigma) \geq n-d-1. \quad \square$$

Definition 3.5. We denote the coefficient of x^d in $E_n(x, q)$ by $E_n[x^d]$ and the coefficient of $x^d q^m$ in $E_n(x, q)$ by $E_n[x^d q^m]$.

We now show that the coefficients of $E_n[x^d]$ display a stabilization phenomenon.

Theorem 3.6. *For all $k, d, m \in \mathbb{N}$ such that $m = d + k + 1$ and $n \geq m$, the value of $E_n[x^d q^{\text{maxwt}(n, d) - k}]$ is independent of n .*

Proof. We show that for sufficiently large n , the number of permutations of length n with a given weight disparity is constant. Let $\sigma \in S_{m+1}$ be such that $m = d + k + 1$ for $k, m, d \in \mathbb{N}$. We determine the form of the permutations σ counted by $E_{m+1}[x^d q^{\maxwt(m+1,d)-k}]$. We have the following cases:

- (i) Suppose σ is of the form $1 \cdot \pi$, where $\pi \in S_m$ and $\text{des}(\pi) = d$ and $w(\pi) = \maxwt(m, d) - k = (d-1)(m-d-1)$. Then we have

$$w(\sigma) = w(\pi) + d = \maxwt(m+1, d) - k.$$

So σ is counted by $E_{m+1}[x^d q^{\maxwt(m+1,d)-k}]$. Note that removing 1 from the beginning of σ and subtracting 1 from every element is a bijection that gives us permutations $\pi \in S_m$ with d descents and of weight equal to $w(\pi) = \maxwt(m, d) - k$. So σ is also counted by $E_m[x^d q^{\maxwt(m,d)-k}]$.

- (ii) Suppose σ is of the form $1 \cdot \pi$, where $\text{des}(\pi) \neq d$ or $w(\pi) \neq d(m-d-1) - k$. It is easy to see that σ is not counted by $E_{m+1}[x^d q^{\maxwt(m+1,d)-k}]$.

- (iii) Suppose σ does not start with 1. Let $p = \text{des}(\sigma)$. By Theorem 3.4, we have $\Delta(\sigma) \geq m - \text{des}(\sigma)$, which implies the following:

$$\begin{aligned} \maxwt(m+1, p) - w(\sigma) + p &\geq m \Rightarrow \\ \maxwt(m+1, p) - w(\sigma) + p &> m - 1 = d + k \Rightarrow \\ p - w(\sigma) &> d - (\maxwt(m+1, p) - k). \end{aligned}$$

So either $p \neq d$ or $w(\sigma) \neq \maxwt(m+1, p) - k$. Thus σ is not counted by $E_{m+1}[x^d q^{\maxwt(m+1,d)-k}]$.

Since the number of permutations with weight disparity k and d descents in S_{m+1} is the same as those in S_m , we have

$$E_m[x^d q^{\maxwt(m,d)-k}] = E_{m+1}[x^d q^{\maxwt(m+1,d)-k}].$$

It is easy to see that, by induction, the above argument shows that

$$E_{m+i}[x^d q^{\maxwt(m+i,d)-k}] = E_{m+i+1}[x^d q^{\maxwt(m+i+1,d)-k}]$$

for any $i \geq 0$. □

Theorem 3.6 shows that the formal power series $W_d(t)$ defined in [2] exists and is given by $W_d(t^k) = E_{d+k+1}[x^d q^{(d-1)k}]$. We can now extract $W_d(t)$ from the stabilized coefficients of $E_n(x, q)$.

Definition 3.7. If $E_n[x^d]$ stabilizes to $q^N + a_1 q^{N-1} + a_2 q^{N-2} + a_3 q^{N-3} + \dots$, where $N = d(n-d-1)$, then the formal power series $W_d(t)$ is defined in [2] as

$$W_d(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

Example 3.1. By Theorem 3.6 and our data for the q -Eulerian polynomials, we have

$$\begin{aligned} W_1(t) &= 1 + 3t + 7t^2 + 15t^3 + 31t^4 + 63t^5 + \cdots \\ W_2(t) &= 1 + 4t + 11t^2 + 31t^3 + 65t^4 + 157t^5 + \cdots \\ W_3(t) &= 1 + 5t + 16t^2 + 41t^3 + 112t^4 + 244t^5 + \cdots \\ W_4(t) &= 1 + 6t + 22t^2 + 63t^3 + 155t^4 + 393t^5 + \cdots \\ W_5(t) &= 1 + 7t + 29t^2 + 92t^3 + 247t^4 + 590t^5 + \cdots \end{aligned}$$

4. A Recurrence Relation for $E_n(x, q)$

The classical Eulerian polynomials $A_n(x)$ in Definition 2.3 enumerate permutations according to their number of descents. The Eulerian polynomials $A_n(x)$ satisfy the following recurrence for $n \geq 1$:

$$\begin{aligned} A_0(x) &= 1, \\ A_n(x) &= \sum_{i=1}^{n-1} \binom{n-1}{i} A_i(x) \cdot A_{n-i-1}(x) \cdot x. \end{aligned}$$

We now derive a recurrence relation for the q -Eulerian polynomials $E_n(x, q)$, similar to the recurrence for $A_n(x)$.

Theorem 4.1. *The q -Eulerian polynomials $E_n(x, q)$ satisfy the recurrence relation*

$$\begin{aligned} E_0(qx, q) &= 1, \\ E_n(x, q) &= \sum_{i=1}^{n-1} \binom{n-1}{i} E_i(x, q) \cdot E_{n-i-1}(qx, q) \cdot x + E_{n-1}(qx, q). \end{aligned}$$

We introduce the following definitions and lemmas used in the proof of Theorem 4.1.

Definition 4.2. We define S'_n to be the set of all permutations in S_n ending in 1. That is, $S'_n = \{\sigma \in S_n \mid \sigma = \pi \cdot 1, \text{ where } \pi \in S_{n-1}\}$.

Lemma 4.3. *There exists a bijection $f : S_n \leftrightarrow S'_{n+1}$ such that, for all $\sigma \in S_n$, $w(f(\sigma)) = w(\sigma)$ and $\text{des}(f(\sigma)) = \text{des}(\sigma) + 1$.*

Proof. We first show that there exists a function $f : S_n \rightarrow S'_{n+1}$ that preserves weight and increases number of descents by 1. Let $\pi \in S_n$ be such that $\pi = \pi_L \cdot 1 \cdot \pi_R$, where the subpermutations π_L and π_R can be empty. Consider the function $f : S_n \rightarrow S'_{n+1}$ defined by

$$f(\pi) = f(\pi_L \cdot 1 \cdot \pi_R) = \pi_R \cdot (n+1) \cdot \pi_L \cdot 1.$$

That is, f replaces 1 with the maximum element $(n+1)$, exchanges π_L and π_R , and appends a minimum element 1 at the end. Notice that for any $\pi \in S_n$,

we have $f(\pi) \in S'_{n+1}$ and $\text{des}(f(\pi)) = \text{des}(\pi) + 1$. The weight of $f(\pi)$ is given by

$$w(f(\pi)) = w(\pi_L \cdot 1) + w(\pi_R) + \text{des}(\pi_R),$$

so $w(f(\pi)) = w(\pi)$. So f preserves weight and increases the number of descents by 1.

Now f is clearly invertible as $(n+1)$ and 1 in $f(\pi)$ uniquely determine π_L and π_R . Moreover, f is a bijection as S_n and S'_{n+1} are equinumerous, where its inverse f^{-1} must preserve weight and reduce the number of descents by 1. $\square$

Definition 4.4. We write $E_n^*(x, q)$ to denote the q -Eulerian polynomial

$$\sum_{\sigma \in S'_n} x^{\text{des}(\sigma)} q^{w(\sigma)}.$$

The following corollary is a direct result of Lemma 4.3.

Corollary 4.5. *We have*

$$xE_n(x, q) = E_{n+1}^*(x, q). \quad (4.1)$$

Lemma 4.6. *The q -Eulerian polynomials $E_n(x, q)$ satisfy the recurrence relation*

$$E_n[x^d] = \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i E_{n-i-1}[x^{d-k}] \cdot E_i[x^{k-1}] \cdot q^{d-k} \right) + E_{n-1}[x^d] q^d.$$

Let $\sigma \in S_n$ be a permutation with d descents. We have two cases:

Case 1. Suppose $\sigma = 1 \cdot \pi_1$ for some $\pi_1 \in S_{n-1}$. We have $w(\sigma) = w(\pi_1) + d$. So the number of permutations $\sigma \in S_n$ with d descents and of weight $w(\sigma)$ is equal to the number of permutations $\pi_1 \in S_{n-1}$ with d descents and of weight $w(\pi_1) + d$. This is counted by $E_{n-1}[x^d] q^d$.

Case 2. Suppose $\sigma = \pi_L \cdot 1 \cdot \pi_R$, where 1 is in position $i+1$ of σ and $1 \leq i \leq n-1$. It is clear that $\pi_L \in S_i$ and $\pi_R \in S_{n-i-1}$. There are $\binom{n-1}{i}$ ways to select the i elements in the subpermutation π_L . Let $k = \text{des}(\pi_L \cdot 1)$, where $1 \leq k \leq i$. Then we have $\text{des}(\pi_R) = d - k$ and $w(\sigma) = w(\pi_L \cdot 1) + w(\pi_R) + \text{des}(\pi_R)$. The permutations $\pi_L \cdot 1$ are counted by $E_{i+1}^*[x^k]$, and the permutations π_R are counted by $E_{n-1-i}[x^{d-k}]$. We know that $E_n[x^d q^w] \cdot E_m[x^e q^v]$ counts permutations of length $n+m$ with $d+e$ descents and of weight $w+v$. Thus permutations of length n with d descents and of weight $w(\pi_L \cdot 1) + w(\pi_R)$ are counted by $E_{i+1}^*[x^k] \cdot E_{n-1-i}[x^{d-k}]$. We add $\text{des}(\pi_R) = d - k$ to the weight of these permutations to count permutations of weight $w(\sigma)$, length n , and d descents. Multiplying by q^{d-k} , we have $q^{d-k} \cdot E_{i+1}^*[x^k] \cdot E_{n-1-i}[x^{d-k}]$. Thus the permutations σ are counted by

$$\sum_{i=1}^{n-1} \binom{n-1}{i} \sum_{k=1}^i E_{i+1}^*[x^k] \cdot E_{n-1-i}[x^{d-k}] q^{d-k},$$

which by Corollary 4.5 is equal to

$$\sum_{i=1}^{n-1} \binom{n-1}{i} \sum_{k=1}^i E_i[x^{k-1}] \cdot E_{n-1-i}[x^{d-k}] q^{d-k}.$$

Combining Case 1 and Case 2, we have

$$E_n[x^d] = \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i E_i[x^{k-1}] \cdot E_{n-1-i}[x^{d-k}] q^{d-k} \right) + E_{n-1}[x^d] q^d. \quad \square$$

Proof of Theorem 4.1. From Lemma 4.6, we have

$$\begin{aligned} E_n(x, q) &= \sum_{d=0}^{n-1} x^d \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i E_{n-i-1}[x^{d-k}] \cdot E_i[x^{k-1}] \cdot q^{d-k} \right) \right. \\ &\quad \left. + E_{n-1}[x^d] q^d \right). \end{aligned}$$

If we distribute the first summation and let $\ell = d - k$, we get

$$\begin{aligned} E_n(x, q) &= \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{\ell=0}^{n-1-i} E_{n-i-1}[x^\ell] \cdot q^\ell \cdot E_i[x^{k-1}] \cdot x^{\ell+k} \right) \right) \\ &\quad + \sum_{j=0}^{n-1} E_{n-1}[x^j] q^j x^j. \end{aligned}$$

We can rearrange terms to get

$$\begin{aligned} E_n(x, q) &= \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i E_i[x^{k-1}] x^{k-1} \right) \left(\sum_{\ell=0}^{n-i-1} E_{n-i-1}[x^\ell] \cdot q^\ell x^\ell \right) \cdot x \\ &\quad + \sum_{j=0}^{n-1} E_{n-1}[x^j] q^j x^j. \end{aligned}$$

Substituting the corresponding q -Eulerian polynomials, we have

$$E_n(x, q) = \sum_{i=1}^{n-1} \binom{n-1}{i} E_i(x, q) \cdot E_{n-i-1}(qx, q) \cdot x + E_{n-1}(qx, q). \quad \square$$

Remark 4.7. We compare our recurrence relation for $E_n(x, q)$ with other recurrences in the literature. Note that if we set $q = 1$, Theorem 4.1 becomes the recurrence for the classical Eulerian polynomials $A_n(x)$. Since there are type B and D Eulerian polynomials that satisfy recurrence relations analogous to the recurrence for the polynomials $A_n(x)$ [7], there may also be other q -analogs like the one we study.

Finally, we note that the recurrence relation in Theorem 4.1 resembles the recurrence derived by Postnikov for the polynomials $f_n(x)$, given by

$$f_n(x) = x \left(\sum_{\ell=1}^{n-1} \binom{n-1}{\ell} \cdot f'_\ell(x) f_{n-\ell-1}(x) + f_{n-1}(x) \right),$$

where $f_n(x) = \sum_k f_{nk} x^k$ and f_{nk} denotes the number of all maxmin trees on the set $[n+1]$ with k maxima [8]. Given that weight 0 maxmin trees are in bijection with permutations [2], it would be interesting to explore this connection between all maxmin trees and permutations further.

Using Lemma 4.6 and Theorem 4.1, we give a recursive formula for the coefficients of $E_n(x, q)$.

Corollary 4.8. *The coefficients of the q -Eulerian polynomials $E_n(x, q)$ satisfy the recurrence relation*

$$E_n[x^d q^m] = \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{j=0}^m E_{n-i-1}[x^{d-k} q^{m-j}] \cdot E_i[x^{k-1} q^{k+j-d}] \right) \right) + E_{n-1}[x^d q^{m-d}].$$

Proof. Since by Lemma 4.6, we have

$$E_n[x^d] = \sum_{i=1}^{n-1} \binom{n-1}{i} \sum_{k=1}^i E_i[x^{k-1}] \cdot E_{n-1-i}[x^{d-k}] q^{d-k} + E_{n-1}[x^d] q^d,$$

we can piece together m in $E_n[x^d q^m]$ by introducing a third summation and iterating through all possibilities. $\square$

5. Alternate Proof of the Stabilization Phenomenon

Using our recursive formula for the coefficients of $E_n(x, q)$ in Corollary 4.8, we determine a lower bound for the exponent of q such that the coefficients $E_n[x^d q^w]$ are stabilized in Theorem 5.1. We then use this result to give a second proof of Theorem 3.6.

Theorem 5.1. *If $m > (d-1)(n-d-1)$ for $0 < d < n-1$, then*

$$E_n[x^d q^m] = E_{n-1}[x^d q^{m-d}].$$

Proof. By Corollary 4.8, we have

$$E_n[x^d q^m] = E_{n-1}[x^d q^{m-d}] + \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{j=0}^m E_{n-i-1}[x^{d-k} q^{m-j}] \cdot E_i[x^{k-1} q^{k+j-d}] \right) \right). \quad (5.1)$$

For fixed i and k , we show that for all j , we have $E_{n-i-1}[x^{d-k} q^{m-j}] \cdot E_i[x^{k-1} q^{k+j-d}] = 0$. We consider the following two cases:

Case 1. Suppose $m = (d-1)(n-d-1) + u$ for some $u \in \mathbb{N}$. We show that $E_n[x^d q^m] = E_{n-1}[x^d q^{m-d}]$. Note that $m-d \geq 0$ since $0 < d < n-1$. By Corollary 4.8, we have

$$E_n[x^d q^{(d-1)(n-d-1)+u}] = E_{n-1}[x^d q^{(d-1)(n-d-1)+u-d}] + \sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{j=0}^{(d-1)(n-d-1)+u} E_{n-i-1}[x^{d-k} q^{(d-1)(n-d-1)+u-j}] \cdot E_i[x^{k-1} q^{k+j-d}] \right) \right). \quad (5.2)$$

The weight of a permutation of length n with d descents is bounded by $\maxwt(n, d) = d(n - d - 1)$. So from the coefficients $E_i[x^{k-1}q^{k+j-d}]$ and $E_{n-i-1}[x^{d-k}q^{(d-1)(n-d-1)+u-j}]$, we have the following inequalities:

$$(d-1)(n-d-1) + 1 + u - j \leq (d-k)(n-i-1-d+k-1) \quad (5.3)$$

$$k + j - d \leq (k-1)(i-k). \quad (5.4)$$

Analyzing the terms and summation bounds in (5.2) gives $1 \leq k \leq d$, $k \leq i$ and $n-i-1 \geq d-k-1$. A straightforward computation shows that these summation bounds and (5.3) and (5.4) contradict each other. Thus for all j , we have $E_{n-i-1}[x^{d-k}q^{m-j}] \cdot E_i[x^{k-1}q^{k+j-d}] = 0$ and

$$\sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{j=0}^{(d-1)(n-d-1)+u} E_{n-i-1}[x^{d-k}q^{(d-1)(n-d-1)+u-j}] \cdot E_i[x^{k-1}q^{k+j-d}] \right) \right) = 0.$$

So $E_n[x^d q^m] = E_{n-1}[x^d q^{m-d}]$.

Case 2. Suppose $m = (d-1)(n-d-1) - v$ for some integer $v \geq 0$. Using the same reasoning as in Case 1, we have the following inequalities:

$$(d-1)(n-d-1) - j - v \leq (d-k)(n-1-i-d+k-1) \quad (5.5)$$

$$k + j - d \leq (k-1)(i-k) \quad (5.6)$$

A straightforward computation shows that there exist j that satisfy both (5.5) and (5.6). Thus we have

$$\sum_{i=1}^{n-1} \binom{n-1}{i} \left(\sum_{k=1}^i \left(\sum_{j=0}^{(d-1)(n-d-1)-v} E_{n-i-1}[x^{d-k}q^{(d-1)(n-d-1)-v-j}] \cdot E_i[x^{k-1}q^{k+j-d}] \right) \right) \neq 0.$$

So $E_n[x^d q^m] \neq E_{n-1}[x^d q^{m-d}]$. $\square$

Using Theorem 5.1, we give an alternate proof of Theorem 3.6.

Proof of Theorem 3.6. By Theorem 5.1, we have $E_n[x^d q^m] = E_{n-1}[x^d q^{m-d}]$ if $m > (d-1)(n-d-1)$, where $0 < d < n-1$. Furthermore, $E_n[x^d q^m]$ is stabilized when $m > (d-1)(n-d-1)$ and is not stabilized when $m \leq (d-1)(n-d-1)$. It is easy to show that $E_n[x^0 q^0]$ is stabilized and $E_n[x^{n-1} q^0]$ is not. $\square$

6. Connection Between $W_d(t)$ and Integer Partitions

Theorem 4.1 gives a recursive formula for the q -Eulerian polynomials $E_n(x, q)$, and Theorem 5.1 states the condition for which the coefficients $E_n[x^d q^w]$ are stabilized. We conclude with a conjecture regarding the coefficients of $W_d(t)$, which are the stabilized coefficients of $E_n(x, q)$.

It was observed in [2] that the coefficients of $W_d(t)$ correspond to numbers in the sequence A256193 in the Online Encyclopedia of Integer Sequences [11].

The numbers $T(n, k)$ count partitions of n with exactly k parts of a second type, denoted by a prime '. For example, we have

$$\begin{aligned} T(3, 0) &= 3, \text{ corresponding to } 111, 21, 3, \\ T(3, 1) &= 6, \text{ corresponding to } 1'11, 11'1, 111', 2'1, 21', 3', \\ T(3, 2) &= 4, \text{ corresponding to } 1'1'1, 1'11', 11'1', 2'1', \\ T(3, 3) &= 1, \text{ corresponding to } 1'1'1'. \end{aligned}$$

Table 1 is a short table of $T(n, k)$ with k constant along the columns

We will give a recursive formula for $T(d + k, d)$, but we first need the following lemma.

Lemma 6.1. *Let $n, k, b \in \mathbb{N}$ such that $b \leq 2k - n$. The numbers $T(n, k)$ satisfy the relation*

$$T(n, k) = \sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j).$$

Proof. Let $n, k, b \in \mathbb{N}$ such that $b \leq 2k - n$. We first show that $T(n, k) \geq \sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j)$. We then show that $T(n, k) \leq \sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j)$.

To each partition counted by $T(n - b, k - j)$, we append all possible partitions consisting of $b - j$ 1s and j 1's and obtain partitions of n with k parts of a second type. For fixed j , there are $\binom{b}{j} \cdot T(n - b, k - j)$ such partitions. Summing over all possible values of j , there are $\sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j)$ such partitions formed in this way. Thus $T(n, k) \geq \sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j)$.

We now show that this summation counts all partitions which are counted by $T(n, k)$. Since $n < 2k$, each partition represented by $T(n, k)$ contains at least one singleton part (1 or 1'). In particular, the partition with the fewest singleton parts is the partition $\underbrace{2'2' \dots 2'}_{n-k} \underbrace{1'1' \dots 1'}_{2k-n}$. So every partition counted

by $T(n, k)$ ends in at least $2k - n$ singleton parts. This means that for $b < 2k - n$, truncating the last b elements from a partition which has $b - j$ 1s and j 1's results in a partition counted by $T(n - b, k - j)$. As there can be at most $\binom{b}{j}$ such partitions, we have $T(n, k) = \sum_{j=0}^b \binom{b}{j} \cdot T(n - b, k - j)$. $\square$

Theorem 6.2. *For fixed $k \in \mathbb{N}$ and $d \geq 2k$, we have*

$$T(d + k, d) = \sum_{i=1}^k \left((-1)^{i+1} \cdot \binom{k}{i} \cdot T(d + k - i, d - i) \right) + 1$$

.

Let $k \in \mathbb{N}$ be fixed and $d \geq 2k$. By Lemma 6.1, we get

$$T(d + k, d) = \sum_{j=1}^k \binom{k}{j} \cdot T(d, d - j) + 1.$$

Note that $\binom{k}{j} = \sum_{i=1}^j \binom{k}{j} \binom{j}{i} (-1)^{i+1}$. We have

$$T(d+k, d) = \sum_{j=1}^k T(d, d-j) \left(\sum_{i=1}^j \binom{k}{j} \binom{j}{i} (-1)^{i+1} \right) + 1.$$

By the identity $\binom{r}{s} \binom{s}{t} = \binom{r}{t} \binom{r-t}{s-t}$, we get

$$T(d+k, d) = \sum_{j=1}^k T(d, d-j) \left(\sum_{i=1}^j \binom{k}{i} \binom{k-i}{j-i} (-1)^{i+1} \right) + 1.$$

If we change the order of summations and let $a = j - i$, we obtain

$$T(d+k, d) = \sum_{i=1}^k \binom{k}{i} (-1)^{i+1} \left(\sum_{a=0}^{k-i} \binom{k-i}{a} \cdot T(d, d-a-i) \right) + 1.$$

Since $k-i \leq 2(d-i) - (d+k-i)$, by Lemma 6.1 we have

$$T(d+k, d) = \sum_{i=1}^k \left((-1)^{i+1} \binom{k}{i} \cdot T(d+k-i, d-i) \right) + 1. \quad \square$$

Conjecture 6.3. For fixed $k \in \mathbb{N}$ and $d \geq 2k$, we have

$$W_d[t^k] = \sum_{i=1}^k \left((-1)^{i+1} \cdot \binom{k}{i} \cdot W_{d-i}[t^k] \right) + 1.$$

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