Shape factor of the turbulent boundary layer on a flat plate and the Reynolds shear stress in the outer region

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It has recently been shown by Wei and Klewicki [Phys. Rev. Fluids **1**, 082401 (2016)] that in a zero-pressure-gradient turbulent boundary layer flow, the product of the nondimensional free-stream velocities in the streamwise (U_∞^+) and wall-normal (V_∞^+) directions is the flow shape parameter (H): $U_\infty^+ V_\infty^+ = H$. It is suggested here that this result is a consequence of the variation of the Reynolds shear stress with $U_\infty V$ in the outer region of the boundary layer.

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I. INTRODUCTION

Recently, Wei and Klewicki [1] and Wei and Maciel [2] have reported very interesting scales for the mean wall-normal velocity and the Reynolds shear stress in a zero-pressure-gradient turbulent boundary layer. Wei and Klewicki [1] have proposed that (1) the appropriate velocity scale for the mean wall-normal velocity, V, is its value at the boundary layer edge, V_∞, and (2) the product of the free-stream speed, U_∞ and V_∞ when normalized by the friction velocity, u_τ, is the shape factor, $H(=\delta^*/\theta)$:

$$U_\infty^+ V_\infty^+ = H. \tag{1}$$

Here $U_\infty^+ = U_\infty/u_\tau$, $V_\infty^+ = V_\infty/u_\tau$, δ^* is the displacement thickness, and θ is the momentum thickness. Another important result due to Wei and Maciel [2] is that (3) the scaling order of the Reynolds shear stress, τ_{xy}, in the outer layer is that of $U_\infty V_\infty$. Their equation (21) in present notations is

$$O(\tau_{xys}) \sim O(U_\infty V_\infty); \tag{2}$$

here an additional subscript "s" denotes the characteristic scale. Utilizing the characteristic scales U_∞ and the boundary layer thickness, δ, these results were obtained from an order-of-magnitude analysis of various nondimensional terms in the continuity equation and the boundary layer equation without the viscous term for the outer region [2] using integral analysis results of the continuity and momentum equation [1]. In this short note, we consider the boundary layer equation for the velocity-deficit in the outer layer [3] to show that $U_\infty^+ V_\infty^+ = H$ is associated with the linear variation of $U_\infty^+ V^+$ with the Reynolds shear stress in the outer layer.

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II. ANALYSIS

We consider a steady and incompressible flow over a semi-infinite flat plate. For this constant-pressure flow, let U and V, respectively, denote the mean streamwise velocity and the wall-normal velocity; u' and v' are the fluctuating velocity components in these directions, respectively.

In the outer region of a turbulent boundary layer, the viscous effect is negligible for large friction Reynolds number, $\mathrm{Re}_\tau(=u_\tau\delta/\nu)$ [3], and one is concerned with the velocity defect, $(U_\infty - U)$. The linear boundary layer equation for the velocity defect [3] is

$$U_\infty \frac{\partial}{\partial x}(U - U_\infty) + (U - U_\infty)\frac{dU_\infty}{dx} - y\frac{dU_\infty}{dx}\frac{\partial}{\partial y}(U - U_\infty) = -\frac{\partial(\overline{u'v'})}{\partial y}. \tag{3}$$

For $U_\infty = \text{const}$, Eq. (3) becomes

$$U_\infty \frac{\partial U}{\partial x} = -\frac{\partial(\overline{u'v'})}{\partial y}. \tag{4}$$

Using the continuity equation,

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{5}$$

Eq. (4) becomes

$$-U_\infty \frac{\partial V}{\partial y} = -\frac{\partial(\overline{u'v'})}{\partial y}. \tag{6}$$

Integrating once, we have

$$U_\infty V = \overline{u'v'} + C, \tag{7}$$

where C is the constant of integration; in fact, C is a function of x, as the dependent variables in Eq. (6) are functions of x as well. This constant is such that, as $\overline{u'v'} \to 0$ away from the wall, $U_\infty V$ is finite. It may be noted that Eqs. (3)–(5) and implicitly Eq. (6) are available in textbooks [4]. However, to our knowledge, Eq. (7) is unique to this paper.

In terms of the inner velocity scale, u_τ, Eq. (7) can be written as

$$U_\infty^+ V^+ = (\overline{u'v'})^+ + C^+, \tag{8}$$

where $C^+ = C/u_\tau^2$. Comparing this with Eq. (2), it can be seen that this scaling is similar to that proposed by Wei and Maciel [2].

Equation (7) shows that $U_\infty V$ in the outer boundary layer varies linearly with $\overline{u'v'}$. This linear variation is shown in Figs. 1(a)–1(c) using the simulation data of Simens *et al.* [5], Schlatter and Örlü [6], and Eitel-Amor *et al.* [7]; $\mathrm{Re}_\theta(=U_\infty\theta/\nu)$ is the Reynolds number based on the momentum thickness. We may note that the data shown here are for $y^+(=yu_\tau/\nu) > 100$. For a given Reynolds number, the value of C^+ was obtained by fitting a straight line to the data. The value of C^+ (corresponding to $\overline{u'v'} = 0$) so inferred is compared with the shape parameter in Figs. 1(d)–1(f). For a finite V_∞^+ (as $\overline{u'v'} \to 0$) far away from the wall, we have

$$U_\infty^+ V_\infty^+ = C^+. \tag{9}$$

Comparing this with Eq. (1), one can say that $C^+ = H$. That this is so can be seen in Figs. 1(d)–1(f) showing a good agreement between C^+ and H. Therefore, $U_\infty^+ V_\infty^+ = H$ is an outcome of the variation of the Reynolds shear stress in the outer layer.

Alfredsson *et al.* [8] have found that u_{rms}/U varies linearly with U/U_∞ in the outer region of the boundary layer; u_{rms} is root-mean-squared value of the fluctuating streamwise velocity component,

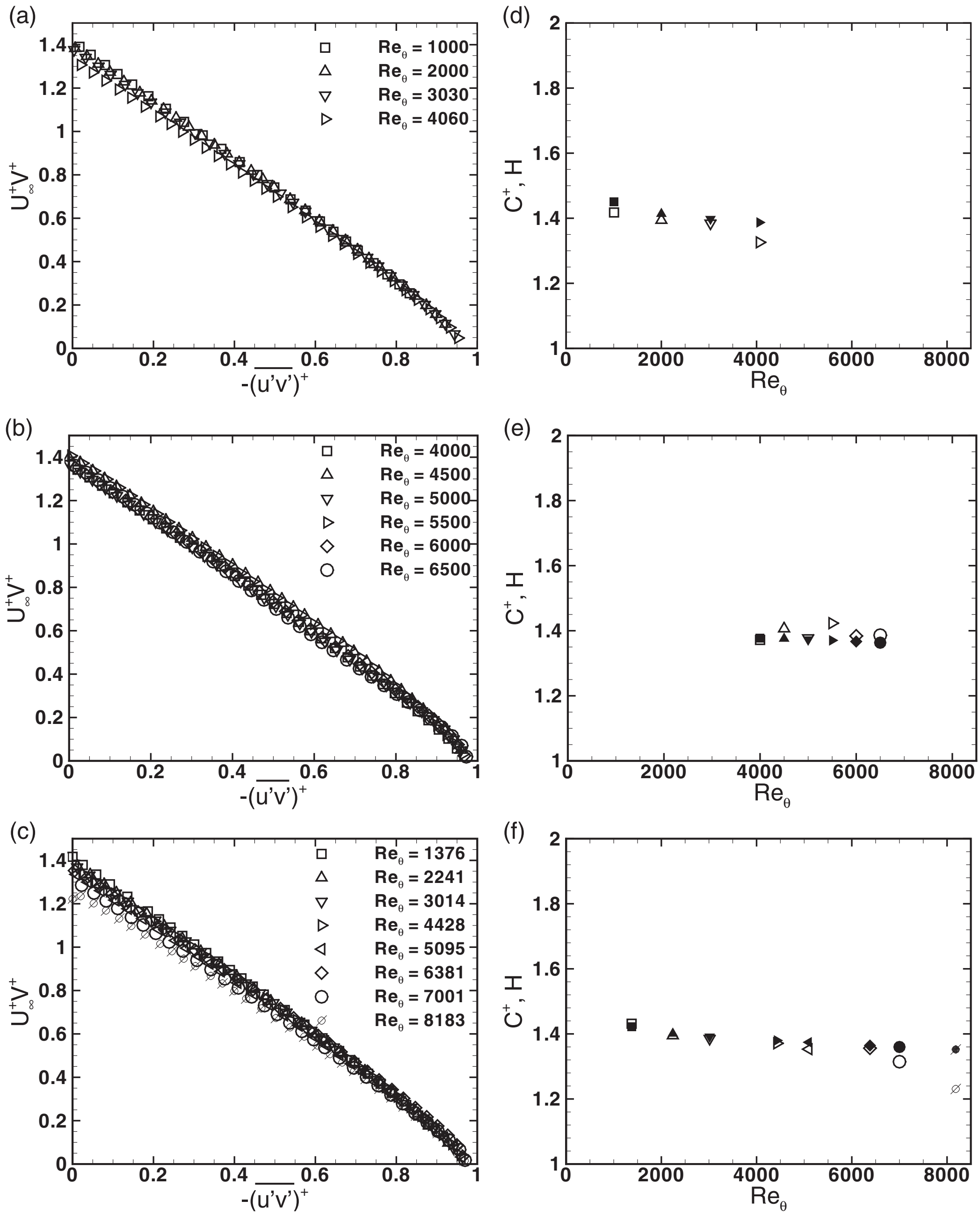

FIG. 1. (a–c) Variation of $U_\infty^+ V^+$ with the Reynolds shear stress. (d–f) Comparison of C^+ (open symbol) with the shape parameter (filled symbol). Data shown are Schlatter and Örlü [6] in panels (a) and (d), Simens *et al.* [5] in panels (b) and (e), and Eitel-Amor *et al.* [7] in panels (c) and (f).

u'. The region of such linear variation increases with an increase in the Reynolds number. For the Reynolds shear stress in the outer layer, on the other hand, it is shown here that one should seek its variation with $U_\infty V$; also, this possibly suggests a simple relationship between these two quantities via a complicated physics.

III. CONCLUSION

For a zero-pressure-gradient boundary layer, it has recently been shown that the nondimensional product of the characteristic velocities in the streamwise direction and the wall-normal direction is the flow shape parameter [Eq. (1)]. It is shown here that this is a consequence of the linear variation of $U_\infty^+ V^+$ with the Reynolds shear stress, and for the far field zero Reynolds stress condition, Eq. (8) leads to Eq. (1).

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