Algorithms for Computing the Capacity of Markov Channels with Side Information

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I. INTRODUCTION

Consider a finite state, irreducible Markov channel $\{S_k\}$. At time k the channel output Y_k depends only on the channel input X_k and the channel state S_k . Let U_k and V_k be the CSIT (channel state information at transmitter) and the CSIR (CSI at receiver) respectively. In this paper we provide algorithms to compute the capacity of the Markov channel under various assumptions on CSIT and CSIR where the assumption $P(S_k = s | U_1^k) = P(S_k | U_k)$ in [1] is not necessarily satisfied. Our algorithms are based on the extension of ideas developed in [3] to the channels with feedback.

II. DISCRETE CHANNEL

In this section we further assume perfect CSIR i.e., $V_n = S_n$ and $U_n = g(S_{n-d+1}^n)$ where g is a deterministic function of $S_{n-d+1}^n \triangleq (S_{n-d+1}, \dots, S_n)$. Under these assumptions, we show that the capacity achieving distribution of X_n can be made a function of the conditional distribution ρ_n of S_n given U_1^n i.e., of

$$\rho_n(s) = P(S_n = s | U_1^n). \quad (1)$$

ρ_n can be computed recursively and $\{\rho_n\}$ is a regenerative process which converges in distribution. We also show that the capacity of the channel is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sup_{P(X_k | \rho_k)} \sum_s I(X_k; Y_k | S_k = s, \rho_k) \rho_k(s) \quad (2)$$

and this limit exists a.s. The sup in (2) is taken over all distributions on the input alphabet.

Theorem 1 *With the above mentioned assumptions on CSIT and perfect CSIR, the channel capacity of the finite state Markov channel is*

$$C = \int_{\rho} \sup_{P(X|\rho)} \sum_s I(X; Y | S = s, \rho) \rho(s) d\mu(\rho) \quad (3)$$

where μ is the limiting distribution of ρ_n and (2) converges a.s. to (3).

III. AWGN CHANNEL WITH FADING

In this section we consider a Markov fading channel with additive white Gaussian noise (AWGN) with mean zero and variance one. The X_n satisfies long term average power constraint $E[X_n^2] \leq \bar{P}$. Here we assume that the receiver has the perfect knowledge of instantaneous SNR, i.e. $V_n = S_n E[|X_n|^2 | U_1^n]$ and the CSIT may be imperfect i.e. $U_n = g(S_{n-d+1}^n, Z_n)$ where g is a deterministic function and $\{Z_n\}$ is an iid noise sequence. As in the previous section here we show that the OPA (Optimal Power Allocation Policy) can be made a function of the conditional distribution ρ_n , defined in (1). We show that

$$\frac{1}{n} \sum_{i=1}^n \sum_{s_i=s} \frac{1}{2} \log(1 + s\gamma(\rho_i)) \rho_i(s) \quad (4)$$

converges a.s. as $n \rightarrow \infty$, for any power allocation policy $\gamma(\cdot)$. This limit equals the capacity if $\gamma(\cdot)$, a function of ρ , is the OPA policy with respect to the limiting distribution μ of ρ_i .

Theorem 2 *With the above assumptions on CSIT and CSIR, the channel capacity of the finite state Markov fading channel with AWGN noise and average power constraint $\bar{P}$ is*

$$C = \max_{\gamma(\cdot): \int_{\rho} \gamma(\rho) d\mu(\rho) \leq \bar{P}} \int_{\rho} \sum_s \frac{1}{2} \log(1 + s\gamma(\rho)) \rho(s) d\mu(\rho) \quad (5)$$

where μ is the limiting distribution of ρ_n and (4) converges a.s. to (5) for the OPA policy.

From the Lagrange multiplier and the Kuhn-Tucker conditions [2] we get that the optimal power allocation policy $\gamma(\cdot)$ should satisfy

$$\sum_s \frac{s}{1 + s\gamma(\rho)} \rho(s) \leq \lambda \quad (6)$$

and $\gamma(\cdot) \geq 0$, for all ρ for which $\mu(\rho) > 0$. Solving (6) for the minimum λ such that the average power constraint $\int_{\rho} \gamma(\rho) d\mu(\rho) \leq \bar{P}$ is satisfied gives the optimal power allocation policy.

IV. EXAMPLES

Consider an AWGN channel with the fade values $s_0 = 1.0, s_1 = 0.5, s_2 = 0.01, s_3 = 0.001$ and let the average power constraint $\bar{P}$ be 20db. The capacity with perfect CSIT and CSIR is 1.81 hits. We will take a 1 bit CSIT ($U_k = 0$ if $S_k \in \{.001, .01\}$ else 1) and perfect CSIR and the delay in the feedback of the CSI to the transmitter d . Now the assumptions in [1] are violated. Then the capacity of this channel with $d = 0$ is $C_0 = 1.786$ bits and with $d = 1$ is $C_1 = 1.699$ bits. Fig. 1 shows the convergence of (4) for $d = 0$ and $d = 1$.

Figure 1: Convergence of Mutual Information to the capacity in Gaussian case

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