Galerkin methods in solving integral equations with applications to scattering problems

A. Chakrabarti
Department of Mathematics, Indian Institute of Science
Bangalore - 560012, India

Abstract
The Galerkin methods of obtaining approximate solutions of integral equations and their applications to problems of scattering of electromagnetic and surface water waves are examined . Two typical problems, one occurring in electromagnetic wave propagation and the other in the propagation of two dimensional surface water waves, are taken up as illustrative examples of the methods.

1. Introduction

Varieties of mixed boundary value problems (see Sneddon [6]), of Mathematical Physics are solved by first reducing them to those of solving integral equations of various types and forms. It is only in some specially simple situations that exact closed form solutions of the integral equations can be determined completely, and, in the cases of integral equations with complicated looking kernels or otherwise, only approximate solutions of certain types can be worked out successfully. Of all such approximate methods for solving integral equations, the Galerkin methods (see Jones [4], Evans and Morris [2],[3], Banerjea and Mandal [1], Mandal and Das [5], and others) appear to be extremely powerful, in the sense that certain practical results of high accuracy can be recovered with appropriate choice of certain sets of independent functions, to be described in section 2 of the present paper.

After explaining the major mathematical ideas behind the Galerkin methods in section 2, we have taken up in section 3, two different mathematical problems of scattering, occurring in Electromagnetic theory and in the theory of water waves respectively, and have reduced each of these problems to those of solving two integral equations of first kind, with two different kernels. In section 4, we have presented the approximate solutions of the integral equations formulated in section 3, by employing just one term Galerkin approximations and in section 5, we have derived approximate results for certain special quantities of practical interest for both the problems considered in section 3.

2. The major Mathematical ideas

In many practical situations, like the ones considered in the present work, the principal mathematical problems turn out to be those of solving some linear operator equations (linear integral equations, for the problems considered here) of the type

$$(Lf)(x) = l(x), \; x \in A, \tag{2.1}$$

where L is a linear operator from a certain inner product space S to itself and $A \subset \mathbb{R}$ (can be $\mathbb{R}^n$, in general), where f and l are real valued functions. It may also be required (as in the problems considered here) to determine the inner product :

$$[l, f] \equiv [f, l] := \int_A f(x)l(x)dx, \tag{2.2}$$

Whenever the mathematical problems at hand are expressible in the forms of the two relations (2.1) and (2.2) we can resolve them, approximately, by utilizing the following senses and ideas:

Definition: A real valued function $F(x) \in S$ is said to solve the equation (2.1), approximately, if and only if

$$[LF, \lambda] \equiv [\lambda, LF] \approx [\lambda, l], \tag{2.3}$$

where the symbol $\approx$ means "approximately equal to" and we shall write: $f \approx F$, in the sense that

$$[Lf, \lambda] \approx [LF, \lambda], \tag{2.4}$$

for all $\lambda(x) \in S$.

Then, using the approximate solution F of the equation (2.1), we can derive an approximate value of the inner product $[f, l]$, as given by the relation (2.2), in the form:

$$[f, l] \approx [F, l]. \tag{2.5}$$

In the Galerkin methods which can be successfully utilized for many problems (especially for the problems considered here), we express the approximate solution $F(x)$, in the form:

$$F(x) = \sum_{j=1}^{n} c_j \phi_j(x), \tag{2.6}$$

where $\{\phi_j(x)\}_{j=1}^{n}$ denotes a set of n linearly independent functions (not necessarily orthogonal) in S and c_j's are n constants to be determined, as desired below.

Using the relation (2.6) in the relation (2.3), after choosing $\lambda(x) = \phi_k(x)$, for a fixed k $(1 \leq k \leq n)$, we obtain the following set of approximate linear relations

$$\sum_{j=1}^{n} c_j [L\phi_j(x), \phi_k(x)] \approx [l(x), \phi_k(x)], \;\; (k = 1, 2, \cdots, n). \tag{2.7}$$

Treating the above approximate relations (2.7) as a set of n linear equations, we can determine the constants c_j's $(j = 1, 2, \cdots, n)$ and then the determination of the approximate solution $F(x)$, can be completed by using the relation (2.6).

Also, the approximate evaluation of the inner product $[f, l]$ can be completed and we obtain

$$[f, l] \approx \sum_{j=1}^{n} c_j [\phi_j, l]. \tag{2.8}$$

As an example, by taking $n = 1$ only, we obtain

$$f(x) \approx F(x) = \frac{[l, \phi_1]}{L\phi_1, \phi_1]} \phi_1(x), \qquad [f, l] = \frac{[l, \phi_1]^2}{[L\phi_1, \phi_1]}. \tag{2.9}$$

It is obvious from the above discussion that varieties of Galerkin methods can be developed by varying ϕ_j's and n.

The methods for which $n = 1$ are called (see Evans and Morris [2,3]) "single-term" methods, whereas for values of $n > 1$ the corresponding methods are referred to as "multi-term" methods (see Banerjea and Mandal [1] and others). In the present work we shall concentrate only on "single-term" Galerkin approximations.

We shall now make the following observations:
We have

$$\begin{array}{ll} (i) & [F, LF] \approx [F, l], \quad (ii) \quad [f, l] = [l, f] = [l, f] + [l, f - F], \\ (iii) & [l, f - F] \approx [Lf, f] - 2[LF, F] + [F, LF], \text{ (by using } (i)) \\ (iv) & [f - F, L(f - F)] \approx [Lf, f] - 2[LF, F] + [F, LF]. \end{array}$$

By using the results (iii) and (iv) we find that

$$[l, f - F] \approx [f - F, L(f - F)], \tag{2.10}$$

and then one of the following two cases hold good.

Case (a): If L is a positive semi-definite linear operator, i.e. if $[h, Lh] \geq 0$, for all $h \in S$, then

$$[l, F] \leq [l, f], \tag{2.11}$$

and
Case (b): If L is a negative semi-definite linear operator, i.e. if $[h, Lh] \leq 0$, for all $h \in S$, then

$$[l, F] \geq [l, f]. \tag{2.12}$$

The above results (2.11) and (2.12) imply that the "approximate" value $[l, F]$, computed with the aid of the "approximate" solution F of the equation (2.1), provides a lower bound for the actual quantity $[l, f]$ in Case (a) whereas $[l, F]$ will provide an upper bound for $[l, f]$ in Case (b).

The above observations clearly help in obtaining estimates of the quantity $[l, f]$ in many practical problems, and in sections 4 and 5 we have demonstrated the applications of these ideas to the two problems of scattering, considered in section 3.

3. Two mathematical problems of scattering theory

Problem 1:
A problem occurring in Scattering of Electromagnetic Waves (See Jones [4])
To solve

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + k^2\phi = 0, \ (k > 0), \text{ for } -\infty < z < \infty, \ 0 < x < a, \ 0 < y < b, \tag{3.1}$$

with $\frac{\pi}{a} < k$, such that

$$(i) \quad \phi = 0, \text{ on } x = 0 \text{ and } x = a, \quad (ii) \quad \frac{\partial\phi}{\partial y} = 0, \text{ on } y = 0 \text{ and } y = b$$

$$(iii) \quad \phi \longrightarrow \begin{cases} \left(e^{-i\lambda z} + Re^{i\lambda z}\right) \sin\left(\frac{\pi x}{a}\right), \text{ as } z \longrightarrow -\infty, \ \lambda > 0 \text{ (a known constant)} \\ Te^{-i\lambda z} \sin\left(\frac{\pi x}{a}\right), \text{ as } z \longrightarrow \infty, \ \lambda > 0 \text{ (a known constant)} \end{cases}$$

(Note that R and T are unknown complex constants to be determined).

$$(iv) \quad \begin{cases} (A) \quad \dfrac{\partial \phi}{\partial z} = 0, \text{ on } z = 0^{\pm}, \text{ for } d < y < b, 0 < x < a, \\ (B) \quad \phi|_{z=0^+} = \phi|_{z=0^-}, \text{ for } 0 < y < d, 0 < x < a, \\ (C) \quad \left.\dfrac{\partial \phi}{\partial z}\right|_{z=0^+} = \left.\dfrac{\partial \phi}{\partial z}\right|_{z=0^-}, \text{ for } 0 < y < d, 0 < x < a, \end{cases}$$

along with the edge condition that $\nabla\phi$ possesses a square-root singularity at the edge $y = d$.
Note: The forms of ϕ as given by (iii), suggest that

$$-\lambda^2 - \frac{\pi^2}{a^2} + k^2 = 0 \implies \lambda^2 = \left(k^2 - \frac{\pi^2}{a^2}\right) > 0,$$

along with the equation (2.1).

Reduction to two integral equations.

Setting

$$\phi(x, y, z) = \psi(y, z) \sin\left(\frac{\pi x}{a}\right), \tag{3.2}$$

with

$$\psi(y, z) = e^{-i\kappa_0 z} - \sum_{n=0}^{\infty} a_n e^{-i\kappa_n |z|} \operatorname{Sgn}(z) \cos\left(\frac{n\pi y}{b}\right), \ (\operatorname{Sgn}(z) = \pm 1, \text{ according as } z > 0 \text{ or } z < 0), \tag{3.3}$$

with

$$\kappa_n = -i\mu_n, \ \mu_n = \left(\frac{n^2\pi^2}{b^2} - \lambda^2\right) > 0, \text{ (assumed), and } \kappa_0 = \lambda, \tag{3.4}$$

where $a_0 = R = (1 - T)$ and a_n's $(n \geq 1)$ are unknown constants, we find that all the conditions of the problem1 are met with, except the two conditions (A) and (B) of (iv), which lead to the following DUAL SERIES RELATIONS, for the determination of the constants a_n:

$$\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi y}{b}\right) = 0, \text{ for } 0 < y < d, \tag{3.5}$$

and

$$-\kappa_0 + \sum_{n=0}^{\infty} \kappa_n a_n \cos\left(\frac{n\pi y}{b}\right) = 0, \text{ for } d < y < b. \tag{3.6}$$

These dual relations can be easily reduced to two integral equations, in the following manner:
Firstly, setting the left side of the relation (3.6) as equal to $\frac{i}{2} a_0 \kappa_0 b g(y)$, and noting that $g(y) = 0$, for $d < y < b$, we can easily determine the Fourier coefficients, in terms of $g(y)$ and then the relation (3.5) easily gives rise to the integral equation:

$$\int_0^d K(y, t) g(t) dt = \frac{1}{\kappa_0}, \ (0 < y < d), \tag{3.7}$$

with

$$K(y,t) := \sum_{n=1}^{\infty} \mu_n^{-1} \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{n\pi t}{b}\right) \tag{3.8}$$

We also find that a quantity H can be defined as :

$$H := \frac{1}{2i}\left(\frac{1-a_0}{a_0}\right) = -\frac{1}{4}\int_0^d g(t)dt. \tag{3.9}$$

As a second approach, setting the left side of the relation (3.5), as equal to $-\frac{i\kappa_0}{2}(1-a_0)bg_1(y)$, and noting that $g_1(y) = 0$, for $0 < y < d$, we can again determine the Fourier coefficients easily in terms of $g_1(y)$ and then the relation (3.6) gives rise to the following integral equation:

$$1 + \int_d^b K_1(y,t)g_1(t)dt = 0, \ (d < y < b), \tag{3.10}$$

with

$$K_1(y,t) := \lim_{\epsilon \longrightarrow 0} \sum_{n=1}^{\infty} e^{-\epsilon n} \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{n\pi t}{b}\right). \tag{3.11}$$

We also find that

$$\frac{1}{H} \equiv \frac{2ia_0}{1-a_0} = \kappa_0 \int_d^b g_1(t)dt. \tag{3.12}$$

We have thus reduced the problem 1 to that of solving either of the integral equations (3.7) and (3.10), along with either of the two relations (3.9) and (3.12), respectively, which determines an "important" quantity H. It should be emphasized that knowing either $g(y)$ or $g_1(y)$, the problem 1 can be solved completely. However, the kernels K and K_1 are complicated and hence, we will adopt approximate methods (Galerkin methods) as explained in section 4. We also make the observations that the functions g, g_1 and the constant H are real, since the functions K and K_1 are so.

Problem 2:

A Problem occurring in Scattering of Surface Water Waves,

(see Evans and Morris [2])

To solve

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \text{ for } -\infty < x < \infty, \ y > 0, \ -\infty < z < \infty, \tag{3.13}$$

such that

(*i*) $K\phi + \frac{\partial\phi}{\partial y} = 0$, on $y = 0$ and $-\infty < x < \infty$, $-\infty < z < \infty$, ($K > 0$, a known constant)

(*ii*)
$$\begin{cases} (A) \ \ \frac{\partial\phi}{\partial x} = 0, \text{ on and } 0 < y < a, \quad (B) \ \ \left.\frac{\partial\phi}{\partial x}\right|_{x=0^+} = \left.\frac{\partial\phi}{\partial x}\right|_{x=0^-}, \text{ for } y > a, \\ (C) \ \ \phi|_{x=0^+} = \phi|_{x=0^-}, \text{ for } y > a \end{cases}$$

$$(iii) \quad \phi \longrightarrow \begin{cases} e^{-Ky+ipz}(e^{-imx} + Re^{imx}), \text{ as } x \longrightarrow \infty, \\ Te^{-Ky+ipz}e^{-imx}, \text{ as } x \longrightarrow \infty, \\ \text{where } p = K\sin(\alpha), m = K\cos(\alpha), (0 < \alpha < \frac{\pi}{2}) \\ (R \text{ and } T \text{ are unknown complex constants to be determined}). \end{cases}$$

$$(iv) \quad \phi, \nabla\phi \longrightarrow 0, \text{ as } y \longrightarrow \infty$$

along with the edge conditions that $\nabla\phi$ possesses a square root singularity at the edge $y = a$, ensuring uniqueness of the solution of the problem.

Reduction to integral equations

Setting

$$\phi(x, y, z) = \psi(x, y)e^{ipz}, \tag{3.14}$$

with

$$\psi(x, y) = e^{-Ky-imx} + R\text{Sgn}(x)e^{-Ky+im|x|} + \int_0^\infty \frac{b(k)\text{Sgn}(x)(k\cos(ky) - K\sin(ky))e^{-k_1|x|}}{k_1(k^2 + K^2)}dk, \tag{3.15}$$

where $k_1 = (k^2 + p^2)^{\frac{1}{2}}$ and $R = (1 - T)$ and $b(k)$ are unknowns, we find that all the conditions of the problem are satisfied, except the conditions (A) and (C) of (ii), giving rise to the following "DUAL INTEGRAL EQUATIONS" for the determination of the function $b(k)$:

$$-\int_0^\infty \frac{b(k)(k\cos(ky) - K\sin(ky))}{k^2 + K^2}dk + im(R - 1)e^{-Ky} = 0, (0 < y < a), \tag{3.16}$$

and

$$\int_0^\infty \frac{b(k)(k\cos(ky) - K\sin(ky))}{k_1(k^2 + K^2)}dk + Re^{-Ky} = 0, (y > a) \tag{3.17}$$

in which R is also an unknown constant.

The above dual integral equations can be reduced to two separate integral equations, by employing a trick, similar to the one used for the problem 1, along with the use of the Havelock's expansion theorem (see Ursell [7]). In fact, this has already been done by Evans and Morris [2].

Firstly, setting the left side of the relation (3.16), as equal to $\frac{\pi R}{2}f(y)$, and noting that $f(y) = 0$, for $0 < y < a$, we can determine $b(k)$ in terms of $f(y)$, by using Havelock's expansion theorem, and then the relation (3.17) gives rise to the integral equation:

$$\int_a^\infty f(t)L(y, t)dt = e^{-Ky}, \text{ for } y > a, \tag{3.18}$$

where

$$L(y, t) = \int_0^\infty \frac{(k\cos(ky) - K\sin(ky))(k\cos(kt) - K\sin(kt))}{k_1(k^2 + K^2)}dk, \tag{3.19}$$

along with the defining relation:

$$A := \frac{im(1 - R)}{\pi KR} = \int_a^\infty f(t)e^{-Kt}dt. \tag{3.20}$$

Secondly, setting the left side of the relation (3.17), as equal to $-\frac{\pi i}{2}(1-R)f_1(y)$, and noting that $f_1(y) = 0$, for $y > a$, we can determine $b(k)$ in terms of $f_1(y)$, by using Havelock's expansion theorem, and then the relation (3.16) gives rise to the integral equation:

$$\int_0^a f_1(t)M(y,t)dt = e^{-Ky}, \text{ for } 0 < y < a, \tag{3.21}$$

where

$$M(y,t) := \lim_{\delta \longrightarrow 0} \int_0^\infty \frac{k_1(k\cos(ky) - K\sin(ky))(k\cos(kt) - K\sin(kt))}{(k^2+K^2)} e^{-k\delta} dk, \tag{3.22}$$

along with the relation

$$\frac{1}{\pi^2 K^2 A} = \int_0^a f_1(t) e^{-Kt} dt. \tag{3.23}$$

We note that f, f_1 and A are all real quantities, since L and M are real.

From the above discussion it follows that the problem under consideration can be solved completely, either by solving the integral equation (3.18) along with the use of the relation (3.20), or by solving the integral equation (3.21), along with the relation (3.23).

4. Approximate solutions of the integral equations.

In this section, we shall employ "single-term"-Galerkin methods to obtain approximate solutions of the integral equations (3.7) and (3.10), derived for the Problem 1, as well as of the integral equations (3.18) and (3.21), derived for the Problem 2. In fact these solutions have also been presented earlier by Jones [4] and Evans and Morris [2], respectively.

For the equation (3.7) we assume a "single-term" Galerkin approximation as given by

$$g(y) \approx b_0 \text{ (a constant)}, \tag{4.1}$$

in the light of the relation (2.6), with

$$b_0 = \frac{d}{\kappa_0 \int_0^d \int_0^d K(y,t)dtdy} = \frac{d}{\kappa_0 \frac{b^2}{\pi^2} \sum_{n=1}^{\infty} (n^2 \mu_n)^{-1} \sin^2\left(\frac{n\pi d}{b}\right)}, \tag{4.2}$$

in the light of the first of the relations (2.9).

Similarly,we assume a "single-term" galerkin approximation to the solution of the integral equation (3.10), as given by

$$g_1(y) \approx C_0 \cos\left(\frac{\pi}{2}\left(\frac{b-y}{b-d}\right)\right), \tag{4.3}$$

with

$$C_0 = -8(b-d)^3\pi \left[\frac{1}{\sum_{n=1}^{\infty} \mu_n \left\{\frac{1}{4(b-d)^2} - \frac{n^2}{b^2}\right\}^{-2} \cos^2\left(\frac{n\pi d}{b}\right)}\right]. \tag{4.4}$$

Again, for the solution of the integral equations (3.18) and (3.21), we assume the following "single-term" Galerkin approximations (see Evans and Morris [2]) :

$$f(y) \approx C_1 \hat{f}(y) = C_1 \frac{d}{dy} \left[e^{-Ky} \int_a^y \frac{ue^{Ku}}{(u^2-a^2)^{\frac{1}{2}}} du \right], \text{ for } a < y < \infty, \tag{4.5}$$

and

$$f_1(y) \approx C_1' \hat{f}_1(y) = C_1' e^{-Ky} \int_y^a \frac{ue^{Ku}}{(a^2-u^2)^{\frac{1}{2}}} du, \text{ for } 0 < y < a, \tag{4.6}$$

where C_1 and C_1' are to be calculated by using the first of the relations (2.9), giving

$$C_1 = \frac{\int_a^\infty e^{-Ky} \hat{f}(y) dy}{\int_a^\infty \int_a^\infty L(y,t) \hat{f}(t) \hat{f}(y) dy dt} \tag{4.7}$$

and

$$C_1' = \frac{\int_0^a e^{-Ky} \hat{f}_1(y) dy}{\int_0^a \int_0^a M(y,t) \hat{f}_1(t) \hat{f}_1(y) dy dt} \tag{4.8}$$

5. Some approximate results

In this section we shall explain about the derivation of some approximate results for the quantities H and A associated with the two problems 1 and 2, considered in section 3, which represent important quantities of practical interest in the theory of electromagnetism (see Jones[4]) and in surface water wave theory (see Evans and Morris [2]), respectively.

By using the relations (3.9) and the approximate solution for $g(y)$ as given by the relations (4.1) and (4.2), we can easily determine H approximately. We find that in the particular situation, when $d = \frac{b}{2}$ and $\mu_n \approx \frac{n\pi}{b}$ (i.e when $\kappa_0 b << \pi$), we obtain

$$H \approx -\frac{\pi^3}{16\kappa_0 b \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}} \approx -0.59 \frac{\pi}{\kappa_0 b}. \tag{5.1}$$

Similarly, by using the relation (3.12), along with the approximate solution for $g_1(y)$ as given by the relations (4.3) and (4.4), we can determine an approximate value for the quantity H. We find that when $d = \frac{b}{2}$ and $\mu_n = \frac{n\pi}{b}$, we have

$$H \approx -\frac{\pi}{\kappa_0 b} \left(\frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{2n}{(4n^2-1)^2} \right) \approx -0.87 \frac{\pi}{\kappa_0 b} \tag{5.2}$$

From the theory that has been explained in section 2, we find that the two results in the relations (5.1) and (5.2) provide some upper and lower bounds respectively, for the quantity H, and, we find that the average of these two bounds gives the value $-0.73\frac{\pi}{\kappa_0 b}$, which, according to Jones [4], is very near the actual value $-0.71\frac{\pi}{\kappa_0 b}$.

Also, by using the approximate solutions as given by the relations (4.5), (4.6), (4.7) and (4.8), into the relations (3.20) and (3.23), we can determine two values of A, approximately, which provide some upper and lower bounds, A_1 and A_2 respectively, for this quantity. Then, with the aid of the defining relation (3.20), we find that

$$|R| = (1 + \pi^2 A^2 \sec^2(\alpha))^{-\frac{1}{2}} \tag{5.3}$$

The numerical values of $|R|$ have been worked out by Evans and Morris [2], by using the two bounds for A_1 and A_2 of A as described above, and we give below a representative table for $\alpha = 30^0$, for the purpose of completion of this article. The table clearly shows the closeness of the bounds of $|R|$, i.e. $|R_1|$ and $|R_2|$, which must be attributed to the particular choice of the "single-term" Galerkin approximations as suggested in the relations (4.5) and (4.6).

Table

$\|R\|$ / μ	0.2	0.4	0.6	0.8	1.0	1.4	1.8
$\|R_1\|$	0.0569	0.2432	0.5389	0.7971	0.9252	0.9900	0.9984
$\|R_2\|$	0.0569	0.2430	0.5382	0.7961	0.9246	0.9898	0.9984

Table of values of $|R_j| = (1 + \pi^2 A_j^2 \sec^2(\alpha))^{-\frac{1}{2}}, \; j = 1, 2,$
for $\alpha = 30^0$, $\mu = Ka$.

References

[1] Banerjea S. and Mandal B. N., "Scattering of water waves by a submerged thin vertical wall with a gap", *J. Aust. Math. Soc., Ser B*, **38**, pp. 1-5, (1997)

[2] Evans D. V. and Morris C. A. N., "The effect of a fixed vertical barrier on obliquely incident surface waves in deep water", *J. Inst. Maths. Applics*, **9**, pp. 198-204, (1972 a)

[3] Evans D. V. and Morris C. A. N., "Complementary approximations to the solution of a problem in water waves", *J. Inst. Maths. Applics*, **10**, pp. 1-9, (1972 b)

[4] Jones D. S., *Theory of Electromagnetism*, Pergamon Press, (1964)

[5] Mandal B. N. and Das P., "Oblique diffraction of surface waves by a submerged vertical plate", *J. Engng. Math.*, **30**, pp. 459-470, (1996)

[6] Sneddon I. N., *Mixed Boundary Value Problems*, North Holland, (1966).

[7] Ursell F., "The effect of a fixed vertical barrier on surface waves in deep water", *Proc. Camb. Phil. Soc.*, **43**, pp. 374-382, (1947)