Quasicyclic Dyadic Codes in Walsh-Hadamard Domain

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Abstract — A dyadic code is a code which is closed under all dyadic shifts. A quasicyclic dyadic (QCD) code is one which is both quasicyclic and dyadic. We obtain a simple characterization of all QCD codes over any field of odd characteristic using Walsh-Hadamard Transform defined over that field.

I. PRELIMINARIES

An s -quasicyclic code is a code with the property that cyclic shift of any codeword by s positions gives another codeword. Cyclic codes are 1-quasicyclic codes. Dyadic codes are defined for length $n = 2^r$ as follows: For any integer $i \in \{0, 1, \dots, 2^r - 1\}$, let $[i] = [i_{r-1}, i_{r-2}, \dots, i_1, i_0]$ denote its radix-2 representation, where

$$i = i_{r-1}2^r + i_{r-2}2^{r-1} + \dots + i_12 + i_0,$$

and $i_j = 0, 1$ for $j = 0, 1, \dots, r-1$. Addition of two numbers $[i]$ and $[j]$, denoted by $[i] \oplus [j]$ is defined by $[i] \oplus [j] = [k]$, where $k_l = (i_l + j_l) \text{ modulo } 2$, for $l = 0, 1, \dots, (r-1)$. By an m -dyadic shift, $m = 0, 1, \dots, n-1$, of a vector $(a_0, a_1, \dots, a_{n-1})$ is meant the vector $(a_{0 \oplus m}, a_{1 \oplus m}, \dots, a_{(n-1) \oplus m})$ [1].

Definition 1: A linear code of length $n = 2^r$ over a field is said to be a dyadic code if m -dyadic shift of every codeword is also a codeword for all $m = 0, 1, \dots, n-1$.

Throughout the correspondence, we will consider linear codes over a finite field with q elements, denoted by F_q , where q is a power of an odd prime p .

Definition 2: A dyadic code of length $n = 2^r$ is said to be a Quasi-Cyclic Dyadic (QCD) code if it is an t -quasicyclic. If t is the smallest such integer, say $t = 2^s$ for some $s = 0, 1, \dots, r-1$, we call the code a " 2^s -QCD code". Note that a 2^0 -QCD code is a cyclic code which is also dyadic.

We shall characterize all 2^s -QCD codes in the transform domain using the Walsh-Hadamard transform (WHT) defined over F_q . Our characterization does not need extension of the field. Notice that if a cyclic code over F_q of length n is to be described in the transform domain using DFT then it is necessary that the DFT is defined in the extension field F_{q^m} , where m is the least integer such that n divides $q^m - 1$ [2]. The main result of this correspondence is that if such a cyclic code is dyadic also, then using WHT it can be described without field extension.

Definition 3: Let $(a_0, a_1, \dots, a_{n-1})$ be a n -length vector over F_q , a field with odd characteristic p and $n = 2^r$. The WHT of this vector is defined to be the n -length vector $(A_0, A_1, \dots, A_{n-1})$ over F_q , given by

$$A_j = \sum_{i=0}^{n-1} (-1)^{\langle j, i \rangle} a_i; \quad j = 0, 1, \dots, n-1 \quad (1)$$

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where $\langle j, i \rangle$ denotes the modulo-2 inner product

$$j_{r-1}i_{r-1} + j_{r-2}i_{r-2} + \dots + j_0i_0 \quad (\text{modulo } 2).$$

II. QCD CODES IN WHT DOMAIN

In this section we present the main result of this correspondence which gives the constraints among the spectral components to be satisfied for a dyadic code to be a 2^s -QCD code. Obviously,

Theorem 1: All length $n = 2^r$ dyadic codes are 2^{r-1} -QCD codes.

Definition 4: A subset $\mathcal{S}$ of spectral components $\{A_0, A_1, \dots, A_{n-1}\}$ is said to be a **constrained set** of spectral components for a dyadic code $\mathcal{C}$, if any one spectral component in $\mathcal{S}$ takes only the zero value then all other spectral components in $\mathcal{S}$ also take only the zero value for $\mathcal{C}$. A constrained set with only one spectral component will be referred as a **free spectral component** or equivalently a spectral component which does not belong to any constrained set with at least two elements is a free spectral component.

Definition 5: For a given s , $0 \leq s \leq r-2$, for every μ , $(s+1 \leq \mu \leq r-1)$ and a fixed $(j_{s-1}, j_{s-2}, \dots, j_0)$, the set of $2^{\mu-s}$ spectral components consisting of those A_j 's where

$$j = [0, 0, \dots, 0, 1 = j_\mu, j_{\mu-1}, j_{\mu-2}, \dots, j_{s+1}, j_s, j_{s-1}, \dots, j_0]$$

is denoted by $J(\mu, s, j_{s-1}, j_{s-2}, \dots, j_0)$.

The main result of this correspondence is

Theorem 2: A dyadic code $\mathcal{C}$ of length $n = 2^r$ is 2^s -QCD, $0 \leq s < r-2$, iff the set of WHT vectors of all the codewords satisfy the following conditions:

(i) For all $0 \leq j \leq 2^{s+1} - 1$, the spectral component A_j is free.

(ii) Among all $j \geq 2^{s+1}$, every $J(\mu, s, j_{s-1}, j_{s-2}, \dots, j_0)$ form a constrained set of spectral components for $\mathcal{C}$.

Example 1: For length 8 codes over any field of odd characteristic, there are $2^4 = 16$ cyclic dyadic codes corresponding to the following 4 constrained sets of spectral components.

$$\{A_0\} \quad \{A_1\} \quad \{A_2, A_3\} \quad \{A_4, A_5, A_6, A_7\}.$$

There are $2^6 = 64$ number of 2-QCD codes corresponding to the following 6 constrained sets of spectral components:

$$\{A_0\} \quad \{A_1\} \quad \{A_2\} \quad \{A_3\} \quad \{A_4, A_6\} \quad \{A_5, A_7\}.$$

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