Asymptotic Elias Bound for Euclidean Space Codes over Uniform Signal Sets

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Abstract — **We extend Piret's upper bound [1] to codes over uniform signal sets (a signal set is referred to be uniform if the Euclidean distance distribution is same from any point in the signal set) which include as a special case codes over symmetric PSK signal sets and all signal sets matched to groups [2]. The probability distribution that gives optimum bound is obtained for codes over simplex, biorthogonal signal sets and hamming spaces.**

I. Introduction

For codes designed for the Hamming distance, Elias bound gives an asymptotic upper bound on the normalized rate of the code for a specified normalized Hamming distance. Let C be a length n code over a q-ary alphabet with minimum hamming distance $d_H(C)$. The asymptotic Elias bound is given by

$$\begin{aligned} R(\delta_H) &\leq 1 - H_q(\theta - \sqrt{\theta(\theta-\delta_H)}) \quad &if\ 0 \leq \delta < \theta \\ R(\delta_H) &= 0 \quad &if\ \theta \leq \delta < 1 \end{aligned} \quad (1)$$

where $\theta = (q-1)/q$, $R = \lim_{n\to\infty} \frac{1}{n}\log_q \mid C \mid$ is the normalized rate, $\delta_H = \lim_{n\to\infty} \frac{1}{n} d_H(C)$ is the normalized Hamming distance and $H_q(x)$ is the generalized Entropy function given by

$$H_q(x) = -x\log_q\left[\frac{x}{q-1}\right] - (1-x)\log_q(1-x) \quad (2)$$

where $0 \leq x \leq \left[\frac{q-1}{q}\right]$

II. Extended Piret's Upper Bound (*EPUB*)

Theorem 1: *EPUB* Let A be a uniform signal with M signal points $\{a_0, a_1, \ldots, a_{M-1}\}$ and S be a $M \times M$ matrix with $(i,j)^{th}$ entry s_{ij} equal to $d_{i,j}^2$, the squared Euclidean distance between a_i and a_j. For C, a length n code over A, let

$$R(M,\delta) = \lim_{n\to\infty} sup_{\substack{|C|\geq n \\ \delta(C)\geq q\delta}} R(C) \quad (3)$$

where $\delta(C) = \frac{1}{n}d^2(C), R(C) = \frac{1}{n}\ln \mid C \mid$ The asymptotic upper bound $R_U(M,\delta)$ on $R(M,\delta)$ is given in terms of a probability distribution $\{\beta_r, r = 0, 1, \ldots, M-1\}$ as a set of parameters, by

$$R_U(M,\delta) = \ln(M) - H(\beta) \ \ and\ \ \delta = \underline{\beta} S \underline{\beta}^T \quad (4)$$

The proof of this theorem follows in spirit the arguments in [1]. But our proof for the general class of codes over uniform sets leads to a simpler proof for codes over *PSK* signal sets. The optimum bound depends on the choice of the probability distribution $\underline{\beta}$. The analytical derivation of the best bound is difficult for arbitrary signal sets.

Theorem 2: *EPUB* **for Simplex Signal Sets:** The distribution $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_M)$ that gives the best bound for codes over an M - ary simplex signal set is given by

$$\beta_r = \frac{1}{M}\left(1 - \sqrt{1 - M\frac{\delta}{(M-1)K}}\right), r = 1, 2, \ldots, M-1 \quad (5)$$

where K is the squared distance between any two signal points. Moreover, for all values of q the asymptotic Elias bound can be obtained from this bound.

Theorem 3: *EPUB* **for Hamming Spaces** Let A be a signal set which is an m-th order q-ary hamming space. Then

$$R_U(q^m, \delta) = m\left(1 - H_q\left[\theta - \sqrt{\theta^2 - \frac{\theta\delta}{K}}\right]\right) \quad (6)$$

where $\theta = \frac{(q-1)}{q}$ and K is the squared Euclidean distance between any two points differing in only one position in the label.

Corollary 1: For N - dimensional cube the *EPUB* is given by

$$R_U(2^N, \delta) = N\left(1 - H_2\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{\delta}{2}}\right)\right) \quad (7)$$

Theorem 4: *EPUB* **for Biorthogonal Signal sets:** The optimum distribution $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \ldots, \beta_M)$ giving the best *EPUB* for codes over a binary signal set is given in terms of a parameter $\mu > 0$, as

$$\beta_r(\mu) = \frac{e^{-\mu d_{0,r}^2}}{\sum_{s=0}^{M-1} e^{-\mu d_{0,s}^2}}, r = 0, 1, \ldots, M-1 \quad (8)$$

III. Summary

We have extended the known upper bound [1] to the case of any uniform signal set with simplified proof for the known bound over symmetric *PSK* signal sets. In case of codes over simplex, hamming spaces and biorthogonal signal sets we obtain the probability distribution that gives the optimum bound.

IV. References

1. P. Piret, 'Bounds for codes over a unit circle,' *IEEE Trans. Information Theory.*, vol. IT-32, No.6, pp. 760–767, Nov. 1986
2. H. A. Loeliger, 'Signal sets matched to groups,' *IEEE Trans. Information Theory.*, vol. IT-37, No.6, pp. 1675–1682, Nov. 1991.

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