

Bell's inequalities, multiphoton states and phase space distributions

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Abstract

The connection between quantum optical nonclassicality and the violation of Bell's inequalities is explored. Bell type inequalities for the electromagnetic field are formulated for general states (arbitrary number or photons, pure or mixed) of quantised radiation and their violation is connected to other nonclassical properties of the field. Classical states are shown to obey these inequalities and for the family of centered Gaussian states the direct connection between violation of Bell-type inequalities and squeezing is established.

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The violation of Bell's inequalities is one of the most striking features of quantum theory [1]. The testing ground for these inequalities has mostly been the states of the electromagnetic field [2–4]. When a state does not obey Bell-type inequalities it definitely has essential quantum features which cannot be reconciled with the classical notions of reality and locality. In most treatments the Bell-type inequalities are formulated for specific quantum states. For the electromagnetic field there have been attempts to generalise the treatment and relate the violation of Bell-type inequalities with other general nonclassical features of the states [5–9].

We develop the machinery for analysing the violation of Bell type inequalities for a general state of the 4-mode radiation field in a set-up of the type shown in Fig. 1. For the direction \mathbf{k} , the two orthogonal polarisation modes are described by the annihilation operators a_1 and a_2 , with a_3 and a_4 being similarly chosen for the direction \mathbf{k}' . Without any loss of generality we choose \mathbf{k} and \mathbf{k}' to be in the plane of the paper. This allows a simple choice for the directions x , x' to be in the same plane while y and y' point out of this plane. The passive, total photon number conserving, canonical transformations (which will play an important role in our analysis) amount to replacing the a_j 's by their complex linear combinations $a'_j = U_{jk} a_k$, with U being a unitary matrix belonging to $U(4)$. P_1 and P_2 are polarisers placed at angles θ_1 and θ_2 with respect to the x and x' axes while D_1 and D_2 are photon detectors.

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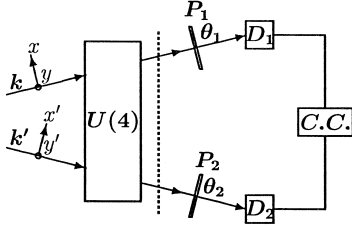


Fig. 1. Set-up to study the violation of Bell type inequalities for arbitrary states of the four mode radiation field.

Usually states with strictly one photon in each direction are considered for violation of Bell-type inequalities; a general state however could have an *arbitrary number of photons*, and could even be a *mixed state*. To handle such states one needs to generalise the concept of coincidence counts, stipulate the polariser action on general quantum states and identify precisely the hermitian operators for which a hidden variable description is being assumed. As a result of this generalisation we will show that a classical state in the quantum optical sense always obeys these inequalities while a non-classical state may violate them, possibly after a passive $U(4)$ transformation. Starting with a general nonclassical state, we subject it to a general unitary evolution corresponding to passive canonical transformations $U(4)$ before we look for the violation of Bell-type inequalities. A coincidence is defined to occur when both the detectors D_1 and D_2 click simultaneously i.e., *one or more photons* are detected by each. The following coincidence count rates are considered:

- (a) $P(\theta_1, \theta_2)$: P_1 at θ_1 and P_2 at θ_2 .
- (b) $P(\theta_1, \quad)$: P_1 at θ_1 and P_2 removed.
- (c) $P(\quad, \theta_2)$: P_1 removed and P_2 at θ_2 .
- (d) $P(\quad, \quad)$: Both P_1 and P_2 removed.

Before further analysis and calculation of these count rates we need to specify the precise action of the polarisers on a given quantum state. Classically, the action of a polariser is straightforward. The component of the electric field along the axis passes through unaffected while the orthogonal component is completely absorbed. The quantum action of the polariser is more complicated: for a given two-mode density matrix ρ (the two polarisation modes for a fixed direction) incident on a polariser placed at an

angle θ , the output (single-mode) state $\rho(\theta)$ is obtained by taking the trace over the mode orthogonal to the linear polarisation defined by θ . Explicitly in the number state basis:

$$\rho(\theta) = \sum_{n=0}^{\infty} \langle n | \rho | n \rangle_{(\theta + \pi/2)}. \quad (1)$$

The output state $\rho(\theta)$ is in general *mixed* even when the input ρ is pure. For the special case when the input state is a two-mode coherent state the output single-mode state is once again a coherent state. This is due to the fact that coherent states are not entangled in any basis:

$$\begin{aligned} |z_1\rangle_x |z_2\rangle_y &= |z_1 \cos \theta - z_2 \sin \theta\rangle_\theta |z_1 \sin \theta \\ &+ z_2 \cos \theta\rangle_{\theta + \pi/2} \rightarrow |z_1 \cos \theta - z_2 \cos \theta\rangle_\theta. \end{aligned} \quad (2)$$

On the other hand, single photon states can in general be entangled states of the two-mode field and thus would lead to mixed one-mode states after passing through the polariser. For example a pure two-mode single photon state $1/\sqrt{2}(|1\rangle_x |0\rangle_y + |0\rangle_x |1\rangle_y)$, after passage through a polariser placed in the x direction reduces to a mixed state with density matrix $\frac{1}{2}(|0\rangle_x \langle 0| + |1\rangle_x \langle 1|)$. For a comparable discussion on the action of a beam splitter see [10].

We define the following hermitian operators, with eigen values 0 and 1

$$\begin{aligned} \hat{A}_1 &= (I_{2 \times 2} - |00\rangle \langle 00|)_k, \\ \hat{A}_2 &= (I_{2 \times 2} - |00\rangle \langle 00|)_{k'}, \\ \hat{A}_1(\theta_1) &= (I_{\theta_1} - |0\rangle_{\theta_1} \langle 0|) I_{\theta_1 + \pi/2}, \\ \hat{A}_2(\theta_2) &= (I_{\theta_2} - |0\rangle_{\theta_2} \langle 0|) I_{\theta_2 + \pi/2}. \end{aligned} \quad (3)$$

The subscripts θ_1 and θ_2 in the last two equations refer to the settings of P_1 and P_2 . $I_{2 \times 2}$ is the two-mode unit operator while $I_{\theta_{1(2)}}$ and $I_{\theta_{1(2)} + \pi/2}$ are one-mode unit operators for the relevant polarisation modes along the propagation directions k or k' . The expectation values of these operators are the probabilities of detecting *at least one photon* of the appropriate kind (for example $\langle \hat{A}_1 \rangle$ is the probability of detecting at least one photon at D_1 with P_1 removed, and $\langle \hat{A}_1(\theta_1) \rangle$ that at D_1 with P_1 set at θ_1).

The quantum mechanical predictions for various coincidence count rates are the expectation values of the products of pairs of these operators:

$$\begin{aligned}
P(\theta_1, \theta_2) &= \langle \hat{A}_1(\theta_1) \hat{A}_2(\theta_2) \rangle, \\
P(\theta_1, \theta_2') &= \langle \hat{A}_1(\theta_1) \hat{A}_2(\theta_2') \rangle, \\
P(\theta_1', \theta_2) &= \langle \hat{A}_1(\theta_1') \hat{A}_2(\theta_2) \rangle, \\
P(\theta_1', \theta_2') &= \langle \hat{A}_1(\theta_1') \hat{A}_2(\theta_2') \rangle.
\end{aligned} \tag{4}$$

We note here that due to the definitions (3) the polariser action (1) is automatically implemented!

In the case when a hidden variable theory is assumed the value of the hidden variable along with the state vector will give us the actual outcomes of the individual measurements for the dynamical variables $A_1, A_2, A_1(\theta_1)$ and $A_2(\theta_2)$. The locality condition of ‘no action at a distance’ can then be readily used to calculate the coincidence count rates. Further, these rates are constrained by the following inequality due to Clauser and Horne [2]

$$\begin{aligned}
-P(\theta_1', \theta_2') &\leq P(\theta_1, \theta_2) - P(\theta_1, \theta_2') + P(\theta_1', \theta_2) \\
&\quad + P(\theta_1', \theta_2') - P(\theta_1', \theta_2) \\
-P(\theta_1', \theta_2) &\leq 0.
\end{aligned} \tag{5}$$

This is the required generalised Bell-type inequality relevant for arbitrary multi-photon states. We emphasise here that the coincidence count rates for general states have a different meaning as opposed to two-photon states. For two-photon states, extensively studied in the literature [4] (for example the state $\frac{1}{2}(a_1^\dagger - a_3^\dagger)(a_4^\dagger - a_2^\dagger)|0\rangle_1|0\rangle_2|0\rangle_3|0\rangle_4$), our formalism reduces to the usual one. More explicitly, the single hermitian operator $\hat{A} = I - |0\rangle\langle 0|$ giving the probability for finding one or more photons reduces effectively to $a^\dagger a$. The simplifying relations $P(\theta_1, \theta_2) = P(\theta_1, \theta_2) + P(\theta_1, \theta_2 + \frac{\pi}{2})$ etc. are also obtained from the reduction of \hat{A} 's and are not valid for general states.

We now turn to the analysis of interesting multi-photon states. Consider the 4-mode coherent states:

$$|z\rangle = \exp\left(-\frac{1}{2}z^T z^*\right) \exp\left(\sum_{j=1}^4 z_j a_j^\dagger\right) |0\rangle, \tag{6}$$

where $z^T = (z_1 \ z_2 \ z_3 \ z_4)$ is a complex row vector. The quantum mechanical values of the coincidence count rates for this case can be computed quite easily:

$$\begin{aligned}
P(\theta_1, \theta_2) &= (1 - e^{-|z_1'|^2})(1 - e^{-|z_3'|^2}), \\
P(\theta_1, \theta_2') &= (1 - e^{-|z_1'|^2})(1 - e^{-|z_3|^2 - |z_4|^2}), \\
P(\theta_1', \theta_2) &= (1 - e^{-|z_1|^2 - |z_2|^2})(1 - e^{-|z_3'|^2}), \\
P(\theta_1', \theta_2') &= (1 - e^{-|z_1|^2 - |z_2|^2})(1 - e^{-|z_3|^2 - |z_4|^2}), \\
\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\
\begin{pmatrix} z_3' \\ z_4' \end{pmatrix} &= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}.
\end{aligned} \tag{7}$$

A typical count rate $P(\theta_1, \theta_2)$ factorises, with the first factor depending solely on θ_1 and the second on θ_2 . This is a consequence of the unentangled nature of coherent states and is sufficient to establish their nonviolation of the Bell-type inequalities (5).

An arbitrary state ρ of the 4-mode radiation field can be expressed in terms of projections onto coherent states [11]:

$$\rho = \frac{1}{\pi^4} \int \varphi(z) |z\rangle\langle z| d^8z, \quad \frac{1}{\pi^4} \int \varphi(z) d^8z = 1. \tag{8}$$

In quantum optics the diagonal coherent state distribution function $\varphi(z)$ describing the state ρ is used to distinguish between classical and nonclassical states [12]. The states with nonnegative nonsingular $\varphi(z)$ are classical while the ones with negative or singular (worse than a delta function) $\varphi(z)$ are non-classical.

The function φ undergoes a point transformation when the state undergoes a unitary evolution corresponding to a passive canonical transformation given by an element of $U(4)$:

$$\varphi(z) \rightarrow \varphi'(z) = \varphi(z'), \quad z' = Uz, U \in U(4). \tag{9}$$

Thus, the classical or nonclassical nature of a state is preserved under such transformations.

In principle, $\varphi(z)$ can be used to calculate coincidence count rates for any given state. In particular for *classical states*, they are just their coherent state

values integrated over the positive distribution function $\varphi(z)$. When such count rates are substituted in the Bell-type inequality (5) it becomes the inequality for coherent states integrated over a normalized positive $\varphi(z)$. Since coherent states obey this inequality the integration over such a distribution obviously preserves this property. Thus we conclude that a 'classical state' will not violate the Bell type inequalities (5). Since the classical or nonclassical status of a 4-mode state is invariant under $U(4)$, the group of passive canonical transformations, a classical state after undergoing such a transformation will still not violate Bell type inequalities. On the other hand, the nonclassical states can violate these inequalities; in fact, the violation of such an inequality implies that the underlying $\varphi(z)$ for the state is negative or singular and the state is nonclassical in the quantum optical sense.

The family of squeezed thermal states, which are in general mixed states and possess a fluctuating number of photons vividly illustrate the strength of our formalism. Consider a 4-mode state with a centered Gaussian Wigner distribution [13]

$$W(\xi) = \pi^{-4} (\text{Det}G)^{1/2} \exp(-\xi^T G \xi),$$

$$\xi^T = (q_1 \quad q_2 \quad q_3 \quad q_4 \quad p_1 \quad p_2 \quad p_3 \quad p_4),$$

$$G = G^* = G^T,$$

$$G^{-1} + i\beta \geq 0, \quad \beta = \begin{pmatrix} 0_{4 \times 4} & 1_{4 \times 4} \\ -1_{4 \times 4} & 0_{4 \times 4} \end{pmatrix}. \quad (10)$$

Here, $q_1 = 1/\sqrt{2}(a_1^\dagger + a_1)$, $p_1 = i/\sqrt{2}(a_1^\dagger - a_1)$, etc. are the quadrature components. The matrix $V = \frac{1}{2}G^{-1}$ is the variance or the noise matrix. For a given state, if the smallest eigenvalue of this matrix is less than $\frac{1}{2}$ then the state is squeezed and therefore nonclassical [14].

We now look at specific examples of such Gaussian states in order to illustrate their violation of Bell-type inequalities. Take

$$G = U^{-1} S^T G_0 S U, \quad G_0 = \kappa I_{8 \times 8},$$

$$0 \leq \kappa \leq 1, \quad \kappa = \tanh \frac{\hbar \omega}{2kT}. \quad (11)$$

Here $\kappa = 1$ implies zero temperature and $\kappa < 1$ corresponds to some finite temperature, S is a 4-mode

squeezing symplectic transformation, an $Sp(8, \mathfrak{R})$ matrix, and U is a passive symplectic $U(4)$ transformation whose role is to produce entanglement. As an example, we start with a state in which the modes a_1 and a_4 are squeezed by equal and opposite amounts u and the modes a_2 and a_3 are squeezed by equal and opposite amounts v , and the entanglement is 'maximum'. This corresponds to the choices

$$S = \text{Diag}(e^{-u}, e^v, e^{-v}, e^u, e^u, e^{-v}, e^v, e^{-u}),$$

$$U = \frac{1}{2} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}, \quad X = \begin{bmatrix} Y & Y \\ -Y & Y \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \quad (12)$$

For this particular class of centered Gaussian Wigner states the function

$$f(\theta_1, \theta_2, \theta'_1, \theta'_2) = P(\theta_1, \theta_2) - P(\theta_1, \theta'_2) \\ + P(\theta'_1, \theta_2) + P(\theta'_1, \theta'_2) \\ - P(\theta_1, \theta'_1) - P(\theta_2, \theta'_2) \quad (13)$$

can be calculated. Though one ought to search over all values of the angles $\theta_1, \theta_2, \theta'_1$ and θ'_2 to locate possible violations of the inequality (5), we choose special values for the angles and demonstrate the violation. Plots for the function $f(\frac{\pi}{8}, \frac{\pi}{4}, 3\frac{\pi}{8}, 0)$ are shown in Fig. 2 clearly demonstrating that the Bell-type inequalities are violated by these states. The solid lines represent the squeezed vacuum while the dotted lines are for the case of finite temperature. We note that even at finite temperatures the inequalities are violated and their violation stems from the nonclassical property of squeezing.

A pure quantum mechanical state of a composite system is said to be entangled if we are not able to express it as a product of two factors. Such states have nontrivial quantum correlations and can lead to

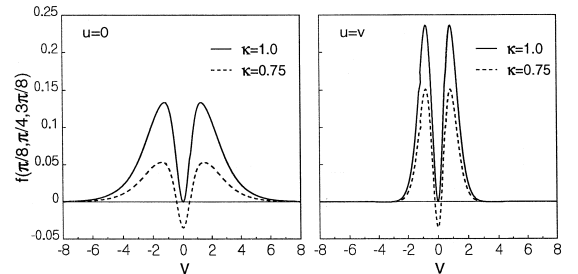


Fig. 2. Violation of Bell type inequality for states with centered Gaussian Wigner distributions representing a 4-mode centered Gaussian states.

the violation of suitable Bell-type inequalities. The passive canonical transformations $U(4)$ have been used to manipulate entanglement properties of 4-mode states. This capacity of passive transformations can already be seen at the level of two-mode fields. The group of passive canonical transformations in this case is $U(2)$; its elements, though incapable of producing or destroying nonclassicality are capable of entangling (disentangling) originally unentangled (entangled) states. For example, the unentangled nonclassical state $(\exp[i\xi(a_1^{\dagger 2} - a_1^2)]|0\rangle)$ $(\exp[i\xi(a_2^{\dagger 2} - a_2^2)]|0\rangle)$ becomes the entangled state $\exp[2\xi(a_1^{\dagger}a_2^{\dagger} - a_1a_2)]|00\rangle$ by the $U(2)$ transformation $\exp[i\pi/4(a_1^{\dagger}a_2 + a_2^{\dagger}a_1)]$. However, coherent states are not entangleable in this way! Classical states are statistical mixtures of coherent states and under $U(2)$ remain classical. Such a mixture can definitely have correlations which are purely classical, but it cannot have truly quantum mechanical entanglement. Thus classical states are to be regarded as nonentangled, and they remain so under passive $U(2)$ transformations. However this is in general not true for a nonclassical nonentangled state which may get entangled under a suitable $U(2)$ transformation. It is a straightforward matter to generalise the above statements to n mode systems where the group of passive canonical transformations is $U(n)$.

The above conclusions have an interesting bearing on violation of Bell-type inequalities with beams originating from independent sources as seen in the work of Yurke et al. and Zukowski et al. [15–17]. Though their inequalities are based on phase shifts and not on polarisation selection, the situation is analogous to the one discussed by us. These experiments take two beams from two independent sources, pass them through some passive optical elements and show that the Bell-type inequalities are violated. The first conclusion we can draw from our analysis is that it must be the quantum optical nonclassicality of one of the beams in their experiment which was converted into entanglement by the $U(4)$ transformation and hence led to the violation of the appropriate inequality. Secondly, if the original beams were quantum optically classical, no matter what one does, no violation would be seen as $U(4)$ is incapable of producing entanglement in classical beams.

In our analysis, we have not distinguished between strengths of coincidences. The coincidence counter registers a count when simultaneously each detector detects one or more photons. This is the reason why we chose the operators A 's to have eigen values 0 and 1. In this sense, the measurements involved here are not refined. It would be interesting to further generalise the analysis by considering somewhat refined measurements where to some extent coincidences are distinguished on the basis of their strengths. However, the relevant operators in this context may be unbounded; and it is well known that the formulation of Bell type inequalities for such operators, though desirable, is nontrivial.

We have compared quantum optical nonclassicality with violation of Bell's inequalities. When a state is nonclassical in the quantum optical sense, it does not allow a classical description based on an ensemble of solutions of Maxwell's equations, which is a very specific classical theory. On the other hand violation by a state of a Bell type inequality rules out any possibility of describing it by any general local 'classical' hidden variable theory. Therefore, it is understandable that quantum optical nonclassicality is a necessary but not a sufficient condition for the violation of Bell's inequalities. This disparity is partially compensated for by the freedom to perform passive canonical transformations on a nonclassical state before looking for violation of Bell's inequalities though it is not obvious whether this freedom completely removes this discrepancy. On the other hand, if a state obeys Bell's inequalities, it may still not allow a 'classical' description. Therefore, we need a complete set of Bell's inequalities capturing the full content of the locality assumption. These and related aspects will be explored elsewhere.

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