Matrix Characterization of Generalized Hamming Weights

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Abstract — **A linear code with the systematic generator matrix $[I \mid P]$ is Maximum Distance Separable (MDS) if and only if every square submatrix of P is nonsingular. In this paper we obtain similar matrix characterization for all linear codes with a specified Hamming Weight Hierarchy (HWH). Using this we characterize Near-MDS codes (NMDS), Near-Near-MDS (N^2MDS) codes and a generalization of these codes called $N^{\mu}MDS$ codes in terms of their systematic generator matrices.**

I. Preliminaries

The support of a linear code C is the set of coordinate positions, where not all codewords of C are zero. The r-th generalized Hamming weight d_r, $1 \leq r \leq k$, of a $[n, k]$ linear code C over a field is defined as the cardinality of the minimal support of an $[n, r]$ subcode of C. The sequence $(d_1, d_2, \cdots, d_k)$ is called the Hamming Weight Hierarchy (HWH) of C [3]. The relation $d_r = n - k + r$ for $r = 1, 2, \cdots, k$ characterizes MDS codes.

NMDS Codes: [1] The class of $[n, k]$ codes where $d_1(C) = (n-k)$, and $d_i(C)$ is $(n-k+i)$ for $i = 2, 3, \ldots, k$.

N^2MDS **Codes:** [2] The class of $[n, k]$ codes where $d_1(C) = (n-k-1)$, $d_2(C) = (n-k+1)$ and $d_i(C)$ is $(n-k+i)$ for $i = 3, 4, \ldots, k$.

We generalize NMDS and N^2MDS codes as

$N^{\mu}MDS$ **Codes:** The class of $[n, k]$ codes where $d_i(C) = (n-k+2i-\mu-1)$ for $i = 1, 2, \ldots, k$.

II. Matrix Characterization of HWH

The MDS discrepancy of an $[n, k]$ code is defined as the smallest μ such that $d_{\mu+1} = n - k + \mu + 1$.

Theorem 1: The systematic parity check matrix of an $[n\ k]$ linear code with MDS discrepancy μ and $d_i(C) = n-k+i-\eta_i$ can be characterized as follows:

- For $i < g \leq min\{d_i - 1, k\}$, every $(g + \eta_i + 1 - i, g)$ sub matrix of P has rank $\geq (g - i + 1)$.
- There exists a g, $i < g \leq min\{d_i, k\}$, such that rank of $(g - i + \eta_i, g)$ sub matrix is $(g - i)$
- For $1 < g \leq min\{(n-k), (k-\mu)\}$ every $(g, g+\mu)$ sub matrix has rank g.

For $N^{\mu}MDS$ codes the defect, η_i, for $1 \leq i \leq \mu$, is $(\mu + 1 - i)$. The defect is zero for all $i > \mu$.

Corollary 1: **($N^{\mu}MDS$ codes)** The systematic G matrix characterization of $N^{\mu}MDS$ codes is as follows:

- For $i = 1, 2, \ldots, \mu$
  1. and $i < g \leq min\{d_i - 1, k\}$, every $(g - 2i + 1 + \mu, g)$ sub matrix has rank $\geq (g - i + 1)$.
  2. there exist a g, $i < g \leq min\{d_i, k\}$, such that $(g - 2i + \mu, g)$ sub matrix has rank equal to $(g - i)$.
- For $i = (\mu + 1), \ldots, k$
  1. For $1 < g \leq min\{(n-k), (k-\mu)\}$ every $(g, g+\mu)$ sub matrix has rank g.

Corollary 2: **(N^2MDS code)** For N^2MDS codes the systematic generator matrix characterization is

1. For $1 < g \leq min\{(n-k-2), k\}$ every $(g+2, g)$ sub matrix has rank $\geq$ g
2. For $2 < g \leq min\{(n-k), k\}$ every (g, g) sub matrix has rank $\geq (g-1)$
3. For $1 < g \leq min\{(n-k-1), k\}$ there exits a $(g+1, g)$ sub matrix with rank $(g-1)$.
4. For $2 < g \leq min\{(n-k+1), k\}$ there exits a $(g-1, g)$ sub matrix with rank $(g-2)$.
5. For $1 < g \leq min\{(n-k), (k-2)\}$ every $(g, g+2)$ sub matrix has rank g.

Corollary 2.1: For $k > q > 3$ and $n > 2q - 1 + k$ the systematic generator matrix of a N^2MDS code is characterized by every $(g+2, g)$ submatrix having rank $\geq$ g.

Corollary 3: **($NMDS$ code)** Systematic generator matrix characterization of $NMDS$ codes is:

- For $1 < g \leq min\{(n-k-1), k\}$ every $(g+1, g)$ sub matrix has rank $\geq$ g
- For $1 < g \leq min\{(n-k), k\}$ there exists a (g, g) sub matrix with rank equal to $(g-1)$
- For $1 < g \leq min\{(n-k), (k-1)\}$ every $(g, g+1)$ sub matrix has rank g.

Corollary 3.1: If $n > (k+q)$ the systematic generator matrix of a $NMDS$ code is characterized by every $(g+1, g)$ submatrix having rank $\geq$ g.

References

[1] S. D. Dodunekov and I. N. Landgev, ' On Near-MDS Codes', Technical Report, No:Lath-ISY-R-1563, Department of Electrical Engineering, Linkoping University, February, 1994.

[2] J. Olsson, ' On Near-Near-MDS Codes', Proceedings of Algebraic and Combinatorial Coding Theory Workshop, June, 1996

[3] V. K. Wei, "Generalized Hamming Weights for linear codes," *IEEE Trans. on Information Theory*, IT-Vol.37, No:5, Sept.1991, pp.1412-1418.

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