ANOTHER LOOK AT BELL'S INEQUALITIES
AND QUANTUM MECHANICS[†]

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ABSTRACT

Feynman's path integrals provide a hidden variable description of quantum mechanics (and quantum field theories). The expectation values defined through path integrals obey Bell's inequalities in Euclidean time, but not in Minkowski time. This observation allows us to pinpoint the origin of violation of Bell's inequalities in quantum mechanics.

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It is common, and often more convenient, to study quantum mechanics using the Schrödinger/Dirac equations and the Heisenberg operator algebra. Feynman's path integral formulation of quantum mechanics [1], nonetheless, is an alternative approach from which all the known results of quantum mechanics can be derived. In fact, quantum field theories are studied, more often than not, using the path integral formulation. For simplicity, let us consider quantum mechanics in one space dimension. The total amplitude (or wavefunction) for the system to be in the state $\psi(x_f, T)$ at time $t = T$, given an initial state $\psi(x, 0)$, is defined in terms of the transition kernel $K(x_f, T; x_i, 0)$:

$$\psi(x_f, T) \; = \; \int_{-\infty}^{\infty} K(x_f, T; x_i, 0) \; \psi(x_i, 0) \; dx_i \; , \tag{1}$$

$$K(x_f, T; x_i, 0) \; = \; \int_0^T [Dx(t)] \; \exp[\tfrac{i}{\hbar} S(x(t))] \; . \tag{2}$$

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The integration measure $[Dx(t)]$ can be defined precisely by discretising the time interval:

$$[Dx(t)] \propto \lim_{N \to \infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} dx_j \quad , \quad x_j \equiv x(t = jT/N) \quad . \tag{3}$$

Transition matrix elements and other expectation values are defined with the straighforward prescription:

$$\langle \mathcal{O}(x(t)) \rangle \equiv \langle \psi(x_f, T) | \mathcal{O}(x, t) | \psi(x_i, 0) \rangle = \int [Dx(t)] \, \rho_M(x(t)) \, \mathcal{O} \quad , \tag{4}$$

$$\rho_M(x(t)) = \exp[\tfrac{i}{\hbar} S(x(t))] / \int [Dx(t)] \, \exp[\tfrac{i}{\hbar} S(x(t))] \quad . \tag{5}$$

These definitions provide a hidden variable description of quantum mechanics. Indeed, x_j $(j = 1, ..., N-1)$ are the hidden variables which are integrated over. Note that $\{x_j\}$ corresponding to one particle are totally independent of those for another; all the correlations are built into specification of the initial state coordinates, x_0. There is no need to worry about ordering of various factors, because there are no non-commuting operators in this Lagrangian description, only complex numbers[1]. The functional integration measure $[Dx(t)]$ represents a sum over all paths connecting the fixed initial and final states and is inherently non-local[2]. $\exp[\tfrac{i}{\hbar} S(x(t))]$ is the statistical weight of the path $x(t)$ which depends on the interaction amongst the particles in the system. It is well-known that the typical paths contributing to the functional integral are highly irregular and non-differentiable. The individual paths characterised by $\{x_j\}$ follow local history/dynamics[3], but they do not obey constraints of causality. The virtual intermediate states, for example, can be "off-shell" with no relation between their energies and momenta, and they can propagate at a speed faster than that of light. They can even propagate backwards in time which is interpreted as pair creation and annihilation in relativistic field theories. The total amplitude (i.e. the sum over all paths), however, obeys all the constraints of causality and conservation laws.

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[1] Fermions can be dealt with along similar lines using Grassmann variables, but I do not consider them here.

[2] Quantum fluctuations can be suppressed and locality can be restored by taking the formal limit, $\hbar \to 0$.

[3] This is true in quantum mechanics only for non-interacting particles, since long range interparticle interaction potentials would produce instantaneous action-at-a-distance. Locality is much more general, however, in quantum field theories describable in terms of local Lagrangian densities.

The measurement process here corresponds to restricting the sum over all possible paths to only those paths which are consistent with the measured observable having a specific value in the final state. This is obviously a contextual process, once all the hidden variables are integrated out; if a second measurement were to be carried out on the system, the sum over all paths would be restricted to paths which are consistent with the results of both the first and the second measurements—the paths which would be consistent with the second measurement but not with the first have been discarded by the act of the first measurement. Thus the measured value of an observable may depend on the results of other measurements carried out prior to it on the same system, when these other measurements correspond to commuting but correlated observables. Note that the order in which the restriction of paths is carried out is immaterial, so there is no ambiguity regarding the final state of the system when measurements are carried out at space-like separations.

All put together, path integrals describe a contextual and non-local hidden variable theory[4]. It is similar in many aspects to the de Broglie-Bohm theory [3], but with the clear advantage that it is describable in a local language and that it is much more amenable to detailed calculations[5]. The crucial feature in this description is that the integration weight, $\rho_M(x(t))$, is a complex number in general. Therefore, although it is bounded, it cannot be interpreted as a probability density. It is this fact which allows the path integral description to bypass Bell's inequalities [4] and give a true definition of quantum mechanics[6]. Feynman actually showed how one can reconcile the EPR paradigm [5] with quantum mechanics using hidden variables, provided that probabilities are allowed to become negative [6]. Wigner functions [7] describing the density matrix distribution in the phase space are well-known examples of this type:

$$W(x, p) \;=\; \int \; \psi^*(x + \tfrac{1}{2}y) \; \exp(\tfrac{i}{\hbar}py) \; \psi(x - \tfrac{1}{2}y) \;\; . \tag{6}$$

They are real and obey all the usual manipulations of probability theory except that they are not always positive[7] everywhere in the phase space. Since the expectation value for

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[4] Bell's theorems ensure that any hidden variable theory describing quantum mechanics has to have such peculiarities [2].

[5] In particular, we know how to generalise from non-relativitistic to relativistic case, and from quantum mechanics to quantum field theory.

[6] For example, there can be cancellation between different paths giving rise to interference effects.

[7] My use of "positive" in this article includes zero, i.e. it really means "non-negative".

any physical observable is just

$$\langle \mathcal{O} \rangle \;=\; \int dx \; dp \; W(x,p) \; O(x,p) \;\; , \qquad (7)$$

where $O(x,p)$ is the (Hermitian) operator weight corresponding to the observable $\mathcal{O}$, it is necessary that for situations violating Bell's inequalities the Wigner function be negative somewhere in the phase space.

Now let us use the familiar trick of rotating to Euclidean (imaginary) time, $\tau = it$. This rotation converts the quantum theory to the language of statistical mechanics. The integration weight,

$$\rho_E(x(\tau)) \;=\; \exp[-\tfrac{1}{\hbar}S(x(\tau))]/ \int [Dx(\tau)] \; \exp[-\tfrac{1}{\hbar}S(x(\tau))] \;\; , \qquad (8)$$

is now real and positive, and can be interpreted as a probability density. This formal property has been exploited before, for vector-like gauge theories such as QCD, to derive rigorous inequalities among correlation functions and particle masses [8] [9]. The point I want to emphasise in this paper is that a positive integration weight must obey Bell's inequalities[8] Something has been lost in the analytic continuation from Minkowski to Euclidean time and the question is to identify what it is.

Let us first note that the above mentioned analytic continuation is routinely employed in quantum field theories in dealing with divergent loop integrals and renormalisation. As a matter of fact, there are strong theorems governing such an analytic continuation, e.g. the Wightman axioms [10] and the Osterwalder-Schrader positivity of the transfer matrix [11]. In particular, in the complex energy plane, these theorems rely on a sufficiently fast decrease of the amplitudes at infinity and on there being no singularities in the region covered by the rotation[9]. The integration contours can then be freely rotated without affecting the value of the integrals.

Let us also note that statistical mechanics has many features in common with quantum theory. The density matrix description allows probabilistic interpretation of superposed and mixed states. The presence of the heatbath[10] gives rise to vacuum fluctuations and unconstrained behaviour of the virtual states. Non-zero commutators (Poisson brackets)

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[8] I use the term "Bell's inequalities" in this article to generically denote the type of inequalities proved by Bell as well as by others.

[9] The poles corresponding to unstable states or resonances lie on the second Riemann sheet.

[10] $\hbar \neq 0$ corresponds to non-zero temperature.

leading to the uncertainty principle, e.g. $[p, x] \neq 0$, exist in the Hamiltonian description of the theory. Non-zero tunnelling amplitudes exist in the Euclidean theory, and so do "grotesque" states with infinitesimal probabilities[11]. These shared features cannot be responsible for the violation of Bell's inequalities in quantum mechanics.

What the Euclidean theory lacks is the concept of unitarity. The Minkowski space transition kernel, $K_M(x_f, T; x_i, 0)$, corresponds to a unitary transformation—the familiar $S-$matrix. It is easy to see that a unitary matrix with only real and positive matrix elements has to be the identity matrix (or its row-wise permutation corresponding to a shuffling of the states). The Euclidean space transition kernel is less restricted—it does not preserve the norm, though it maintains orthogonality of the states—and can be represented by a diagonal positive definite matrix. The loss of normalisation is not critical, since it can be taken care of following the LSZ prescription [12]. The information about the relative phases of the states, however, is lost. In fact this is the crucial quantum mechanical feature which makes the requirements of unitarity and a real positive integration weight mutually incompatible[12]. One can pick situations where the information contained in the relative phases cannot be made arbitrary small, because there are constraints on the complex amplitudes following, for example, from analyticity and dispersion relations. In such cases a statistical mechanics description cannot provide an arbitrarily close approximation to quantum mechanics. Bell's inequalities are explicit examples of this type[13].

Let us look at the problem again from a slightly different angle. The Euclidean correlation functions defined along the imaginary time axis are real, and a mere analytic continuation of them cannot produce non-trivial complex phase shift factors that are an an essential part of Minkowski scattering amplitudes[14]. *A priori* one does not know whether in

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[11] For instance, there is a non-zero probability for all the molecules of air in a room to collect themselves in one corner, but it would be silly to wait for such a thing to happen spontaneously.

[12] Feynman's choice of negative probabilities corresponds to an orthogonal transformation instead of a unitary one. In special circumstances, constraints of orthogonality are sufficient to show that Bell's inequalities are violated in quantum mechanics.

[13] Normally Bell's inequalities are not looked upon in this manner. They are instead specified by certain correlations in the initial state of a non-interacting system. In the real world, say in an $S-$matrix example, the correlations are not present in the initial state, but are generated dynamically by interaction phases.

[14] It is still possible to extract phase shifts from Euclidean results using indirect methods, e.g. finite volume shifts in energy levels of a multi-particle system [13].

analytical continuation of the Euclidean results the Euclidean time axis should be rotated by $+90°$ or $-90°$ to reach the Minkowski time axis. This ambiguity automatically gets resolved for internal loop variables of Feynman diagrams, just due to the necessity of not crossing any singularities while rotating the integration contours. For the variables corresponding to the external legs (incoming and outgoing states), however, the choice must be enforced as an additional condition; it amounts to the difference between choosing advanced or retarded propagators[15].

More explicitly, the principle of causality is embedded in the use of $i\epsilon-$prescription for retarded propagators. For example, the naive analytic continuation of the Euclidean scalar boson propagator, $(k^2 + m^2)^{-1}$, to $(k^2 - m^2)^{-1}$ is incorrect. The correct physical prescription is:

$$\frac{1}{k^2 + m^2} \rightarrow \lim_{\epsilon \to 0} \frac{1}{k^2 - m^2 + i\epsilon}$$
$$= \mathrm{P}\Big(\frac{1}{k^2 - m^2}\Big) - i\pi\delta(k^2 - m^2) \quad . \tag{9}$$

The non-unitary Euclidean theory fixes the off-shell amplitude (the principle value part) completely, but it is necessary to add an on-shell contribution (exemplified by the $\delta-$function at the pole) to comply with unitarity. This subtlety is unimportant in many cases: location of poles and branch cuts, matrix elements with external legs amputated according to the LSZ prescription, scattering amplitudes at threshold which are real etc. can be determined without recourse to the $i\epsilon-$prescription. On the other hand, $S-$matrix phase shifts, discontinuities across branch cuts etc. are not directly accessible in the Euclidean framework.

The conventional set up illustrating Bell's inequalities is the case where there is just free propagation of particles after initial creation of the state. Let us first look at the original EPR example [5]—a system of two identical non-relativistic particles freely propagating in one space dimension, where the initial state of the two particles is perfectly correlated in space (hence anti-correlated in momentum) with the constraint

$$\delta(x_i - y_i) = \int \frac{dp}{2\pi\hbar} \, \exp\big(\tfrac{i}{\hbar}p(x_i - y_i)\big) \quad . \tag{10}$$

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[15] Given Eucliidean time results, one can construct either advanced or retarded (or a linear combination thereof) amplitudes. The choice of retarded amplitudes only, though physically motivated by principle of causality, is absent in the Euclidean theory and has to be used as an additional input.

For a single free particle of mass m, the propagation kernel for Minkowski time T is

$$K_M(x_f, T; x_i, 0) = \sqrt{\frac{m}{2\pi i\hbar T}} \, \exp\left(i\frac{m(x_f - x_i)^2}{2\hbar T}\right) \quad, \tag{11}$$

while for Euclidean time T it is

$$K_E(x_f, T; x_i, 0) = \sqrt{\frac{m}{2\pi\hbar T}} \, \exp\left(-\frac{m(x_f - x_i)^2}{2\hbar T}\right) \quad. \tag{12}$$

Thus the final state wavefunction for the two particle system is

$$\psi_M(x_f, y_f, T) = \int \frac{dp}{2\pi\hbar} \exp\left(\tfrac{i}{\hbar}p(x_f - y_f)\right) \exp\left(\frac{ip^2 T}{\hbar m}\right) \tag{13}$$

in Minkwoski time, and

$$\psi_E(x_f, y_f, T) = \int \frac{dp}{2\pi\hbar} \exp\left(\tfrac{i}{\hbar}p(x_f - y_f)\right) \exp\left(-\frac{p^2 T}{\hbar m}\right) \tag{14}$$

in Euclidean time. In both cases the theory maintains its contextual character; the non-locality of the kernel ensures that if one particle is detected with mromentum p_f, the other is bound to be found with momentum $-p_f$.

The relative probability of observing various values of momentum is different in the two cases, however, which is just due to the difference in normalisation of states. This difference is not small as can be seen by appealing to the uncertainty relation: In order to detect the two particles distinctly, their separation has to be much larger than their de Broglie wavelengths,

$$|x_f - y_f| \sim p_f T/m \gg \hbar/p_f \quad, \tag{15}$$

implying that $p^2 T \gg \hbar m$ in the exponent. In this particular example, the exponent can be removed by just following the LSZ prescription, and there is no conflict with any inequality.

The reason behind no conflict with any inequality in the above example is that the δ−function correlation is positive. A more general case will have initial state correlations such that the Wigner function becomes negative somewhere in the phase space[16] [14]. The density matrix evolves linearly in time according to:

$$\begin{aligned}
\frac{dW_M}{dt} &= -\tfrac{i}{\hbar}[H, W_M] \implies W_M(t) = \exp(-\tfrac{i}{\hbar}Ht)W_M(0) \quad, \\
\frac{dW_E}{d\tau} &= -\tfrac{1}{\hbar}[H, W_E] \implies W_E(\tau) = \exp(-\tfrac{1}{\hbar}H\tau)W_E(0) \quad,
\end{aligned} \tag{16}$$

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[16] A density matrix with negative entries is perfectly alright for quantum systems, though it would be anathema in statistical physics.

where H is the Hamiltonian for the system. In case of a positive time evolution kernel, interference effects can only annihilate the negative density matrix regions in the phase space; a limit is reached when the Wigner function becomes positive everywhere and thereafter no regions of negative Wigner function can be regenerated[17]. Explicitly

$$f(\tau) \; = \; \int dx \; dp \; |W(x,p;\tau)| \; / \; \int dx \; dp \; W(x,p;\tau) \;\; , \;\; f(\tau) \geq 1 \;\; , \;\; df(\tau)/d\tau \leq 0 \;\; . \;\; (17)$$

No such reduction of negative density matrix regions is expected in case of a unitary time evolution; in fact the Minkowski time evolution of the density matrix obeys the continuity equation [7]:

$$\frac{\partial W(x,p)}{\partial t} + \frac{p}{m}\frac{\partial W(x,p)}{\partial x} \; = \; 0 \;\; . \;\; (18)$$

Thus, after factoring out the normalisation of states, the residual correlations which typify Bell's inequalities, become substantially different in the Euclidean case from the corresponding Minkowski values.

In brief, the Euclidean time evolution cannot dynamically create correlations violating Bell's inequalities if such correlations are absent in the initial state. Moreover, it wipes out such correlations even when they are inserted by hand in the initial state.

Violation of Bell's inequalities in quantum mechanics is often demonstrated using two non-relativistic spin-$\frac{1}{2}$ particles in a singlet state [15]. The Wigner function for this state has negative elements [6]. In this case, there are two ways of modifying the problem so that Bell's inequalities are satisfied. The first is to rewrite the spin operator in terms of the integration variables[18]. (e.g. as the Pauli-Lubanskì 4$-$vector, $ms^\mu = \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}p_\nu J_{\sigma\tau}$, where $J_{\sigma\tau}$ are the Lorentz generators), and then rotate from Minkowski to Euclidean time as described above[19]. An alternative is to replace the unitary spin-$SU(2)$ group by a "real positive counterpart". Both these choices give up unitarity, though in different manners. All this is, however, a subject for future work.

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[17] Though we are talking about the density matrix here, the situation is quite analogous to the evolution of the wavefunction in Euclidean time: It is common knowledge that the ground state wavefunction in quantum mechanics has no nodes. Euclidean time evolution concentrates the wavefunction towards its ground state component, cutting down the excited state contributions responsible for nodes in the wavefunction.

[18] Spin after all is a property of the Poincarè group transformations.

[19] Note that vector-like gauge theories have positive path integral measure in Euclidean space.

Finally, it is impertinent to ask why nature opted for unitarity or the Minkowski metric. What this paper aims to demonstrate is that it is the ubiquitous appearance of "i" together with a non-zero value of "$\hbar$", so characteristic of quantum physics, that sets it apart from classical physics: "i" causes violations of Bell's inequalities, while "$\hbar \neq 0$" is responsible for non-locality. A lesson to be learnt is that care must be exercised in converting results of Euclidean field theory (in particular attempts towards the quantum theory of gravity) to physical amplitudes; the restrictions following from unitarity have to be kept in mind. In fact the imposition of unitarity constraints on Euclidean results may provide a way to quantify the violations of Bell's inequalities.

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