Optimal Power Control for Convolutional and Turbo Codes over Fading Channels

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Abstract **- We consider the problem of optimal power allocation to a user transmitting on a fading channel. The user is using some specific, convolutional or Turbo code. The cost function to be minimized is the bit error rate. The user needs to satisfy an average power constraint. We find that the optimal power policy is code and fading distribution dependent and can be significantly different from the commonly considered policies e.g, water filling and (truncated) channel inversion. The gain in BER can be substantial.**

I. INTRODUCTION

Wireless networks are becoming an important part of this networked world. To be able to provide any service (data or real-time) any time, anywhere, wireless networks will be essential in certain parts of network. These could be in the form of cellular or satellite links. However the wireless channel can be very hostile due to multipath fading, propagation loss, low bandwidth and the time-varying nature of the channel. Therefore, to provide certain Quality of Service (QoS) to data or real-time applications, one needs to utilize the resources efficiently. To do that various techniques e.g, the rake receiver, space-time codes, adaptive modulation, power control, etc., are employed. (see eg. [13, 11]). This paper is concerned about the problem of power control for wireless fading channels.

Power control for wireless channels has been employed with various criteria in mind e.g., to nullify the near-far effect [13, 14], for channel scheduling to provide certain QoS to different users [1], to maximize channel capacity [2, 12], to maximize channel capacity while simultaneously providing certain QoS [4, 12] and to increase spectral efficiency [3]. In this paper we try to minimize the bit-error rate (BER) when a specific coding scheme is used over a fading channel. This problem has been considered in the context of CDMA in [13, 14] and for an uncoded system in [10]. But, we will be concerned about specific convolutional codes, turbo codes and trellis coded modulation (TCM) systems. We are not aware of any work studying power control policies to minimize the BER in fading or non-fading environments for such specific systems. As against our problem, [3], consider adaptive rate coding and modulation schemes which depend on the fading state of the channel. Practical systems ususally fix a specific code and a rate for a system. In that case, the problem considered in this work is relevant.

Although our ideas extend to the multi user case, in this paper we restrict ourselves to a single user case. For the single user case, if we do not restrict to any particular coding scheme, then waterfilling in time provides the optimal power control [2]. Other control schemes commonly considered are channel inversion and truncated channel inversion [2, 3]. However we show in this paper that if we use a particular coding scheme, then certain power control schemes may actually be worse than employing no power control. Therefore, it is important to consider the power control scheme, which is optimal for the given coding scheme and the fading process. We also show that such optimization can indeed improve the BER of the system significantly for the same average power constraint on the transmitter.

To obtain the optimal power policy analytically for a given code, we need expressions for the BER for any given code and power control policy. However it is known that the probability of selecting the least distance path in the trellis provides a reasonably good indicator of the performance of a convolutional code for an additive white Gaussian noise (AWGN) channel [9]. Therefore we obtain the control policy, which minimizes this probability for a convolutional code for a given fading channel distribution. We show that the policy so obtained provides the BER which is very close to the actual optimal policy (obtained via simulations). Exactly the same procedure can be adapted for Trellis Coded modulation (TCM) schemes. We conducted some limited experiments which show usefulness of this approch for a TCM system also. In contrast to convolutional codes, for Turbo codes no such performance measure seems to be available. Therefore we considered the following approach. Instead of an *a posteriori* probability (APP) decoder [8], we employ the soft-output Viterbi algorithm (SOVA) decoder [6]. Then we obtain the power policy that minimizes the probability of occurence of the most likely path in the trellis of the constituent convolutional code. Later we use this control policy on the APP decoder itself and show that it improves significantly the BER of the decoder for the system as compared to the system without power control. This is an interesting result in itself.

The rest of the paper is organized as follows. Section II describes the modeling issues and defines the problem considered in general terms. It also describes some of the commonly recommended power control policies. Section III provides the optimal power control for uncoded binary-phase shift keying (BPSK), convolutional codes and turbo codes. For the latter two cases, optimal power allocation is illustrated with specific examples, though the approach is general. Section IV provides

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a comparison of the optimal power control policies via simulation.

II. The System Model And Problem Definition

We consider a fading wireless channel on which a user transmits his information. We study the discrete real-valued models of the system (generalization to complex valued is obvious). At time t, $t = 0, 1, 2, \ldots$, the transmitter transmits the coded symbol x_t, and the fading gain of the channel is a_t, An additive white Gaussian noise n_t corrupts the signal received at the receiver. We further assume that the channel state information (CSI) a_t is available at the transmitter and the receiver without any delay. This is a commonly made assumption and can be satisfied in varying degrees in practical systems. We further assume that the process a_t is stationary ergodic, taking values in a finite set A and having the stationary distribution π. The actual fading process is nonstationary due to slow shadow fading effects, but this component can be removed and then we obtain an approximately stationary fading process. Finiteness of set A is also violated in practice (e.g., Rayleigh fading is not finite-valued). However one can suitably quantize the fading gains to obtain a finite-valued process.

If the transmitter obtains the channel gain a_t and uses power gain $P(a_t)$ for transmitting the symbol x_t, then the received signal y_t is given by,

$$y_t = a_t \sqrt{P(a_t)}\, x_t + n_t. \tag{1}$$

We will take $E[x_t^2]$=1.We assume the average power constraint

$$\sum_{a \in A} P(a)\pi(a) \leq \alpha \tag{2}$$

where α is a suitable constant. Even though other power constraints may be applicable in practice, (2) is the most commonly used assumption. One can easily modify our scheme to include other power constraints.

Our problem will be to obtain a power control policy $P(a), a \in A$, which minimizes the (information) BER, while satisfying (2). We will consider this optimization problem in Section III for different coding schemes. Now we describe some of the power control algorithms commonly used in practice.

A. Commonly Used Control Schemes

In this section we mention some of the commonly used control policies [2, 3]. In section IV, we will compare these policies with the optimal policies obtained in Section III.

Waterfilling [2, 3] : It has been shown in [2] that the following algorithm, called waterfilling in time, maximizes channel capacity:

$$\frac{P(a_t)}{\alpha} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, & \gamma \geq \gamma_0, \\ 0, & \gamma < \gamma_0, \end{cases} \tag{3}$$

where $\gamma = a_t^2$ and γ_0 is a constant chosen so that (2) is satisfied with equality.

Channel Inversion [2, 3, 13] : This policy ensures that the receiver sees a constant SNR, i.e., $a_t^2 P(a_t) = \beta$, where β is chosen to satisfy (2) with equality. This is a commonly used policy in practice.

Truncated Channel Inversion [2, 3] : When channel inversion is applied for a channel whose fading level could be very low or zero (like the Rayleigh fading channel), one may require an infinite average power for channel inversion. Then (2) cannot be satisfied for any finite α. Thus inversion of channel only for $a_t \geq \beta_1$, for some appropriate positive constant β_1 has also been considered and in fact, can be significantly better than channel inversion.

In Section III we explain the optimal policies for various coding schemes. In Section IV these optimal schemes are compared with waterfilling and channel inversion. We have not compared the optimal policies with truncated channel inversion because in this policy, one does not transmit in some bad channel states. This changes the rate of transmission. In case of a fixed coding scheme, for a particular convolutional code one needs to transmit the coded bits at each time instant. Similarly, in waterfilling, if the channel state a is very bad, the policy may allocate $P(a) = 0$ (see (3)). For such cases for our comparison purposes, we will assume that there is transmission with zero power and hence there is error with probability $\frac{1}{2}$ (assuming all symbols are transmitted with equal probability and BPSK is used).

III. Optimal Power Control Policies

In this section, we solve the optimization problem, which involves minimizing the BER under the power constraint (2). In Section III A we considered the uncoded system. Section III B and III C study the problems for convolutional codes and turbo codes respectively.

A. Uncoded System

Consider the uncoded BPSK system, with $x_t \in \{-1, +1\}$. If the power control policy is $P(a)$, when channel gain is a, then from (1) the probability of error for a bit is (assuming symmetry we can take x_t to be +1)

$$\begin{aligned} P_e &= P(y_t \leq 0) \\ &= \sum_{a \in A} \pi(a) \int_{-\infty}^{-a\sqrt{P(a)}} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dx. \end{aligned} \tag{4}$$

Our aim is to find a power control policy, which minimizes (4) subject to (2) and $P(a) \geq 0$. Of course, it is easy to show that for the optimal policy, (2) is satisfied with equality.

To solve this optimization problem, using Lagrange multipliers, we differentiate

$$\sum_{a \in A} \pi(a) \int_{-\infty}^{-a\sqrt{P(a)}} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dx + \lambda\Big(\sum_{a \in A} P(a)\pi(a) - \alpha\Big) \tag{5}$$

with respect to $P(a)$ and λ and solve for $P(a)$. We obtain the optimal solution

$$\sqrt{P(a)} = -\frac{2\sigma^2}{a^2}\ln\left(\frac{2\lambda\sqrt{2\pi\sigma^2 P(a)}}{a}\right). \qquad (6)$$

The Lagrange multiplier λ is obtained by equating $E_\pi[P(a)]$ so obtained to α. We observe that the optimal $P(a)$ obtained satisfies $P(a) \geq 0$, for all a, and hence we did not have to explicitly include this constraint.

We provide an example to illustrate the effect of different policies.

Example: The fading process $\{a_t\}$ is Markov with state space $A = \{0.9, 0.5, 0.2\}$. Transition probability matrix of a_t is

$$P = \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.2 & 0.6 & 0.2 \\ 0.15 & 0.25 & 0.6 \end{bmatrix}$$

with stationary distribution, $\pi = \{\frac{4}{13}, \frac{5}{13}, \frac{4}{13}\}$. The noise varience is 0.0625. The power allocation and the BER under different policies is provided in Table I. One observes that the waterfilling and channel inversion are actually worse than using no power control. However, the optimal policy improves the BER sufficiently enough to be useful.

As illustrated by the above example, usually the optimal solution (6) is different from waterfilling or channel inversion. This uncoded system was also studied in [10] in a somewhat different form, but no comparison of the optimal policy with waterfilling and channel inversion was made.

B. Convolution codes

To illustrate our method for convolutional codes, we will use the encoder shown in fig 1. It is a rate half encoder with constraint length 3 and the free distance of the code is 5. The trellis corresponding to it is shown in figure 2. The information sequence entering the encoder is $\{z_t\}$ and the encoder output is $\{x_t\}$, $x_t = (x_t^1, x_t^2)$. For simplicity we assume that the channel fading state remains constant during transmission of x_t.

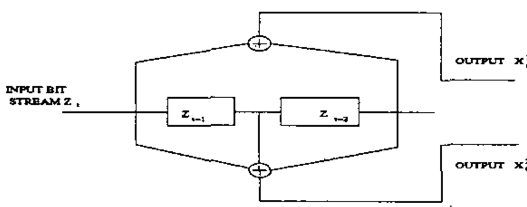

Figure 1: Convolutional Encoder used

As mentioned earlier for these codes there is no explicit expression available for BER. Therefore, we consider the most likey error path (first error event) shown in fig 2 for this decoder when all zeros have been transmitted. We compute the probability of error P_{e1} for this path to occur, for a given fading process and a power control policy. Now to compute the probability of error, it is not enough to know the marginal stationary probability $\pi(a)$ of the fading process. We need the three dimensional stationary probability $\pi(a_1, a_2, a_3)$. Then P_{e1} is given as

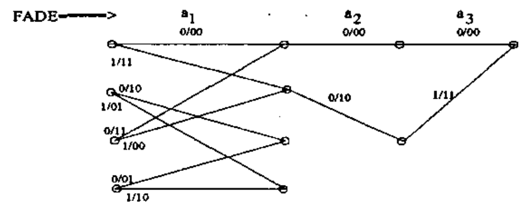

Figure 2: Trellis for the Convolutional Encoder, d_{free}=5

$$\begin{aligned} P_{e1} &= \sum_{(a_1,a_2,a_3)\in A^3} \pi(a_1,a_2,a_3) \\ & P\{\sum_{i=1}^{3}(y_i^1 + a_i\sqrt{P(a_i)})^2 + (y_i^2 + a_i\sqrt{P(a_i)})^2 \\ & \qquad \geq \{(y_1^1 - a_1\sqrt{P(a_1)})^2 + (y_1^2 - a_1\sqrt{P(a_1)})^2 \\ & \qquad + (y_2^1 - a_2\sqrt{P(a_2)})^2 + (y_2^2 + a_2\sqrt{P(a_2)})^2 \\ & \qquad + (y_3^1 - a_3\sqrt{P(a_3)})^2 + (y_3^2 - a_3\sqrt{P(a_3)})^2\}\} \end{aligned}$$

where $y_t = (y_t^1, y_t^2)$ is the signal vector received at the t^{th} transmission. We can easily show that

$$P_{e1} = \sum_{(a_1,a_2,a_3)\in A^3} \pi(a_1,a_2,a_3)\int_{\frac{-EX}{\sigma_x}}^{\infty} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx \qquad (7)$$

where,

$$EX = -8a_1^2P(a_1) - 4a_2^2P(a_2) - 8a_3^2P(a_3), \qquad (8)$$

and

$$\sigma_x = \sigma\sqrt{(32a_1^2P(a_1) + 16a_2^2P(a_2) + 32a_3^2P(a_3))}. \qquad (9)$$

One can obtain the power policy which minimizes P_{e1} subject to the average power constraint (2) via the method of Lagrange multipliers. We will show in the next section, via simulations that the BER of this optimal scheme is less than the BER corresponding to the power policies explained earlier.

From the above discussion it is obvious that the same ideas can be directly extended to TCM systems. The only problem is that since TCM systems are not linear, it is not enough to consider an all zero sequence as the input to the channel. One needs to search for the most likely error path more exhaustively.

C. Turbo Codes

In this section we explain our procedure to obtain an 'optimal' power scheme for a Turbo coded system. We consider a Turbo coded system consisting of two recursive systematic convolutional encoders, the trellis for which is shown in figure 3. If Turbo decoding is performed using SOVA [6], we can apply power control by minimizing the *dominant error event* which is again the first error event for a particular constituent convolutional decoder. This is because the constituent decoders are again Viterbi decoders. The dominant error event for the constituent RS Convolution code is shown in figure 3. All the concepts remain as in convolutionally encoded systems of Section III B except that we are using a recursive systematic Trellis in this case. Consider again the transmission model (1). At time t the output of the encoder is $x_t = (x_t^1, x_t^2, x_t^3)$ where x_t^1 is the information bit, x_t^2 is the output of the first encoder and x_t^3 is the output of the second encoder.

For simplicity we assume that the fading value remains constant for the three transmissions *i.e.* for each time index t, transmission of x_t^1, x_t^2 and x_t^3 are faded by the same fading value. This assumption greatly simplifies the computation of the first error event probability although the more general case can also be handled.

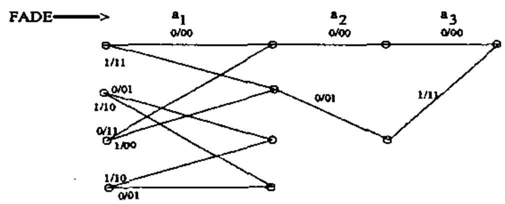

Figure 3: Trellis for constituent RS convolutional encoder

For the trellis considered in Fig. 3 d_{free}=5. The first error event probability (P_{e1}), which is on an average, the dominant error term for the path with distance d_{free} from the all zero path. $y_t = \{y_t^1, y_t^2\}$ is the received vector for the first decoder. With power control introduced in transmission, P_{e1} is given by the equations in Section III B.

We consider P_{e1} as the objective function to be minimized with the average power constraint (2). The optimal policy can be obtained via the method of Lagrange multipliers.

We will see in the next section that the optimal power allocation obtained by minimizing the above objective function P_{e1} gives improvement in terms of BER compared to any of the conventional power control policies. The effect of power control stays for a large number of iterations and hence is worth providing for a specific channel. These advantages stay even when we use this optimal policy with a Turbo decoder using APP decoders.

IV. Simulation Results

In this section we simulate the systems with the convolutional encoder and the Turbo encoder described in section III. We obtain the BER for the systems with the following power control policies: waterfilling, channel inversion, the optimal policies obtained in section III and the globally optimal policies (obtained via simulations) which optimize the BER. The simulations were run long enough to stabilize the BER estimates. The fading process was generated as a three state Markov chain. The average power constraint α was taken as 1.

Table II provides the results for the convolution coding system described in section III B. The fading process $\{a_t\}$ is a Markov process with the state space and the transition probability matrix as in section III A. One observes that water filling and the channel inversion are much worse than even the system without any power control. The optimal policy obtained by minimizing P_{e1} provides sufficient benefits to justify including power control. The globally optimal policy (obtained via simulations) is also included in Table II. We observe that the policy obtained by minimizing P_{e1} is quite close in BER to the optimal policy. This indicates that it is sufficient to have a control policy that optimize P_{e1}.

We consider the Turbo coded system, studied in section III C. The fading process has states $A = \{0.9, 0.5, 0.02\}$ with the transition probabilities as above. The other parameters also stay as above (noise variance = 0.0625). The interleaver length used is 1024. The BER for this system for the various power control policies are provided in figure 4 for various number of iterations. We have used SOVA as the decoding algorithm for constituent decoders in this system.

Finally, we have studied Turbo coded system with constituent decoders as APP decoders. In this system we have $A = \{0.9, 0.5, 0.02\}$, noise variance = 0.1, rest of the system parameters remain as above. The BER is obtained for all the power policies mentioned earlier and is presented in Table III. The optimal power allocation for this case is obtained by using the same concept as used for SOVA i.e. assuming in this case that each constituent decoder is a Viterbi decoder and has a dominant error event (first error event) associated with it. We used such an approach because a similar analysis for an APP decoder is not known. Again we observe that waterfilling and channel inversion perform much worse than the optimal policy and also than the system without power control. The optimal policy is better than the system with no power control for each iteration even after fifteen iterations. In Table IV we provide the simulation results when the Turbo encoder in section III C is punctured and APP decoder is used. The parity bits are alternately punctured to make the a rate $\frac{1}{2}$ turbo code. We observe similar trends as in Table III.

TABLE I
PERFORMANCE OF UNCODED SYSTEM

Control Policy	Power factor $\sqrt{P(0.9)}$	$\sqrt{P(0.5)}$	$\sqrt{P(0.2)}$	BER Theoretical
No Power Control	1.0	1.0	1.0	0.0726
Channel inversion	0.3583	0.6458	1.6125	0.09813
Water Filling	1.72	0.4648	0.0	0.2216
Optimal allocation	0.65	0.96	1.3121	0.0596

TABLE II
PERFORMANCE OF CONVOLUTIONALLY ENCODED SYSTEM

Control Policy	Power factor $\sqrt{P(0.9)}$	$\sqrt{P(0.5)}$	$\sqrt{P(0.2)}$	Simulated BER
No Power Control	0.707	0.707	0.707	0.039619
Channel inversion	0.2533	0.4565	1.14	0.113595
Water Filling	1.216	0.3286	0.0	0.183824
Optimal allocation	0.5	0.75	0.8366	0.0298095
Simulated Optimal	0.45	0.7	0.913	0.028731

TABLE III
SIMULATION STUDY: TURBO CODES USING APP

Control Policy	BER after 5 iterations	BER after 10 iterations	BER after 15 iterations	Power factor $\sqrt{P(0.9)}$	$\sqrt{P(0.5)}$	$\sqrt{P(0.02)}$
No Power Control	0.1068	0.0997	0.09502	0.577	0.577	0.577
Water filling	0.2013	0.1995	0.1957	0.992	0.2681	0.0
Channel Inversion	0.343	0.343	0.343	0.0232	0.0416	1.0386
Optimal allocation	0.0842	0.0774	0.0755	0.4162	0.6773	0.5987

TABLE IV
TURBO CODES USING APP WITH PUNCTURING

Control Policy	BER after 5 iterations	BER after 10 iterations	BER after 15 iterations	Power factor $\sqrt{P(0.9)}$	$\sqrt{P(0.5)}$	$\sqrt{P(0.02)}$
No Power Control	0.093	0.0816	0.0686	0.707	0.707	0.707
Water filling	0.127	0.103	0.0863	1.216	0.3286	0.0
Channel Inversion	0.333	0.333	0.332	0.0284	0.051	1.2726
Optimal allocation	0.078	0.06687	0.0531	0.51	0.83	0.7336

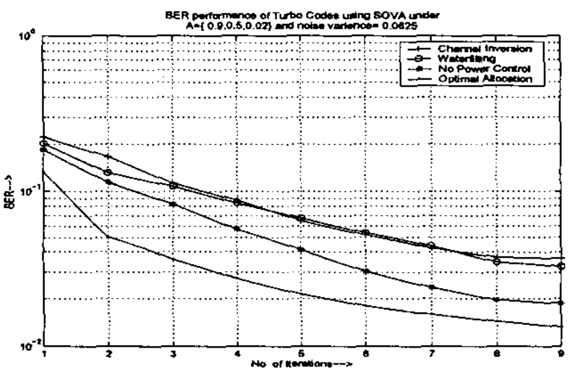

Figure 4: Performance of Turbo Codes using SOVA

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