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# A Riemannian Geometric Approach to Output Tracking for Nonholonomic Systems

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**Abstract:** The problem of designing coordinate-invariant output tracking control laws for nonholonomic mechanical systems is addressed. The velocity constrained Euler-Lagrange equations of motion are expressed through a constrained affine connection which is compatible with the kinetic energy Riemannian metric. This formalism is used in designing an output tracking control law via backstepping, which is shown to guarantee exponential stability when the initial distance between the output and reference trajectory is within injectivity radius of the output manifold. In particular this enables almost-global tracking when the output manifold is a rank-1 symmetric space. The control law is intrinsic to the Riemannian structure, and is explicitly constructed. The control law is applied to the problem of tracking the reduced attitude of a rigid body with a nonholonomic velocity constraint. Numerical simulations illustrating the tracking performance are presented.



*Keywords:* Nonholonomic mechanics, Riemannian geometry, Tracking control

## 1. INTRODUCTION

Nonholonomic mechanical systems are geometrically characterized as systems whose velocity is restricted to a distribution which is not involutive. The notion of a nonholonomic structure on a Riemannian manifold was first introduced in Vranceanu (1928). This theory was extended by Vagner who constructed the Schouten-Vagner curvature tensor for such systems in 1937, published with modern notation in Gorbatenko (1985). More recently, this has been further explored in the variational context where the solution trajectories have been expressed as geodesics of the Schouten-Vranceanu connection, which has been shown to preserve a restricted symmetry condition (Krym (2018)). Some preliminary work on nonholonomic systems in the control paradigm can be found in Brockett (1982) where the solution trajectories are studied as geodesics of a singular Riemannian metric. More recently there has been a significant body of work such as Bloch et al. (1992, 2003, 2015) and Bullo and Lewis (2004) in the area of optimal control, stabilization and motion planning of nonholonomic systems, and in Zuyev (2016) where a time-varying control is constructed for approximate stabilization. In Bullo and Lewis (2004), the equations of motion have been studied in a differential-algebraic framework on Riemannian manifolds, motivated by the fact that a large class of complex mechanical systems can be understood as individual components evolving on free configuration manifolds, whose dynamics are related with one another due to inter-connection, contact or relative motion constraints. The equations of motion have been derived by solving for the constraint force, and have been expressed as (forced) geodesics of a connection which is not necessarily torsion-free. In fact, there is an interesting relationship between this connection and the Schouten-Vranceanu connection under certain restrictions on the differential structure of the manifold (see *pr-symmetric condition* in Krym (2018)).

One of the techniques in geometric control theory that has gained wide popularity in recent times is the extension of classical proportional-derivative (PD) controllers to systems of Riemannian manifolds. An important advantage of such an approach is that the control law is globally defined, and useful properties of PD control such as exponential stability, robustness etc are carried over to the global setting on the manifold (i.e. circumventing the need for local linearization). This idea was first developed in Koditschek (1989) for stabilization of fully-actuated mechanical systems by defining the configuration error using *Morse functions*, and later extended for tracking control in Bullo and Lewis (2004) and Fuentes et al. (2011). Similar approaches in the context of Riemannian observers can be found in Aghannan and Rouchon (2003); Bonnabel (2010); Anisi and Hamberg (2005). Some other significant works in this direction are Chaturvedi et al. (2006a,b), Maithripala et al. (2006); Maithripala and Berg (2015) for Lie groups, and Nayak and Banavar (2019) where the stability has been shown to be almost-global for manifolds which admit a particular class of error functions.

The aforementioned techniques can not be applied to nonholonomic systems for state tracking, primarily due to the well-known Brockett condition (Brockett et al. (1983)).


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However, it is still possible to employ the Riemannian framework for tracking certain outputs. This problem has been addressed in Maithripala et al. (2016); Madhushani et al. (2017) and De Marco et al. (2018) for systems on Lie-groups. In this paper, the equations of motion of nonholonomic mechanical systems on general Riemannian manifolds are expressed through a constrained connection which preserves the kinetic energy metric. The metric-compatibility property is exploited in designing an output tracking controller based on integrator backstepping. The position and velocity errors in the output space (and consequently the control law) are explicitly constructed using only the Riemannian structure (and no additional objects such as perfect Morse functions). The tracking control is shown to be exponentially stable as long as the *cut-locus* of the reference trajectory is not encountered. In particular, the stability is deduced to be *almost-global* when the output manifold is a rank-1 symmetric space.

## 2. THE CONSTRAINED CONNECTION

The invariant representation of Euler-Lagrange equations of motion for mechanical systems with linear differential constraints is described. The reader is referred to Sakai (1996) for Riemannian geometric concepts and Bullo and Lewis (2004), for mechanical systems in this framework.

Let $(M, g)$ be a smooth Riemannian manifold where the metric $g$ defines the kinetic energy tensor of the mechanical system, and $\nabla$ its associated *Levi-Civita* connection. Let $\mathcal{D} \subset TM$ be a regular $p$-dimensional distribution of admissible velocities, with $g$-orthogonal complement $\mathcal{D}^{\perp} := \big(ann\ (\mathcal{D})\big)^{\sharp}$ [1]. Let $P_{\mathcal{D}}$ and $P_{\mathcal{D}^{\perp}}$ denote the $g$-orthogonal projections onto $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively such that $TM \ni v = P_{\mathcal{D}}(v) \oplus P_{\mathcal{D}^{\perp}}(v)$.

The coordinate invariant Euler-Lagrange equations of motion of a nonholonomic constrained mechanical system on $(M, g)$ with admissible velocities in $\mathcal{D}$ is

$$\begin{aligned} \nabla_{\dot{q}}\dot{q} &= \lambda(t) + F_0 + F, \\ P_{\mathcal{D}^{\perp}}(\dot{q}) &= 0, \end{aligned} \tag{1}$$

where $F_0$ and $F$ are the tangent representations [2] of the intrinsic and control forces such that $F \in \mathcal{F} \subset TM$. The evolution of $\lambda(t) \in \mathcal{D}^{\perp}$ is implicitly determined so as to satisfy the velocity constraint. The system (1) is said to be *nonholonomic* if the distribution $\mathcal{D}$ is *non-involutive* (Lee (2003)).

One may formally consider projection operators $P$ as $(1, 1)$ tensors $\bar{P}$, defined as

$$\bar{P}(\omega, v) = \omega(P(v)),\ \forall \omega \in T^*Q,\ v \in TQ. \tag{2}$$

The covariant derivative of the constraint equation, along the trajectories of the system, can be computed as

$$\nabla_{\dot{q}}(P_{\mathcal{D}^{\perp}}(\dot{q})) = (\nabla_{\dot{q}}P_{\mathcal{D}^{\perp}})(\dot{q}) + P_{\mathcal{D}^{\perp}}(\nabla_{\dot{q}}\dot{q}) = 0. \tag{3}$$

(Here $\nabla_{\dot{q}}P_{\mathcal{D}^{\perp}}$ is regarded as the covariant derivative of a $(1, 1)$ tensor field.)

[1] $\sharp : T^*M \to TM$ denotes the $g-$dual

[2] Forces are formally represented in the co-tangent bundle, here we use their $g$-duals

The dynamics $\nabla_{\dot{q}}\dot{q}$ is substituted from (1) to solve for $\lambda$ in the equation above as

$$\lambda = -(\nabla_{\dot{q}}P_{\mathcal{D}^{\perp}})\dot{q} - P_{\mathcal{D}^{\perp}}(F_0 + F). \tag{4}$$

The dynamics of the nonholonomic mechanics can now be written as the affine-connection system

$$\overset{\mathcal{D}}{\nabla}_{\dot{q}}\dot{q} = P_{\mathcal{D}}(F_0) + U, \tag{5}$$

where $U = P_{\mathcal{D}}(F) \in \mathcal{C} \subset \mathcal{D}$, where $\mathcal{C} = P_D(\mathcal{F})$ is a smooth control distribution and $\overset{\mathcal{D}}{\nabla}$ is an affine connection on $Q$, defined by

$$\overset{\mathcal{D}}{\nabla}_X Y = \nabla_X Y + (\nabla_X P_{\mathcal{D}^{\perp}})(Y). \tag{6}$$

**Note:** One may easily deduce that $\mathcal{D} \subset TQ$ is invariant under the flow associated with (5).

*Lemma 2.1.* The constrained connection $\overset{\mathcal{D}}{\nabla}$ is compatible with the metric over $\mathcal{D}$ i.e.

$$L_X g(Y, Z) = g(\overset{\mathcal{D}}{\nabla}_X Y, Z) + g(\overset{\mathcal{D}}{\nabla}_X Z, Y)\ \forall Y, Z \in \Gamma(\mathcal{D}). \tag{7}$$

**Proof** Let $Y \in \Gamma(\mathcal{D})$ be a smooth section of $\mathcal{D}$. Then

$$\nabla_X(P_{\mathcal{D}^{\perp}}(Y)) = P_{\mathcal{D}^{\perp}}(\nabla_X Y) + (\nabla_X P_{\mathcal{D}^{\perp}})(Y) = 0. \tag{8}$$

Therefore, $(\nabla_X P_{\mathcal{D}^{\perp}})(Y) \in \mathcal{D}^{\perp}$. Therefore $\forall Z \in \Gamma(\mathcal{D})$,

$$g(\overset{\mathcal{D}}{\nabla}_X Y, Z) = g(\nabla_X Y + (\nabla_X P_{\mathcal{D}^{\perp}})(Y), Z) = g(\nabla_X Y, Z). \tag{9}$$

Similarly, $g(\overset{\mathcal{D}}{\nabla}_X Z, Y) = g(\nabla_X Z, Y)$, and consequently using the fact that $\nabla$ is a Levi-Civita connection, we obtain the result. □

The distinction between differentially constrained systems and unconstrained systems is that the integral curves are (forced) geodesics of a connection which is not necessarily torsion-free. If the constraint is *holonomic* (i.e. $\mathcal{D}$ is integrable), then it can be shown that the constrained connection is torsion-free as in the following example (see Bullo and Lewis (2004)).

*Example :* Consider a unit-length, unit mass simple pendulum subjected to a control force $F$ and potential and dissipative forces $F_0$, whose constrained equations of motion in $\mathbb{R}^3$ is given by the following differential algebraic equation Bullo and Lewis (2004).

$$\begin{aligned} \ddot{x} &= \lambda(t)x + F_0 + F, \\ P_{\mathcal{D}^{\perp}}\dot{x} &= 0,\ ||x(0)||_2 = 1. \end{aligned} \tag{10}$$

where $x \in \mathbb{R}^3$, $\mathcal{D}^{\perp} = span\{x_i \frac{\partial}{\partial x_i}\}$, and $\lambda(t)x$ acts orthogonal to the spherical constraint. The associated constrained connection on $\mathbb{R}^3$ is obtained by solving for $\lambda$ by differentiating the constraint equation as in (4) as

$$\overset{\mathcal{D}}{\nabla}_{\dot{x}}\dot{x} := \ddot{x} + ||\dot{x}||^2 x, \tag{11}$$

where $\overset{\mathcal{D}}{\nabla}$ is the Levi-Civita connection of the induced metric from $\mathbb{R}^3$ on the leaves of the foliation of $\mathcal{D}$ i.e. scaled spheres. These leaves are invariant under the constrained flow, and the reaction force $\lambda$ implicitly ensures this.

## 3. OUTPUT TRACKING CONTROL

### 3.1 Output space structure

Let $\mathcal{Y}$ be a smooth closed manifold of dimension $m \leq n$ and $\Phi : Q \to \mathcal{Y}$ a smooth submersion such that $y = \Phi(q)$ is a *globally defined output* of (5), satisfying the following assumptions.

**Assumption 1:**

$$D\Phi(\mathcal{D}_q) = T_{\Phi(q)}\mathcal{Y}, \ \forall q \in Q. \tag{12}$$

Define $K = ker(D\Phi) \cap \mathcal{D}$ and $K^{\perp}$ as its $g$ orthogonal compliment in $\mathcal{D}$. Since $D\Phi : TQ \to T\mathcal{Y}$ is a constant rank bundle homomorphism, it can be deduced (see Lee (2003)) that $Ker\ (D\Phi)$ is a smooth vector bundle, and therefore so are $K$ and $K^{\perp}$. Denote $P_K$ and $P_{K^{\perp}}$ as the smooth $g$-orthogonal projections onto $K$ and $K^{\perp}$ respectively, such that $\mathcal{D} \ni v = P_{K^{\perp}}(v) \oplus P_K(v)$. Further since $D\Phi$ is a smooth bundle homomorphism, one may observe from the above assumption that $D\Phi : K^{\perp} \to T\mathcal{Y}$ is a smooth bundle isomorphism (i.e. $D\Phi$ is smoothly invertible over fiber elements that are orthogonal to its kernel).

**Assumption 2:** $P_{K^{\perp}}(\mathcal{C}) = K^{\perp}$, and $dim\ (\mathcal{C}) = dim\ (K^{\perp}) = dim\ (\mathcal{Y}) = m$.

**Assumption 3:**

Endow the output manifold $\mathcal{Y}$ with a Riemannian metric $g_y$, and denote its associated distance by $d_y$ and norm by $||.||_y$. For any initial condition $(q_0, v_0) \in TQ$, let $u(t)$ be an input such that $(y(t), \dot{y}(t))$ is bounded as $||(y(t), \dot{y}(t))||_y < \epsilon,\ \forall t > 0$ (where $||.||_y$ is with respect to the metric on $T\mathcal{Y}$ induced by $g_y$), then there exists $b(\epsilon)$ such that $||(q(t), \dot{q})(t)|| < b(\epsilon),\ \forall t > 0$. This assumption is required to ensure that the trajectories of the system are bounded when tracking bounded reference outputs.

### 3.2 Tracking error dynamics

We now design a control law for a consistent and exactly trackable output $y(t)$ to track a reference trajectory $y_d(t)$.

Define the tracking error $\Psi : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ as

$$\Psi(y, y_d) := \frac{1}{2} d_y(y, y_d)^2, \tag{13}$$

which is smooth when $y \notin C_{y_d}$, the *cut-locus* [3] of $y_d$ Sakai (1996).

In order to obtain the tracking velocity error, one needs to compare $\dot{y}$ with $\dot{y}_d$. The following naturally defined map allows one to do so. Denote $\mathcal{T}_{(y,y_d)} : T_{y_d}\mathcal{Y} \to T_y\mathcal{Y}$ as the *parallel transport* map along the minimal geodesic from $y_d$ to $y$, defined by $\mathcal{T}_{(y,y_d)}v = \xi(1)$ where $\xi : [0,1] \to T\mathcal{Y}$ is the solution of the following boundary-value problem

$$\begin{aligned} &\nabla_{\dot{\gamma}}\dot{\gamma} = 0,\ \gamma(0) = y_d,\ \gamma(1) = y, \\ &\nabla_{\dot{\gamma}}\xi = 0,\ \xi(0) = v. \end{aligned} \tag{14}$$

It can be shown that the parallel transport map is smooth and uniquely defined as long as $y \notin C_{y_d}$ Sakai (1996).

[3] The cut-locus $C_p$ of a point $p$ is the set of all points from which there are multiple minimal geodesics to $p$. The squared distance function to $p$ is non differentiable in this region

*Lemma 3.1.* The error function and transport map $\mathcal{T}$ satisfy

$$d_2\Psi(y, y_d) = -\mathcal{T}^*_{(y,y_d)} d_1\Psi(y, y_d),\ \forall y, y_d \in \mathcal{Y} \mid y \notin C_{y_d}.$$

Note: here $d_1\Psi(y, y_d)$ (resp. $d_2\Psi(y, y_d)$) denotes the differential of the function $\Psi(y, y_d)$ with respect to $y$ (resp. $y_d$), with $y_d$ (resp. $y$) fixed.

**Proof** Let $\gamma : [0, d_y(y, y_d)] \to \mathcal{Y}$ be the unique unit speed distance minimizing geodesic such that $\gamma(0) = y_d$ and $\gamma(d_y(y, y_d)) = y$. We know that $\nabla_1 d_y(y, y_d) = -\dot{\gamma}(0)$ and $\nabla_2 d_y(y, y_d) = \dot{\gamma}(d(p, q)$. (Here $\nabla_1$ and $\nabla_2$ denote the gradients with respect to the first and second argument). Further since $\gamma$ is a geodesic, it is *auto-parallelly transported* and therefore,

$$\nabla_1 d_y(y, y_d) = -\mathcal{T}_{(y,y_d)} \nabla_2 d_y(y, y_d). \tag{15}$$

From this we obtain

$$g(\nabla_2 d_y(y, y_d), v) = g(-\mathcal{T}^{-1}_{(y,y_d)} \nabla_1 d_y(y, y_d), v),\ \forall v \in T_{y_d}\mathcal{Y}. \tag{16}$$

Since the parallel transport map is an isometry, we obtain

$$\begin{aligned} d_2 d_y(y, y_d)(v) &= g(\nabla_2 d_y(y, y_d), v) = g(-\mathcal{T}^{-1}_{(y,y_d)} \nabla_1 d_y(y, y_d), v) \\ &= g(\nabla_1 d_y(y, y_d), -\mathcal{T}_{(y,y_d)} v) = d_1 d_y(y, y_d)(-\mathcal{T}_{(y,y_d)} v) \\ &= (-\mathcal{T}^*_{(y,y_d)} d_1 d_y(y, y_d))(v),\ \forall v \in T_{y_d}\mathcal{Y}. \end{aligned} \tag{17}$$

Further, observe that $d_1\Psi(y, y_d) = d_y(y, y_d) d_1 d_y(y, y_d)$ and $d_2\Psi(y, y_d) = d_y(y, y_d) d_2 d_y(y, y_d)$, thereby proving the Lemma. □

The error dynamics is obtained in the region $y \notin C_{y_d}$ as

$$\frac{d}{dt}\Psi(y, y_d) = d_1\Psi(y, y_d)\dot{y} + d_2\Psi(y, y_d)\dot{y}_d, \tag{18}$$

where $d_1\Psi$ and $d_2\Psi$ are differentials with respect to the first and second argument respectively. Using Lemma 3.1 one can write

$$\frac{d}{dt}\Psi(y, y_d) = d_1\Psi(y, y_d)(\dot{y} - \mathcal{T}_{(y,y_d)}\dot{y}_d), \tag{19}$$

Substitute $\dot{y} = D\Phi(\dot{q})$ to obtain

$$\frac{d}{dt}\Psi(y, y_d) = d_1\Psi(y, y_d)\big(D\Phi(\dot{q}) - \mathcal{T}_{(y,y_d)}\dot{y}_d\big). \tag{20}$$

Since $D\Phi\big|_{K^{\perp}}$ is a smooth bundle isomorphism, one can define a smooth function $\tau : Q \times T\mathcal{Y} \to K^{\perp}$ as

$$K^{\perp} \cap T_qQ \ni \tau_{(q,y_d)} := \left(D\Phi_q\big|_{K^{\perp}}\right)^{-1} \big(\mathcal{T}_{(\Phi(q),y_d)}\dot{y}_d\big). \tag{21}$$

The error dynamics is now obtained as

$$\begin{aligned} \frac{d}{dt}\Psi(y, y_d) &= d_1\Psi(y, y_d)\big(D\Phi_q(\dot{q} - \tau_{(q,y_d)})\big) \\ &= D\Phi_q^* d_1\Psi(y, y_d)\big(\dot{q} - \tau_{(q,y_d)}\big). \end{aligned} \tag{22}$$

Further since $D\Phi(P_K(\dot{q})) = 0$,

$$\frac{d}{dt}\Psi(y, y_d) = D\Phi_q^* d_1\Psi(y, y_d)\big(P_{K^{\perp}}(\dot{q}) - \tau_{(q,y_d)}\big). \tag{23}$$

Define $v_d \in T_qQ$ by

$$v_d(q, y_d) = -K_1(D\Phi_q^* d_1\Psi(y, y_d))^{\sharp} + \tau_{(q,y_d)}, \tag{24}$$

such that

$$\frac{d}{dt}\Psi(y, y_d)\bigg|_{P_{K^{\perp}}(\dot{q}) = v_d} = -K_1||D\Phi_q^* d_1\Psi(y, y_d)||_y^2. \tag{25}$$

*3.3 Tracking control law*

A control law for $U$ is now designed, through an integrator backstepping approach where $\dot{q}$ is driven to $v_d$ (with the intermediate Lyapunov function being $\alpha\Psi(y, y_d)$), in the domain where $y \notin C_{y_d}$, by the following equation.

$$\begin{aligned} P_{K^\perp}(U) = &-\alpha(D\Phi_q^* d_1\Psi(y,y_d))^\sharp - K_2(P_{K^\perp}(\dot{q}) - v_d) \\ &+ P_{K^\perp}\bigg(\overset{\mathcal{D}}{\nabla}_{\dot{q}} v_d\Big|_{y_d=const} + \frac{dv_d}{dt}\Big|_{q=const} \\ &- P_{\mathcal{D}}(F_0) - (\overset{\mathcal{D}}{\nabla}_{\dot{q}} P_{K^\perp})(\dot{q})\bigg) =: U_{y,y_d} \end{aligned} \tag{26}$$

It can be easily deduced from **Assumption 2** that $U$ is exactly solvable from the above equation.

*Theorem 3.2.* Let $y = \Phi(q)$ be a globally defined output of (5) that satisfies **Assumption 1**, **Assumption 2** and **Assumption 3**, and $y_d$ a reference trajectory with bounded derivatives up to second order. Then with the control law $U$ obtained from (26), the integral curves of (5) satisfy

$$\lim_{t\to\infty} \Psi(y(t), y_d(t)) = 0, \tag{27}$$

and $y(t) \notin C_{y_d(t)}$ as long as the initial conditions satisfy

$$||P_{K^\perp}\dot{q}(0) - v_d(q(0), y_d(0))||^2 < \alpha\big(I(\mathcal{Y})^2 - 2\Psi(y(0), y_d(0))\big), \tag{28}$$

where $I(\mathcal{Y})$ is the *injectivity radius* [4] of $(\mathcal{Y}, g_y)$ Further, the tracking is exponentially stable.

**Proof** From the **Assumptions** 1 and 2 and the fact that $Q$ and $\mathcal{Y}$ are compact, we deduce that there exist constants $d_1$, $c_1$ and $c_2$ such that

$$\begin{aligned} c_1||v|| \le ||D\Phi(v)||_y \le c_2||v||,\ \forall\ v \in K^\perp, \\ c_1||\omega||_y \le ||D\Phi^*(\omega)|| \le c_2||\omega||,\ \forall\ \omega \in T^*\mathcal{Y}, \\ ||\big(D\Phi\big|_{K^\perp}\big)^{-1}|| < d_1. \end{aligned} \tag{29}$$

Therefore, as long as $(y, \dot{y})$ are bounded and $y \notin C_{y_d}$, from **Assumption 3** and the above inequalities we observe that the control law is bounded.

Consider the Lyapunov function

$$V := \alpha\Psi(y, y_d) + \frac{1}{2}||P_{K^\perp}(\dot{q}) - v_d||^2. \tag{30}$$

Its derivative is obtained using Lemma 2.1 as

$$\begin{aligned} \dot{V} = &\alpha D\Phi_q^* d_1\Psi(y,y_d)\big(P_{K^\perp}(\dot{q}) - \tau_{(q,y_d)}\big) \\ &+ g\bigg(P_{K^\perp}(\dot{q}) - v_d, -\frac{d}{dt}\Big|_{q=const} v_d\bigg) \\ &+ g\bigg(P_{K^\perp}(\dot{q}) - v_d, \overset{\mathcal{D}}{\nabla}_{\dot{q}}\big(P_{K^\perp}(\dot{q})\big) - \overset{\mathcal{D}}{\nabla}_{\dot{q}} v_d\Big|_{y_d=const}\bigg). \end{aligned} \tag{31}$$

Using (5) one can write

$$\begin{aligned} \overset{\mathcal{D}}{\nabla}_{\dot{q}}\big(P_{K^\perp}(\dot{q})\big) &= \overset{\mathcal{D}}{\nabla}_{\dot{q}}(P_{K^\perp})(\dot{q}) + P_{K^\perp}\big(\overset{\mathcal{D}}{\nabla}_{\dot{q}}\dot{q}\big) \\ &= (\overset{\mathcal{D}}{\nabla}_{\dot{q}} P_{K^\perp})(\dot{q}) + P_{K^\perp}\big(P_{\mathcal{D}}(F_0) + U\big) \end{aligned} \tag{32}$$

[4] $I(\mathcal{Y})$ is the infimum of the set of distances between any point and its cut-locus in $\mathcal{Y}$.

Observe that $(D\Phi_q^* d_1\Psi(y,y_d))^\sharp \in K^\perp$ because

$$\begin{aligned} g((D\Phi_q^* d_1\Psi(y,y_d))^\sharp, u) &= D\Phi_q^* d_1\Psi(y,y_d)(u) \\ &= d_1\Psi(y,y_d)(D\Phi_q(u)) = 0,\ \forall u \in K. \end{aligned}$$

Therefore $v_d \in K^\perp$ in (24) and consequently $P_{K^\perp}(\dot{q}) - v_d \in K^\perp$. One can now rewrite (31) as

$$\begin{aligned} \dot{V} = &\alpha g\big((D\Phi_q^* d_1\Psi(y,y_d))^\sharp, P_{K^\perp}(\dot{q}) - \tau_{(q,y_d)}\big) \\ &+ g\bigg(P_{K^\perp}(\dot{q}) - v_d, P_K^\perp\bigg(\overset{\mathcal{D}}{\nabla}_{\dot{q}}\big(P_{K^\perp}(\dot{q})\big) \\ &- \overset{\mathcal{D}}{\nabla}_{\dot{q}} v_d\Big|_{y_d=const} - \frac{dv_d}{dt}\Big|_{q=const}\bigg)\bigg). \end{aligned} \tag{33}$$

Substituting $P_{K^\perp}(U)$ from (26) in (32) and the result in (33) we obtain

$$\dot{V} = -\alpha K_1||D\Phi_q^* d_1\Psi(y,y_d)||^2 - K_2||P_{K^\perp}(\dot{q}) - v_d||^2, \tag{34}$$

and from (29) we obtain

$$\begin{aligned} \dot{V} &\le -\alpha c_1 K_1||d_1\Psi(y,y_d)||^2 - K_2||P_{K^\perp}(\dot{q}) - v_d||^2 \\ &\le -2\min(c_1K_1, K_2)V,\ \forall q, y_d \mid \Phi(q) \notin C_{y_d}. \end{aligned} \tag{35}$$

From the negative-definiteness of $\dot{V}$, using (30) and condition (28) we obtain

$$\begin{aligned} &\alpha\Psi(y(t), y_d(t)) \le V(t) \le V(0) \le \alpha\Psi(y(0), y_d(0)) \\ &+ \frac{1}{2}||P_{K^\perp}(\dot{q}(t) - v_d||^2 < \alpha I(\mathcal{Y})^2/2. \end{aligned} \tag{36}$$

Therefore $d_y(y, y_d) < I(\mathcal{Y})$ which implies that $y(t) \notin C_{y_d(t)}$, $\forall t > 0$, thereby proving the theorem. □

**Remark 1:** Condition (28) which restricts the initial output within injectivity radius of the reference trajectory is required in order to ensure that $y$ never encounters the the cut-locus of $y_d$, where the distance gradient and parallel transport map are no longer well defined, and the control law could become unstable.

**Remark 2:** If the output manifold $\mathcal{Y}$ is a *rank*-1 *symmetric space*, then $I(\mathcal{Y}) = diam(\mathcal{Y})$ [5], and therefore Theorem 4 ensures *almost-global* tracking by choosing large enough $\alpha$.

## 4. ANGULAR VELOCITY CONSTRAINED REDUCED ATTITUDE TRACKING

Consider the Euler-Poincaré attitude dynamics of a rigid body as along with a nonholonomic constraint given by the following differential algebraic equation [6].

$$\begin{aligned} \dot{R} &= R\hat{\Omega}, \\ J\dot{\Omega} &= J\Omega \times \Omega + U + \lambda e_3, \\ 0 &= \langle\Omega, e_3\rangle, \end{aligned} \tag{37}$$

Where $R = [\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3] \in SO(3)$, $\Omega \in \mathbb{R}^3$ is the body angular velocity, $U = [U_1, U_2]^T$ is the control torque,

[5] The *Blaschke conjecture* states that rank-1 symmetric spaces are the only manifolds satisfying $I(\mathcal{Y}) = diam(\mathcal{Y})$.

[6] This can be related to the standard form (1) using the left-invariant connection defined by $\nabla_{DL_R.\eta} DL_R.\eta = DL_R.(-J^{-1}ad^*\eta J\eta)$, where the dynamics in the Lie algebra is $J\dot{\eta} = J^{-1}ad^*\eta J\eta + F$ (see Bullo and Lewis (2004)).

$J = diag([J_1, J_2, J_3])$ is the inertia tensor and $\hat{\cdot}$ is the identification map from $\mathbb{R}^3 \to \mathfrak{so}(3)$. The constraint distribution is obtained here as $\mathcal{D} = span\ \{R\hat{e}_1, R\hat{e}_2\}$, which means that the angular velocity about the body $z$ axis is to be maintained at zero, using the reaction torque $\lambda e_3$. It can be verified that this constraint is non-integrable by showing that $\mathcal{D}$ is not involutive. The constrained distribution may be expressed in the Lie algebra $\mathfrak{so}(3) \sim \mathbb{R}^3$ as $D = span\ \{e_1, e_2\}$, where $e_i$ are standard basis vectors.

A consistent and trackable output $q(t) \in \mathbb{S}^2$ is chosen as $q = \Phi(R) = Re_3$, required to track a randomly generated smooth reference $q_d(t)$.

In order to maintain the constraint, the reaction torque is obtained as $\lambda = -\frac{(J_1-J_2)}{J_3}\Omega_2\Omega_1$. Compute

$$K_R = ker\ D\Phi_R = span\ \{R\hat{e}_3\},\ K_R^{\perp} = span\ \{R\hat{e}_1, R\hat{e}_2\}. \tag{38}$$

Since the tangent space $T_R SO(3)$ is identified with $\mathfrak{so}(3) \sim \mathbb{R}^3$, one can define the map $D\Phi_R \circ DL_R : \mathfrak{so}(3) \to T_{\Phi(R)}\mathbb{S}^2$, given by $D\Phi_R \circ DL_R(v) = [-\boldsymbol{r}_2, \boldsymbol{r}_1]v$. Its kernel is $K = span\ \{e_3\}$ and $K^{\perp} = span\ \{e_1, e_2\}$. Its inverse (when restricted as in **Assumption 1**) from $K^{\perp} \to T_{\Phi(R)}\mathbb{S}^2$ is

$$\left(D\Phi_R \circ DL_R\Big|_{K^{\perp}}\right)^{-1} v = \left[\langle v, -\boldsymbol{r}_2\rangle, \langle v, \boldsymbol{r}_1\rangle, 0\right]^T \tag{39}$$

Indeed, this can be deduced from the fact that the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $q = Re_3 = \boldsymbol{r}_3$ from an orthonormal frame and $T_{\Phi(R)}\mathbb{S}^2 = span\ \{\boldsymbol{r}_1, \boldsymbol{r}_2\}$. Further for $w \in T^*_{\Phi(R)}\mathbb{S}^2 \subset \mathbb{R}^3$, the quantity $\left((D\Phi_R \circ DL_R)^* w\right)^{\sharp}$ coincides with (39) (after replacing $v$ with $w$).

A tracking error is chosen according to the geodesic distance of the metric induced on $\mathbb{S}^2$ from $\mathbb{R}^3$ i.e. $\Psi(q, q_d) = \frac{1}{2}\arccos(q.q_d)^2$, and its gradient is obtained as

$$\nabla_1\Psi(q, q_d) = d_1\psi(q, q_d) = \frac{\arccos(q.q_d)q \times (q \times q_d)}{\sqrt{1-(q.q_d)^2}}. \tag{40}$$

Given a unit vector $\hat{n}$ and an angle $\theta \in [-\pi, \pi]$, define

$$Rot(\hat{n}, \theta) = \cos(\theta)I + \sin(\theta)\hat{n}. \tag{41}$$

The parallel transport map along minimal geodesics (great circle segments) is given by

$$\mathcal{T}_{(q_2,q_1)} = Rot\left(\frac{q_1 \times q_2}{||q_1 \times q_2||}, d(q_1, q_2)\right),\ \forall q_1 \neq \pm q_2, \tag{42}$$

and $\mathcal{T}_{(q_1,q_2)} = I$ when $q_1 = q_2$.

The term $v_d$ is computed as in (24) as

$$v_d = \begin{bmatrix} \langle -\boldsymbol{r}_2, (\mathcal{T}_{(q,q_d)}.\dot{q}_d - K_1 d_1\Psi(q, q_d))\rangle \\ \langle \boldsymbol{r}_1, (\mathcal{T}_{(q,q_d)}.\dot{q}_d - K_1 d_1\Psi(q, q_d))\rangle \end{bmatrix}. \tag{43}$$

Denote $\tilde{\Omega} := P_{K^{\perp}}(\Omega) = [\Omega_1, \Omega_2]^T,\ U = [U_1, U_2]^T$, $f_J(\Omega) = [e_1, e_2]^T J\Omega \times \Omega$,

The torque feedback law is obtained along the lines of (26) as

$$U = \dot{v}_d - \alpha \begin{bmatrix} \langle -\boldsymbol{r}_2, d_1\Psi(q, q_d))\rangle \\ \langle \boldsymbol{r}_1, d_1\Psi(q, q_d))\rangle \end{bmatrix} - K_2(\tilde{\Omega} - v_d) - f_J(\Omega).$$

In the above equation, the term $f_J(\Omega)$ is similar to $(\overset{\mathcal{D}}{\nabla}_{\dot{q}} P_{K^{\perp}})\dot{q}$ in (26), and $v_d$ is ordinarily differentiated since it is a Lie-Algebraic quantity. These simplifications are possible due to the left-invariant property of the connection governing the Euler-Poincaré equations.

The control law is simulated using a variational integrator on $SO(3)$ (see Marsden and West (2001)).

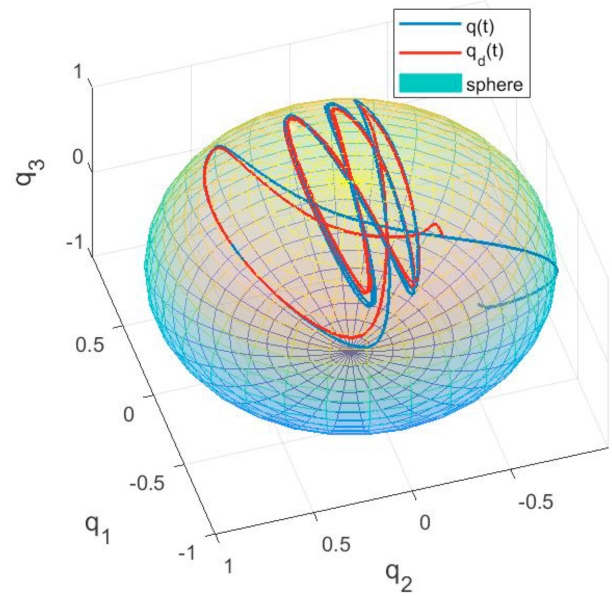


Fig. 1. Output trajectory $q$ and reference trajectory $q_d$, on $\mathbb{S}^2$

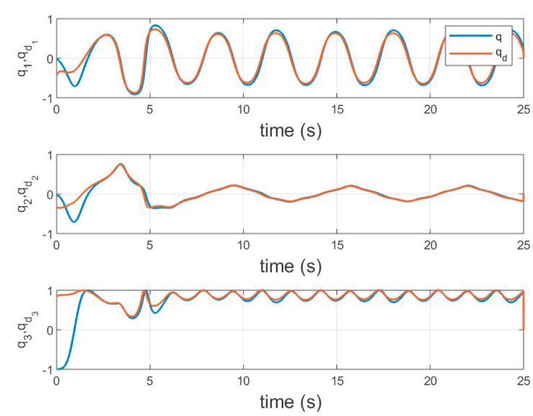


Fig. 2. Component wise tracking of $q$ along $q_d$

Fig.1 shows the output tracking on $\mathbb{S}^2$, with initial angle error $d(q, q_d) = 170°$. The tracking is almost globally exponentially stable since $\mathbb{S}^2$ is a rank-1 symmetric space (see **Remark 2** after the proof of Theorem 4). Fig.2 shows the component wise tracking of the output. Fig.3 shows the control torque $U$ of the tracking controller and reaction torque $\lambda$ which maintains the nonholonomic constraint $\langle\Omega, e_3\rangle = 0$.

## 5. CONCLUDING REMARKS

An output tracking control law for a general class of nonholonomic systems with consistent and trackable outputs was developed. The Euler-Lagrange equation of motion

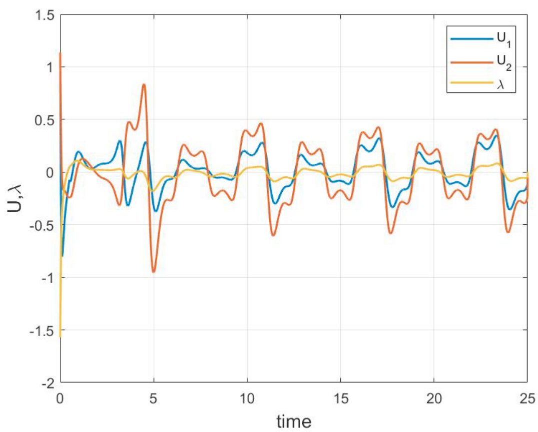


Fig. 3. Control torque $U$ and reaction torque $\lambda$

were presented in a Riemannian geometric framework via an affine connection compatible with the kinetic energy metric. This framework enabled construction of globally defined, coordinate invariant force feedback laws for constrained systems. Exponentially stable output tracking was guaranteed as long as singularities at the cut-locus of the reference trajectory were not encountered. In particular, this guarantees almost-global stability when the output space is a rank-1 symmetric space (or a Blaschke manifold). The tracking controller was applied to velocity-constrained rigid body dynamics on $SO(3)$, for tracking a reduced attitude output on $\mathbb{S}^2$. An important avenue for further research is to globalize the region of stability by exploiting certain special structures of the output-space, or by designing velocity feedback laws which intrinsically avoid singularities such as the cut-locus.

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