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An alternative path integral for quantum gravity

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Abstract: We define a (semi-classical) path integral for gravity with Neumann boundary conditions in D dimensions, and show how to relate this new partition function to the usual picture of Euclidean quantum gravity. We also write down the action in ADM Hamiltonian formulation and use it to reproduce the entropy of black holes and cosmological horizons. A comparison between the (background-subtracted) covariant and Hamiltonian ways of semi-classically evaluating this path integral in flat space reproduces the generalized Smarr formula and the first law. This “Neumann ensemble” perspective on gravitational thermodynamics is parallel to the canonical (Dirichlet) ensemble of Gibbons-Hawking and the microcanonical approach of Brown-York.

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Contents

1 Introduction

A full definition of quantum gravity by starting with a Euclidean path integral for Einstein gravity is beset with various problems [1, 2]. This is possibly not surprising because the path integral over metrics is unlikely to be a full description of gravity if its UV completion is something like string theory. Instead of worrying about the subtleties in defining the path integral, what one tries instead is to interpret mostly only the physics around its saddles. In other words, most of the physics of Euclidean quantum gravity is obtained semi-classically. This semi-classical definition turns out to be quite rich and it has lead to many insights about black hole thermodynamics [3] as well as holography [4].

This standard semi-classical definition of the gravitational path integral assumes that the metric that is being integrated over satisfies a Dirichlet boundary condition at the boundary of the spacetime manifold.[1] To make the variational principle well-defined and to obtain well-defined saddles with these boundary conditions, one has to then add a boundary term to the Einstein action, and this is the Gibbons-Hawking-York (GHY) boundary term [3, 6]. In this paper we will be interested in alternate definitions for the boundary conditions of the gravitational path integral, with our specific focus being the recently introduced Neumann gravity [7].

[1]There are further problems associated to infrared divergences that arise in flat space, but we will ignore such issues when they are not important, or overcome them via suitable background subtraction. Or one can imagine working in asymptotically AdS spaces where the AdS length scale acts as an infrared regulator. In this case, we have to add further counter-terms. Some of this will be discussed in a companion paper [5].

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We will discuss the Neumann boundary condition in detail, for the gravity path integral. The saddles are well-defined under a variational principle that holds the canonical conjugate of the boundary metric (namely the boundary[2] stress tensor density) fixed. An alternate to the GHY boundary term that can make such a Neumann action well-defined was introduced in [7], and we will use that as our main tool. We will also work out a Hamiltonian formulation for such a theory and use it to compute the entropy of black holes and cosmological horizons. For this, we will put the discussion of boundary terms in a somewhat unified footing. Changing the boundary term corresponds to changing the ensemble, and we will see how the generalized Smarr formula and first law can be seen using a background subtracted version of our Neumann path integral[3] and comparing the covariant and canonical results. We will also discuss the canonical ensemble of Gibbons-Hawking and the microcanonical path integral of Brown and York [8, 9], to emphasize the relation between various boundary terms.

We will work with pure gravity but it seems evident that our approach should generalize straightforwardly when usual matter (i.e., scalar or gauge) fields are added, with their own boundary conditions. In an accompanying paper, we will discuss the Neumann path integral in AdS spaces [5].

2 Path integral for Neumann gravity

Our starting point is the schematically defined path integral for the gravitational action with Neumann boundary conditions

$$Z_N = \int Dg\, e^{-S_N[g]}, \tag{2.1}$$

where S_N is the Einstein-Hilbert action with an appropriately constructed boundary term so that the saddles are well-defined with Neumann boundary conditions. We will first review a derivation of this action following [7]. See also [10–22] for discussions on various boundary-related aspects of relevance to our problem.

We first consider the standard Dirichlet problem in D-dimensions which is given by the Einstein-Hilbert action for the bulk and the Gibbons-Hawking-York boundary term. The variation of this action can be written as[4]

$$\begin{aligned}\delta S_D = \delta S_{EH} + \delta S_{GHY} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{-g}(G_{ab}+\Lambda g_{ab})\delta g^{ab} \\ -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^{D-1}y\sqrt{|\gamma|}\varepsilon\left(\Theta^{ij}-\Theta\gamma^{ij}\right)\delta\gamma_{ij}.\end{aligned} \tag{2.2}$$

Here $\kappa = 8\pi G$, G_{ab} is the Einstein tensor, $\gamma_{ij} = g_{ab}e^a_i e^b_j$ is the induced metric on the boundary $\partial\mathcal{M}$ and $e^a_i = \frac{\partial x^a}{\partial y^i}$ is the coordinate transformation relating the boundary coordinates y^i to the bulk coordinates x^a, and $\Theta = \gamma^{ij}\Theta_{ij}$ is the trace of the extrinsic curvature. ε distinguishes the space-like and time-like hypersurfaces and takes values $\varepsilon = \pm 1$ for time-like

[2] sometimes also referred to as the quasi-local.
[3] We will only be working with flat space in the this paper.
[4] Our notations and conventions are that of [23].

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and space-like boundaries respectively. We also assume that the boundaries are not null. The extrinsic curvature is defined as

$$\Theta_{ij} = \frac{1}{2}(\nabla_a n_b + \nabla_b n_a)e^a_i e^b_j, \tag{2.3}$$

where n_a is the unit normal to the boundary. From the variation we find that the action S_D is stationary under arbitrary variations of the metric in the bulk provided we satisfy the bulk equations of motion and the variations vanish on the boundary. We introduce the canonical conjugate of the boundary metric as,

$$\pi^{ij} \equiv \frac{\delta S_D}{\delta \gamma_{ij}} = -\frac{\sqrt{|\gamma|}}{2\kappa}\varepsilon(\Theta^{ij} - \Theta\gamma^{ij}), \tag{2.4}$$

using which the variation (2.2) can be written in a suggestive manner

$$\delta S_D = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{-g}(G_{ab} + \Lambda g_{ab})\delta g^{ab} + \int_{\partial\mathcal{M}} d^{D-1}y\; \pi^{ij}\delta\gamma_{ij}. \tag{2.5}$$

We have now expressed the variation of Dirichlet action in a form which is suitable for moving to Neumann problem. We take the view that a well-defined Neumann problem is one where instead of holding the boundary metric, its canonical conjugate is held fixed (while satisfying the bulk equations of motion). This boundary condition can be accomplished by adding a term to the Dirichlet action whose form is suggested by (2.5) as follows

$$S_N = S_{EH} + S_{GHY} - \int_{\partial\mathcal{M}} d^{D-1}y\; \pi^{ij}\gamma_{ij}. \tag{2.6}$$

The variation of S_N then yields,

$$\delta S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{-g}(G_{ab} + \Lambda g_{ab})\delta g^{ab} - \int_{\partial\mathcal{M}} d^{D-1}y\; \delta\pi^{ij}\gamma_{ij}. \tag{2.7}$$

In terms of the extrinsic curvature, the action now takes the form

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{-g}(R - 2\Lambda) + \frac{(4-D)}{2\kappa}\int_{\partial\mathcal{M}} d^{D-1}y\sqrt{|\gamma|}\varepsilon\Theta. \tag{2.8}$$

This is the Neumann action that we will use to give a semi-classical definition to the Neumann path integral for gravity.

Before moving on to other things, we make one comment about how one can sew path integrals together to build a new path integral in this picture. The Dirichlet path integral for gravity enjoys the sewing property, wherein if we cut the spacetime manifold $\mathcal{M}$ into two pieces $\mathcal{M}_1$ and $\mathcal{M}_2$ joined at a hypersurface Σ, then the total path integral can be viewed as being "sewed" together from the two separate path integrals,

$$Z_D^{\mathcal{M}} = \int [dg_0] Z_D^{\mathcal{M}_1}[g_0] Z_D^{\mathcal{M}_2}[g_0]. \tag{2.9}$$

The crucial fact here is that the extrinsic curvature has a sign that is controlled by the normal to the surface Σ, it occurs with compensating sign on the two Z pieces on the right

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hand side. It is for the same reason, similar construction holds for Neumann theory also and the sewing property is satisfied.

$$Z_N^{\mathcal{M}} = \int [d\pi_0] Z_N^{\mathcal{M}_1}[\pi_0] Z_N^{\mathcal{M}_2}[\pi_0]. \tag{2.10}$$

where the canonical conjugate is held fixed at Σ. It will be interesting to evaluate this path integral explicitly in detail (perhaps in 2+1 dimensions where the Neumann action translates into a pure Chern-Simons theory [7]) to see whether the boundary conditions force the presence of our boundary term. Similar computations in lower dimensions and in other contexts have been done in [24, 25].

3 Hamiltonian formulation of Neumann gravity

The Arnowitt-Deser-Misner (ADM) approach is a space+time split of the field variables in gravity that is useful as a natural starting point for the Hamiltonian formulation of general relativity. We wish to write down the Neumann action in this language, with applications in later sections in mind.

We consider manifolds which are like box, i.e. cut-off at finite spatial distance so that the boundary is time-like, denoted $\mathcal{B}$. The spatial section of the boundary $\mathcal{B}$ is denoted B. The covariant action is given by

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\, \sqrt{-g}({}^{(D)}R - 2\Lambda) + \frac{(4-D)}{2\kappa}\int_{\mathcal{B}} d^{D-1}y\, \sqrt{-\gamma}\Theta, \tag{3.1}$$

where Θ is the extrinsic curvature of $\mathcal{B}$ and γ_{ij} is the induced metric. ADM approach relies on foliating the D-dimensional spacetime $(\mathcal{M}, g_{\alpha\beta})$ by $(D-1)$-dimensional spatial hypersurface (Σ_t, h_{ab}), labelled by the time parameter t. The timelike unit normal to the hypersurface Σ_t is denoted u^α and satisfies, $u^\alpha u_\alpha = -1$. The spacetime metric can be expressed as

$$ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt), \tag{3.2}$$

where N is the lapse function, N^a is the shift vector and h_{ab} is the induced metric on the hypersurface Σ_t. The induced metric γ_{ij} can also be split as

$$ds^2 \equiv \gamma_{ij}dx^i dx^j = -N^2 dt^2 + \sigma_{AB}(d\theta^A + N^A dt)(d\theta^B + N^B dt), \tag{3.3}$$

where σ_{AB} is the induced metric on B. The space-like boundary which is at the initial time t_i and final time t_f is ignored. The reason for this as follows. We can always choose to work with a box where the hypersurfaces Σ_t and $\mathcal{B}$ are mutually orthogonal, i.e. $u^\alpha r_\alpha = 0$, where r^α is the radial outward pointing unit vector. This allows us to split the boundary of the spacetime manifold into three parts $\partial\mathcal{M} = \mathcal{B}\cup\Sigma_{t_i}\cup\Sigma_{t_f}$. Since we are ultimately interested in horizon entropy calculations, where the (Wick rotated) time is periodically identified (as we will explain), the surface contributions coming from Σ_{t_i} and Σ_{t_f} plays no role. The Ricci scalar of the bulk spacetime can be decomposed as

$${}^{(D)}R = {}^{(D-1)}R + K^{ab}K_{ab} - K^2 - 2\nabla_\alpha\left(u^\beta\nabla_\beta u^\alpha - u^\alpha\nabla_\beta u^\beta\right), \tag{3.4}$$

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where K_{ab} is the extrinsic curvature of Σ_t. Substituting the above expression into (3.1), we obtain[5]

$$
\begin{aligned}
(2\kappa)S_N =& \int_{\mathcal{M}} d^D x\sqrt{-g}\left[{}^{(D-1)}R - 2\Lambda + K^{ab}K_{ab} - K^2\right] \\
&-2\int_{\mathcal{B}}\left(u^\beta\nabla_\beta u^\alpha - u^\alpha\nabla_\beta u^\beta\right)d\Sigma_\alpha + (4-D)\int_{\mathcal{B}} d^{D-1}y\sqrt{-\gamma}\Theta.
\end{aligned}
\tag{3.5}
$$

On the hypersurface $\mathcal{B}$, the measure $d\Sigma_a$ is given by $d\Sigma_a = r_a\sqrt{-\gamma}d^{D-1}y$. The surface integral on $\mathcal{B}$ thus gives

$$
-2\int_{\mathcal{B}}\left(u^\beta\nabla_\beta u^\alpha - u^\alpha\nabla_\beta u^\beta\right)d\Sigma_\alpha = 2\int_{\mathcal{B}} d^{D-1}y\sqrt{-\gamma}u^\alpha u^\beta\nabla_\beta r_\alpha, \tag{3.6}
$$

where we have used the fact that $u^\alpha r_\alpha = 0$. The action (3.5) now becomes

$$
\begin{aligned}
(2\kappa)S_N =& \int_{\mathcal{M}} d^D x\sqrt{-g}\left[{}^{(D-1)}R - 2\Lambda + K^{ab}K_{ab} - K^2\right] \\
&+2\int_{\mathcal{B}} d^{D-1}y\sqrt{-\gamma}u^\alpha u^\beta\nabla_\beta r_\alpha + (4-D)\int_{\mathcal{B}} d^{D-1}y\sqrt{-\gamma}\Theta.
\end{aligned}
\tag{3.7}
$$

The two surface terms in the above expression can be rearranged as follows. From the definition of extrinsic curvature [23], it follows that

$$
\Theta + u^\alpha u^\beta\nabla_\beta r_\alpha = \sigma^{AB}(\nabla_\beta r_\alpha e^\alpha_A e^\beta_B) = k_{AB}\sigma^{AB} \equiv k, \tag{3.8}
$$

where σ_{AB} is the induced metric on the boundary $\partial\Sigma_t$ (see (3.3)), $k_{AB} = (\nabla_\beta r_\alpha)e^\alpha_A e^\beta_B$ is the extrinsic curvature of $\partial\Sigma_t$ embedded in Σ_t. $e^\alpha_A \equiv \frac{\partial x^\alpha}{\partial\theta^A}$ is the projector relating the bulk coordinates[6] x^α to the $\partial\Sigma_t$ coordinates θ^A.

Using the expression for determinants $\sqrt{-g} = N\sqrt{h}$ and $\sqrt{-\gamma} = N\sqrt{\sigma}$, (3.7) can be expressed as

$$
\begin{aligned}
(2\kappa)S_N =& \int_{t_i}^{t_f} dt\left[\int_{\Sigma_t} d^{D-1}y\; N\sqrt{h}\left({}^{(D-1)}R - 2\Lambda + K^{ab}K_{ab} - K^2\right)\right. \\
&\left.+\,2\int_B d^{D-2}\theta\; N\sqrt{\sigma}k + (2-D)\int_{\partial\Sigma_t} d^{D-2}\theta\; N\sqrt{\sigma}\Theta\right].
\end{aligned}
\tag{3.9}
$$

To obtain the action in terms of canonical variables, we have to introduce the conjugate momentum for h_{ab}. This is given by

$$
p^{ab} \equiv \frac{\partial}{\partial\dot{h}_{ab}}\left(\sqrt{-g}\mathcal{L}_G\right) = \frac{\sqrt{h}}{2\kappa}(K^{ab} - Kh^{ab}). \tag{3.10}
$$

The extrinsic curvature Θ can be be split into two pieces as

$$
\Theta = \Theta^{ij}\gamma_{ij} = \Theta^{ij}(\sigma_{ij} - u_i u_j\Theta^{ij}) = k - u_i u_j\Theta^{ij} = k + \frac{r^a\partial_a N}{N}. \tag{3.11}
$$

[5]For an analogous discussion of Dirichlet problem in ADM decomposition, see [23].
[6]See eq. (3.2) for definition of x^α.

With this expressions for p^{ab} and Θ, the action can be written as

$$\begin{aligned} S_N &= \int_{\mathcal{M}} d^D x \left(p^{ab}\dot{h}_{ab} - NH - N_a H^a \right) + \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}\left(\frac{Nk}{\kappa} - \frac{2N^a r^b p_{ab}}{\sqrt{h}} \right) \\ &+ \frac{(2-D)}{2\kappa}\int_{\mathcal{B}} d^{D-1}y\, N\sqrt{\sigma}\left(k + \frac{r^a\partial_a N}{N} \right), \end{aligned} \tag{3.12}$$

where H and H^a are the Hamiltonian and momentum constraints respectively, whose exact expressions are given by

$$\begin{aligned} H &= \frac{\sqrt{h}}{2\kappa}\left(K^{ab}K_{ab} - K^2 - {}^{(D-2)}R + 2\Lambda \right), \\ H^a &= -\frac{\sqrt{h}}{\kappa} D_b (K^{ab} - K h^{ab}). \end{aligned} \tag{3.13}$$

The two boundary integrals can be now combined and expressed in terms of canonical variables as

$$S_N = \int_{\mathcal{M}} d^D x \left(p^{ab}\dot{h}_{ab} - NH - N_a H^a \right) + \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}\left(\frac{N\varepsilon}{2} - N^a j_a + \frac{N}{2}s^{ab}\sigma_{ab} \right), \tag{3.14}$$

where $\sqrt{\sigma}\varepsilon$, $\sqrt{\sigma}j_a$ and $N\sqrt{\sigma}s^{ab}/2$ are the momenta conjugate to N, N^a and σ_{ab}, respective [8]. They are defined as

$$\begin{aligned} \varepsilon &= \frac{k}{\kappa}, \quad j_a = \frac{2}{\sqrt{h}} r_b p^b_{\ a} \\ s^{ab} &= \frac{1}{\kappa}\left[k^{ab} - \left(\frac{r^a\partial_a N}{N} + k \right)\sigma^{ab} \right]. \end{aligned} \tag{3.15}$$

4 Neumann ensemble

We will view consistent boundary terms for Einstein-Hilbert action as definitions of thermodynamic ensembles arising in Euclidean quantum gravity. In this spirit, we should be able to reproduce the thermodynamics of horizons with our Neumann path integral/partition function. To put things in perspective, we have reviewed the Brown-York and the standard Gibbons-Hawking (aka grand canonical) ensembles in the appendix. Our discussion of Neumann ensemble is directly influenced by the Brown-York approach [8].

The basic idea that we have used to obtain the Gibbons-Hawking path integral from the Brown-York path integral is the fact that in statistical mechanics, given the microcanonical density of states, other thermodynamic potentials can be obtained by suitable Laplace transforms. However, the Neumann ensemble is best thought of as a mixed ensemble. We can see this as follows.

First, note that the variation of the action (3.14) with respect to the canonical variables gives

$$\begin{aligned} \delta S_N &= (\text{eq. of motion}) \\ &+ \int_{\mathcal{B}} d^{D-1}y \left[\delta N(\sqrt{\sigma}\varepsilon/2) - N\delta(\sqrt{\sigma}\varepsilon/2) - \delta N^a(\sqrt{\sigma}j_a) + \delta(N\sqrt{\sigma}s^{ab}/2)\sigma_{ab} \right]. \end{aligned} \tag{4.1}$$

The first two terms can be combined into $\delta N(\sqrt{\sigma}\varepsilon/2) - N\delta(\sqrt{\sigma}\varepsilon/2) = N^2\delta\left(\frac{\sqrt{\sigma}\varepsilon}{2N}\right)$. This means that one can view the Neumann action and the associated partition function loosely[7] as an ensemble in which $\frac{\sqrt{\sigma}\varepsilon}{N}$, N^a and $N\sqrt{\sigma}s^{ab}$ are held fixed. This should be contrasted to the Brown-York and Gibbons-Hawking ensembles which are defined by (A.3) and (A.15) respectively, see [8].

We noted before that in the covariant formalism, the Gibbons-Hawking action and Neumann actions are related by a boundary Legendre transform. As reviewed in the appendix, the Brown-York action is related to the Gibbons-Hawking action by a Legendre transform in the *canonical* variables. Not surprisingly, one can check that the Neumann action, (3.14), is *not* a Legendre transform of the Brown-York action, (A.1), in terms of *canonical* variables. To see this, let us first note that [8]

$$\frac{\delta S_{BY}}{\delta(\sqrt{\sigma}\varepsilon)} = -N, \quad \frac{\delta S_{BY}}{\delta(\sqrt{\sigma}j_a)} = N^a, \quad \frac{\delta S_{BY}}{\delta\sigma_{ab}} = -\frac{N\sqrt{\sigma}s^{ab}}{2}. \tag{4.2}$$

The Neumann action, (3.14), can be now written as

$$S_N = S_{BY} - \int d^{D-1}x\left[\frac{1}{2}\frac{\delta S_{BY}}{\delta(\sqrt{\sigma}\varepsilon)}(\sqrt{\sigma}\varepsilon) + \frac{\delta S_{BY}}{\delta(\sqrt{\sigma}j_a)}(\sqrt{\sigma}j_a) + \frac{\delta S_{BY}}{\delta\sigma_{ab}}\sigma_{ab}\right]. \tag{4.3}$$

The factor of $1/2$ spoils the Legendre transform and is the reason why the variation of the Neumann action, (4.1), has mixed terms proportional to variations of $\sqrt{\sigma}\varepsilon$ as well as N. So the Neumann ensemble is best thought of a mixed ensemble where there is dependence both on N and $\sqrt{\sigma}\varepsilon$ or equivalently, on temperature and energy density.

This means that a naive Laplace transform of the Brown-York partition function (aka microcanonical density of states) of the form (say)

$$Z_N \neq \int D(\sqrt{\sigma}\varepsilon)D(\sqrt{\sigma}j_a\phi^a)D(\sigma_{ab})\nu[\varepsilon, j_a, \sigma_{ab}]\exp\left[\int_B d^{D-2}\theta\sqrt{\sigma}\beta\left(\frac{\varepsilon}{2} - \omega j_a\phi^a + \frac{1}{2}\mu^{ab}\sigma_{ab}\right)\right], \tag{4.4}$$

cannot work: this is because if we integrate the right hand side over $D(\sqrt{\sigma}\varepsilon)$ the right hand side will purely be a function of temperature and will not have any dependence on ϵ, which cannot be the case for the Neumann partition function. To get to the true Neumann partition function, in principle one must identify the function of temperature and energy that characterizes the ensemble and set it to a constant in the path integral via a delta functional. For our purposes of showing the emergence of the correct Smarr formula and first law, fortunately these subtleties will not be necessary.

For our purposes, we merely have to determine the complex saddles of

$$Z_N = \sum_M \int D[H]\ \exp(iS_N), \tag{4.5}$$

which when we demand smoothness, forces the correct periodicity of the time circle. Together with the addition of the horizon term (A.9) to ensure the right boundary condition

[7] Loosely, because we are not being careful about the boundary symplectic structure in making this variable redefinition. Our discussion will not rely on this subtlety.

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at the "bolt" and a regularization scheme in the asymptotic region, this will make our partition function well-defined and computable (in the saddle point approximation). The regularization scheme will depend on the asymptotics of the geometry: in this paper we will consider asymptotically flat situations and use a form of background subtraction, in a companion paper we will consider the asymptotically AdS case and develop a version of holographic renormalization.

Analogous to the Brown-York/Gibbons-Hawking cases, one can choose to think of the black hole in the co-rotating frame. But we will see that this is not strictly necessary (in any of the ensembles) when interpreted correctly, because the horizon piece in the action (A.9) will get a contribution that automatically implements this as a sort of "datum subtraction". Also for stationary flat space (Kerr) black holes, we will find that the $\mu^{ab} \equiv \frac{N\sqrt{\sigma}s^{ab}}{2}$ fall off sufficiently fast that they don't contribute in the discussion of the Smarr formula.

This Neumann ensemble for gravity, we will use in the next section to discuss horizon thermodynamics. Before we proceed however, we make one comment. We will use notations like β, M to denote quantities in the Neumann ensemble as well, even though they are, strictly speaking, defined as thermodynamical quantities only in the GH/BY ensembles respectively. However, both these objects are well-defined geometrically: β is fixed by the periodicity of the time circle via smoothness of the "bolt" as we discussed. Also, since $N \to 1$ in asymptotically flat space, $\sqrt{\sigma}\varepsilon/2N$ (which is what one holds fixed in Neumann), $\to \sqrt{\sigma}\varepsilon/2$. The integral of this quantity over $d^{D-2}\theta$ is what we call $M/2$ (after suitable background subtraction), and this is a well-defined geometric quantity in any saddle as well. So we will express our Neumann thermodynamic relations in terms of them. This philosophy should be compared to the discussion of how the time periodicity is fixed in section IV (and thermodynamics in section VI) of [8]: the ensemble one starts with is microcanonical, but β is a nonetheless useful quantity.

5 Horizon thermodynamics

In this section we will compute Neumann actions both covariantly and canonically (after doing an appropriate background subtraction). Equating the covariant and canonical results to each other will reproduce the generalized Smarr formula. We will do this for Schwarzschild black holes in all dimensions and the 4-dimensional Kerr black hole. This should be contrasted with the Dirichlet case, as in [3], where the free energy obtained from covariant on-shell action is equated to $M - TS - \Omega J$.

5.1 Schwarzschild in D dimensions

The appropriate background subtracted Neumann action in covariant form by analogy with the Dirichlet case [3], is given by

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{-g}R + \frac{(4-D)}{2\kappa}\int_{\mathcal{B}}\sqrt{-\gamma}(\Theta - \Theta_0), \tag{5.1}$$

where Θ_0 is the extrinsic curvature of the boundary embedded in Minkowski space. The Schwarzschild metric in D dimensions is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{(D-2)}, \qquad f(r) = 1 - \frac{2M}{r^{D-3}}, \tag{5.2}$$

where M is the black hole mass parameter, related to the ADM mass of the black hole as [26]

$$M_{ADM} = \frac{(D-2)S_{D-2}M}{8\pi}, \tag{5.3}$$

where $S_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$ is the area of the unit n-sphere. The horizon is at

$$r_H = (2M)^{\frac{1}{D-3}}, \tag{5.4}$$

and the inverse temperature is given by

$$\beta = \frac{4\pi}{f'(r_H)} = \frac{4\pi(2M)^{\frac{1}{D-3}}}{(D-3)}. \tag{5.5}$$

The associated complex metric is given by the identification $N = -i\tilde{N}$ and has periodically identified time with periodicity β. We consider the boundary of the manifold to be at $r = R_c$ which shall be pushed to infinity eventually. For the Schwarzschild black hole, we have

$$\begin{aligned} \Theta &= \frac{\sqrt{1 - \frac{2M}{R_c^{D-3}}}\left((D-2)R_c^{D-3} - (D-1)M\right)}{R_c^{D-2} - 2MR_c}, \\ \Theta_0 &= \frac{D-2}{R_c}. \end{aligned} \tag{5.6}$$

Evaluating the action over the complex metric and taking $R_c \to \infty$ limit, we get

$$S_N = -i\frac{(D-4)S_{D-2}(2M)^{\frac{D-2}{D-3}}}{8(D-3)}. \tag{5.7}$$

This is related to the Neumann free energy by

$$-\beta F_N \equiv \log Z_N \approx iS_N, \tag{5.8}$$

which gives

$$F_N^{\text{cov}} = -\frac{(D-4)S_{D-2}M}{16\pi} = -\frac{(D-4)}{2(D-2)}M_{ADM}. \tag{5.9}$$

Now, the background subtracted Neumann action in ADM variables is given

$$\begin{aligned} S_N = &\int_{\mathcal{M}} d^D x \left[p^{ab}\dot{h}_{ab} - NH - N^a H_a\right] + \int_{\mathcal{H}} d^{D-1}x\sqrt{\sigma}\left(\frac{r^a\partial_a N}{\kappa} + \frac{2N^a r^b p_{ab}}{\sqrt{h}}\right) \\ &+ \int_{\mathcal{B}} d^{D-1}x\sqrt{\sigma}\left[\frac{N}{2}(\varepsilon - \varepsilon_0) - N^a(j_a - j_{a\,0}) + \frac{N}{2}(s^{ab} - s_0^{ab})\sigma_{ab}\right]. \end{aligned}$$

Evaluating the action on the complex metric, the horizon integral gives a contribution

$$S_{\mathcal{H}} = -i\frac{A}{4}. \tag{5.10}$$

The boundary integral evaluates to

$$S_{\mathcal{B}} = i\left(\frac{D-2}{D-3}\right) S_{D-2}(2M)^{\frac{D-2}{D-3}}. \tag{5.11}$$

Using (5.8) the free energy takes the form

$$F_N = \frac{(D-2)}{16\pi} S_{D-2}M - \frac{A}{4\beta}, \tag{5.12}$$

which is of the form

$$F_N^{\text{canon}} = \frac{1}{2}M_{ADM} - TS. \tag{5.13}$$

Equating $F^{\text{canon}} = F^{\text{cov}}$ leads to the correct Smarr formula

$$\frac{D-3}{D-2}M_{ADM} = TS. \tag{5.14}$$

5.2 Kerr in 3+1 dimensions

The $D = 4$ Kerr metric in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 &= \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2 \\ &\quad -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2, \end{aligned} \tag{5.15}$$

where $a = \frac{J}{M}$ is the angular momentum parameter and $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2+a^2-2Mr$. This is an asymptotically flat solution to the vacuum Einstein's equation with zero cosmological constant and describes the geometry of a rotating black hole. The horizon is located at the largest positive root of $\Delta(r_H) = 0$ and is given by $r_H = M + \sqrt{M^2 - a^2}$ and the angular velocity at the horizon is given by

$$\Omega_H = \frac{a}{r_H^2 + a^2}. \tag{5.16}$$

Comparing the metric (5.15) with the ADM split metric, the lapse and shift functions can be extracted as

$$\begin{aligned} N &= \sqrt{\frac{\rho^2\Delta}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}}, \\ N^\phi &= -\frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}. \end{aligned} \tag{5.17}$$

The Neumann action in covariant form in 4D is given by

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}R. \tag{5.18}$$

Since Kerr solution is Ricci flat, the on-shell action for the complex metric vanishes. Furthermore, this leads to zero Neumann free energy

$$F_N^{\rm cov} = 0. \tag{5.19}$$

The Neumann action in terms of the ADM variables in 4D is given

$$\begin{aligned} S_N =& \int_{\mathcal{M}} d^4x \left[p^{ab}\dot{h}_{ab} - NH - N^a H_a\right] + \int_{\mathcal{H}} d^3x\sqrt{\sigma}\left(\frac{r^a\partial_a N}{\kappa} + \frac{2N^a r^b p_{ab}}{\sqrt{h}}\right) \\ &+ \int_{\mathcal{B}} d^3x\sqrt{\sigma}\left[\frac{N}{2}(\varepsilon - \varepsilon_0) - N^a j_a + \frac{N}{2}(s^{ab} - s_0^{ab})\sigma_{ab}\right]. \end{aligned}$$

Evaluating the action on the complex metric, the horizon integral gives a contribution

$$S_{\mathcal{H}} = -i\frac{A}{4} - i\Omega_H a M\beta. \tag{5.20}$$

On the boundary, we have

$$s^{ab} - s_0^{ab} = -\frac{M^2}{2\kappa}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta}\end{pmatrix}\frac{1}{R_c^5} + O(1/R_c^6), \tag{5.21}$$

and

$$\sigma_{ab} = \begin{pmatrix} \rho^2 & 0 \\ 0 & \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)\end{pmatrix}. \tag{5.22}$$

Only the first term in the boundary integral contributes, while the other terms fall-off rapidly when the boundary is pushed to infinity.

$$S_{\mathcal{B}} = i\frac{M}{2}\beta. \tag{5.23}$$

The free energy is computed via (5.8) to be

$$F_N^{\rm canon} = \frac{1}{2}M - TS - \Omega_H J. \tag{5.24}$$

Setting this equal to $F_N^{\rm cov} = 0$ obtained above correctly reproduces the Smarr formula in $D = 4$,

$$\frac{1}{2}M = TS + \Omega_H J. \tag{5.25}$$

5.3 Cosmological horizons

A similar approach can also be applied to de-Sitter geometry, whose metric in the static coordinates is given by $ds^2 = -\left(1 - \frac{r^2}{\alpha^2}\right)dt^2 + \frac{dr^2}{\left(1-\frac{r^2}{\alpha^2}\right)} + r^2 d\Omega^2_{(D-2)}$. The crucial difference between the de-Sitter and the black hole geometries considered previously is that the relevant complexified section of the geometry does not have any boundaries. Therefore all the contributions comes from the horizon term.[8] As in the black hole case, regularity of

[8]We believe the action for de Sitter mentioned in eq. (3.15) of [3] has a numerical factor of 4 missing. Our result matches the one quoted in [27, 28].

the complexified de-Sitter at the horizon at $r = \alpha \equiv \sqrt{3/|\Lambda|}$ (where Λ is the cosmological constant), fixes the periodicity of the time coordinate to be $P = 2\pi\alpha$. Finally, the on-shell action evaluates to $S_{dS} = -\frac{2\pi i}{\kappa}\int d^{D-2}\theta\ \sqrt{\sigma} = -3\pi i\Lambda^{-1}$. Thus we find that the action used to compute black hole density of states is also suitable for computing the entropy of de-Sitter space. This is because in both the cases the crucial argument is the periodicity of the time circle which relies on the fact that the horizon is a bifurcate Killing horizon which is true in both cases.

5.4 First law

We will conclude this section by deriving the first law from Neumann path integral around its saddles. We will work by analogy with the microcanonical discussion in [8]. The variations of the Neumann action are given by (4.1), which when restricted around the complex saddles takes the form

$$\delta S_N = -i\int d^{D-1}y\left[\delta\tilde{N}\left(\frac{\sqrt{\sigma}\varepsilon}{2}\right) - \tilde{N}\delta\left(\frac{\sqrt{\sigma}\varepsilon}{2}\right) - \delta\tilde{N}^a\left(\sqrt{\sigma}j_a\right) - \delta\left(\frac{\tilde{N}\sqrt{\sigma}s^{ab}}{2}\right)\sigma_{ab}\right]. \tag{5.26}$$

Using the relation between free energy and on-shell action and noting that on the solutions we consider, asymptotically at the boundary, the conditions

$$\int dt\tilde{N} = \beta, \qquad \int dt\tilde{N}^\phi = \beta\Omega, \tag{5.27}$$

holds,[9] we get the functional form

$$\begin{aligned}\delta(iS_N) &= \delta(-\beta F_N) \\ &= -\int_B d^{D-2}\theta\left[-\delta\beta\left(\frac{\sqrt{\sigma}\varepsilon}{2}\right) + \beta\delta\left(\frac{\sqrt{\sigma}\varepsilon}{2}\right) + \delta(\beta\omega)(\sqrt{\sigma}j_a\phi^a) + \delta\left(\frac{\beta\sqrt{\sigma}s^{ab}}{2}\right)\sigma_{ab}\right].\end{aligned} \tag{5.28}$$

The above expression is of the form (see [8] for the definition of pressure p),

$$d(-\beta F_N) = -d\beta\frac{E}{2} + \frac{\beta}{2}dE - d(\beta\Omega)J - d(\beta p)V. \tag{5.29}$$

Using the Hamiltonian form for the Neumann free energy that we obtained in the previous sections,

$$F_N = \frac{E}{2} - TS - \Omega J + pV, \tag{5.30}$$

we get the familiar form of the first law

$$TdS = dE - \Omega dJ + pdV. \tag{5.31}$$

[9] $N \to 1$ at the boundary, and so we treat β as the periodicity of the time circle, which is fixed by the smoothness of the bolt at the horizon. Note also that our conventions for extrinsic curvature are opposite to those in [8], so the charges are defined with an extra negative sign. See also the discussion at the end of section 4, for the meaning assigned to β in the Neumann ensemble.

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A Boundary terms and thermodynamic ensembles

To give context for a thermodynamic ensemble interpretation of Neumann path integral, we will discuss the canonical ensemble of Gibbons-Hawking and the microcanonical path integral of Brown-York.[10] The philosophy of the latter is essential for our purposes and since this is not widely known we will review it here. Our discussion of the Gibbons-Hawking ensemble will not follow [3]. Instead we will view it as a Laplace transform from the Brown-York ensemble [8]. We will review that as well. Many of these results are directly useful for the computations we do in the main body of the paper.

It is convenient to start with the Brown-York path integral [8] whose notations we largely follow, except for the sign conventions for extrinsic curvature, for which we follow [23]. We will first show how the Brown-York path integral for the gravitational field can be used to compute the entropy, and then relate it to the (grand) canonical approach of Gibbons-Hawking.

We will work with the ADM formulation to discuss the various ensembles. In this language the microcanonical action of Brown-York takes the form [8]

$$S_{BY} = \int_{\mathcal{M}} d^D x \, \left(p^{ab}\dot{h}_{ab} - NH - N_a H^a \right). \tag{A.1}$$

This is just the bulk piece of the ADM action we wrote down in section 3. The claim is that this action (i.e., with no ADM boundary terms) can be used to compute a (microcanonical) density of states via the path integral expression

$$\nu[\epsilon, j, \sigma] = \sum_M \int D[H] \, \exp(iS_m). \tag{A.2}$$

Here, the density of states is a functional of the boundary data

$$(\epsilon, \quad j_a, \quad \sigma_{ab}), \tag{A.3}$$

that were defined in section 3. In other words, the action (A.1) is chosen because its variation vanishes when these quantities are held fixed at the boundary.

[10] In what follows we will refer to the latter as the Brown-York path integral, because the term "microcanonical" can give rise to confusion in an AdS/CFT context: fixing the total CFT energy is different from fixing the energy density at the boundary of AdS.

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A crucial point in this construction is that the time direction needs to be periodically identified, but the periodicity is arbitrary. The periodicity in time arises as a standard consequence of the trace involved in the definition of the quantum density of states, see e.g. [8, 29]. We will not discuss the origin of this construction in detail even though it is straightforward.

The Lorentzian black-hole metrics have the boundary topology $\mathcal{B} = S^{(D-2)} \times I$, but to connect with the above-mentioned periodicity in time means that we need to consider a manifold with boundary topology $S^{(D-2)} \times S^1$. Following [8], there is a related "complexified" Lorentzian black hole metric with periodically identified time which can be used for entropy calculations. Even though the metric is not "real", the motivation for introducing it is that it can be used for steepest descents approximation to the functional integral by distorting the N and N^a contours in the complex plane.

A general Lorentzian black hole metric is of the form

$$ds^2 = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \tag{A.4}$$

and the accompanying complex metric is given by

$$ds^2 = -(-i\tilde{N})^2 dt^2 + \tilde{h}_{ab}(dx^a - i\tilde{N}^a dt)(dx^b - i\tilde{N}^b dt). \tag{A.5}$$

Thus the complex metric is related to the Lorentzian metric with the identification $N = -i\tilde{N}$ and $N^a = -i\tilde{N}^a$. We assume that the black hole metric is stationary so that $\tilde{N}$ and $\tilde{N}^a$ are independent of time. Since Einstein's equations are analytic differential equations, if (A.4) is a solution, then so is (A.5). The locus of points $\tilde{N} = 0$ describes a hypersurface called "bolt" and the $t =$ constant foliations degenerate at that point.[11] If we choose "corotating" spatial coordinates then $\tilde{N}_a$ vanish on the horizon and thus near the bolt, the metric takes the form

$$ds^2 \approx \tilde{N}^2 dt^2 + \tilde{h}_{ab} dx^a dx^b, \tag{A.6}$$

and demanding the absence of conical singularities at the bolt gives the relation

$$r^a \partial_a \tilde{N} = \frac{2\pi}{P}, \tag{A.7}$$

where P is the periodicity of the time coordinate. The $t =$ *constant* hypersurface has the topology $\Sigma_t = I \times S^{(D-2)}$ and the singularity can be interpreted as a puncture on Σ_t. The bolt seals the spatial manifold but it needs an appropriate boundary condition to be specified there.

The argument of [8, 30] is that no extra boundary piece be added at the horizon other than what already arises from the Einstein-Hilbert term in the Hamiltonian form. They argue that this is natural because in the Lorentzian section the horizon is smooth: there exists a free-fall coordinate system (Kruskal, say) where the principle of equivalence explicitly holds and there is no (coordinate) singularity. We will find that this boundary condition indeed yields the right black hole thermodynamics in all the ensembles, including

[11] For Schwarzschild, this just becomes the usual cigar geometry where one demands regularity at the tip.

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Neumann. In any event, this yields the following Brown-York action for the black hole geometry:

$$S_{BY} = \int_{\mathcal{M}} d^D x \, \left(p^{ab}\dot{h}_{ab} - NH - N_a H^a\right) + S_{\text{horizon}}, \tag{A.8}$$

where S_{horizon} is given by [8, 30]

$$S_{\text{horizon}} \equiv \int_{\mathcal{H}} d^{D-1}y \, \sqrt{\sigma} \left(\frac{r^a \partial_a N}{\kappa} + \frac{2 r_a N_b p^{ab}}{\sqrt{h}}\right). \tag{A.9}$$

This action, just as the (A.1), has the property that the boundary data at $\mathcal{B}$ is specified by fixed ε, j_a and σ_{ab}. In this section, we will assume that the boundary metric σ_{ab} is axi-symmetric.

Now we are ready to evaluate the action integral on the solution. Notice that the bulk integral is identically zero by virtue of being a stationary metric and the fact that it satisfies the Hamiltonian and momentum constraints. Thus the Brown-York action for the black hole yields

$$S_{BY}^{BH} = -\frac{i}{\kappa} \int_0^P dt \int d^{d-2}\theta \, \sqrt{\sigma} r^a \partial_a \tilde{N}. \tag{A.10}$$

Since we are working in the co-rotating frame (A.6), N^a at the horizon is zero, the second term in (A.9) doesn't contribute. Therefore this discussion that reproduces the entropy as in [8] is not really a strong check of the second piece in (A.9). However, we will see that when we move on to other ensembles (both Gibbons-Hawking and Neumann), this term will precisely reproduce the thermodynamics of black holes: one can view this as a further piece of evidence that the horizon boundary condition suggested in [8, 30] is a reasonable one. We will see that in the co-rotating frame, the contributions of some of the thermodynamic quantities arise solely from the boundary, whereas in other frames it is shared between the boundary and the horizon. In both cases, it happens in such a way what the appropriate generalized Smarr formula holds.

Upon using the periodicity condition (A.7) the above expression for the entropy leads to

$$S_{BY}^{BH} = -\frac{2\pi i}{\kappa} A_H = -\frac{i}{4} A_H. \tag{A.11}$$

The advantage of this form is that since the functional integral (A.2) that is defined by S_{BY} has the interpretation as a microcanonical density of states. Therefore, from the standard ideas of statistical mechanics, a saddle point argument yields the entropy of the system to be

$$\mathcal{S}[\varepsilon, j_a, \sigma_{ab}] \approx i S_{BY}. \tag{A.12}$$

This gives the requisite relation between the entropy and the area of the horizon:

$$\mathcal{S}[\varepsilon, j_a, \sigma_{ab}] \approx i S_{BY}^{BH} = \frac{A_H}{4}. \tag{A.13}$$

This is the Brown-York derivation of the microcanonical entropy, presented here with an eye towards generalization to other ensembles, especially Neumann.

We make one comment before moving on to the canonical ensemble. The time periodicity in the Brown-York approach arose because the microcanonical density of states involves a trace over states. But instead of putting this periodicity in from the start, one can also consider the Brown-York path integral (in Euclidean, or more precisely complexified, space) without the periodicity as the starting point. If one demands that the relevant saddles of this path integral are smooth spacetimes, then one again gets the time-periodicity from the smoothness of the "bolt" as discussed above. This is the approach we will adopt for other ensembles. The saddles of the Euclidean path integrals will always fix a periodicity in imaginary time, irrespective of the boundary terms involved: this is because the smoothness at the horizon is what fixes it.

Einstein Hilbert action with the GHY boundary term can be computed in the ADM formulation to be[12]

$$S_g = \int_{\mathcal{M}} d^D x \, \left(p^{ab}\dot{h}_{ab} - NH - N_a H^a\right) + \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}(N\epsilon - N^a j_a). \tag{A.14}$$

In the (grand) canonical ensemble, the boundary data is specified by fixing the potentials at the boundary, for example, an axi-symmetric black hole is specified by its inverse temperature β and (co-rotating) angular velocity Ω_H. The various black hole quantities are then to be thought of as functions of these potentials. One can think of this ensemble as one where

$$(N,\ N^a,\ \sigma_{ab}) \text{ or equivalently } (\beta,\ \omega, \sigma_{ab}), \tag{A.15}$$

are held fixed; ω will be defined momentarily.

One can relate the Brown-York approach to the (grand) canonical approach of Gibbons-Hawking as follows (see [8]). We define ϕ^a to be the axial Killing vector[13] for the boundary metric σ_{ab}. The momentum density along this Killing direction is then given by $\sqrt{\sigma}j_a\phi^a$. The (grand) canonical partition function can be obtained from the microcanonical partition function (A.2) via

$$Z_g[\beta, \beta\omega, \sigma] = \int D[\sqrt{\sigma}\epsilon]D[\sqrt{\sigma}j_a\phi^a]\ \nu[\epsilon, j, \sigma]\ \exp\left[\int d^{D-2}\theta\sqrt{\sigma}\beta(\varepsilon - \omega j_a\phi^a)\right]. \tag{A.16}$$

The steepest descents of Z_g is then given by the simultaneous solutions of

$$\frac{\delta\mathcal{S}}{\delta(\sqrt{\sigma}\varepsilon)} = -\beta, \qquad \frac{\delta\mathcal{S}}{\delta(\sqrt{\sigma}j_a\phi^a)} = \beta\omega. \tag{A.17}$$

In the saddle point approximation, this yields

$$iS_g \approx \ln Z_g \approx \mathcal{S} - \int d^{D-2}\theta\sqrt{\sigma}\beta(\epsilon - \omega j_a\phi^a). \tag{A.18}$$

[12]We give it the subscript g for grand canonical instead of D for Dirichlet here to emphasize its role in thermodynamics.

[13]The existence of such a Killing vector is implicit in the Gibbons-Hawking paper, and our goal is to make connection with it.

Using the (A.2) one can write

$$Z_g[\beta, \beta\omega, \sigma] = \sum_M \int D[H] \; \exp(iS_g), \tag{A.19}$$

where the S_g is evaluated with

$$\int dtN|_B = -i\beta, \quad \int dtN^\phi|_B = -i\beta\omega, \tag{A.20}$$

where N^ϕ is the shift along the ϕ^a direction and B is the spatial slice of $\mathcal{B}$ on which we have the metric σ_{AB}. So instead of $\sqrt{\sigma}\varepsilon$ and $\sqrt{\sigma}j_a$ (which are to be thought of as conjugate fields living in the phase space) being fixed at B, in the (grand) canonical picture we have their potentials fixed at B.

Notice that for a stationary black holes we can assume that β is a constant on the boundary.[14] This is because stationary black holes have time-like Killing vector which gets identified as the generator of U(1) isometry under the Wick rotation. Note also that as the boundary B is taken to radial infinity, since $N \to 1$ for stationary flat space black holes, β is fixed by the periodicity of the time circle. This means that since we fixed this periodicity in the Brown-York ensemble via the smoothness of the "bolt", before doing our Laplace transform, it will be the same as the P that we discussed in the previous subsection.

Similarly, ω is defined as the angular velocity measured by the so-called Zero Angular Momentum Observers (ZAMOs) [8]. Note however, that asymptotically ω goes to zero, for the Kerr black hole [23]. For a rotating black hole, as we saw in the previous subsection when discussing the smoothness of the "bolt", as well as for dynamical and thermodynamical reasons [32, 33], it is more reasonable to think of the black hole in a box that is co-rotating-with-the-horizon.[15] In such a frame, the second equation above gets modified to

$$\int dtN^\phi|_B = i\beta(\Omega_H - \omega) \to i\beta\Omega_H, \tag{A.21}$$

where the last step takes into account the fact that asymptotically ω is zero. Ω_H is the horizon angular velocity [23].

One can also obtain the grand canonical path integral directly by defining a Euclidean path integral with (A.14) as the action. Since this is the approach we will take when defining the Neumann partition function[16] let us emphasize it. Demanding that the (complex)

[14]See however, Lewkowycz and Maldacena [31] who investigate situations where this assumption does not hold. Indeed, the construction of this subsection as well as the entire spirit of this paper can be generalized to non-constant fields at the boundary. The Gibbons-Hawking (aka standard grand canonical) ensemble is a special case of the Lewkowycz-Maldacena ensemble.

[15]There are some problems here, related to the fact that a co-rotating frame becomes superluminal at a finite radius, and therefore it is not really possible to define a fully consistent thermodynamics for rotating black holes in flat space. A related observation is that there is no fully satisfactory Hartle-Hawking state for the flat space Kerr black hole [34, 35]. But these problems are usually glossed over and a formal treatment of thermodynamics is unaffected by them. Our discussion should be taken in that spirit, we will see in a follow up paper [5] that for AdS black holes these problems have natural solutions.

[16]As we will explain in section 3, in terms of canonical variables, the Neumann boundary term is not a simple Legendre transform like it was in covariant variables. So the transformation relating the two ensembles is non-trivial, unlike in the transformation to Gibbons-Hawking from Brown-York. We will not pursue this further here.

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saddles of the action are regular again fixes the periodicity of the time circle to be the inverse Hawking temperature. After complexification we get the tilde versions of (A.20), (A.21) as definitions of β and ω: note that N and N^ϕ are fixed at the boundary.

In the (grand) canonical ensemble then, it is natural to have the formula

$$\frac{-S_g}{\beta} = M - T\mathcal{S} - \Omega_H J, \tag{A.22}$$

which directly lead to useful relations like $\mathcal{S} = (\beta\partial_\beta - 1)S_g$. We have defined $T = 1/\beta$. Gibbons and Hawking motivate (A.22) by noting that the free energy associated to a grand canonical partition function is *defined* by the right hand side of (A.22). Then they make a saddle-point approximation to (A.19) to obtain the left hand side. At this stage, they note that covariantly evaluating the left hand side of (A.22) on black hole solutions after background subtraction leads to the generalized Smarr formula. This is how [3] relate the classical general relativity of black holes with black hole thermodynamics.

But instead of taking the right hand side of (A.22) as a definition of the thermodynamic potential like [3] does, one can instead look at it as an explicit evaluation of the action in the *canonical* (aka ADM aka Hamiltonian) approach. This means that one can view the emergence of the generalized Smarr formula from (A.22) as a result of comparing the covariant and Hamiltonian ways of (saddle point) evaluating the Gibbons-Hawking partition function (A.19).

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