# SNR Estimation in Nakagami-$m$ Fading With Diversity Combining and Its Application to Turbo Decoding

A. Ramesh, A. Chockalingam, and Laurence B. Milstein, *Fellow, IEEE*

***Abstract*—We propose an online signal-to-noise ratio (SNR) estimation scheme for Nakagami-$m$ fading channels with $L$ branch equal gain combining (EGC) diversity. We derive the SNR estimate based on the statistical ratio of certain observables over a block of data, and use the SNR estimates in the iterative decoding of turbo codes on Nakagami-$m$ fading channels with $L$ branch EGC diversity. We evaluate the turbo decoder performance using the SNR estimate under various fading and diversity scenarios ($m = 0.5, 1, 5$ and $L = 1, 2, 3$) and compare it with the performance using perfect knowledge of the SNR and the fade amplitudes.**

***Index Terms*—Diversity, Nakagami fading, SNR estimation, turbo codes.**

## I. Introduction

TURBO CODES have been shown to offer near-capacity performance on additive white Gaussian noise (AWGN) channels and exceptional performance on fully interleaved flat Rayleigh fading channels [1], [2]. Optimum decoding of turbo codes on AWGN channels requires the knowledge of the channel signal-to-noise ratio (SNR) [1]. Summers and Wilson have recently addressed the issue of the sensitivity of the turbo decoder performance to imperfect knowledge of the channel SNR on AWGN channels, and proposed an accurate online SNR estimation scheme [3]. Performance of turbo codes on flat Rayleigh fading has been addressed in [2], [4], and [5]. It is noted that optimum decoding of turbo codes on fading channels requires the knowledge of the channel SNR and the fade amplitudes [2]. In the performance evaluation of turbo codes in [2], perfect knowledge of both $E_s/N_o$ (channel SNR) and the fade amplitudes of each symbol are assumed to be available at the decoder. In practice, the channel SNR needs to be estimated at the receiver for use in the turbo decoding. A channel estimation technique suitable for decoding turbo codes on flat Rayleigh fading channels is presented in [4]. The technique is based on sending known pilot symbols at regular intervals in the transmit symbol sequence. In [5], a channel estimator based on a low pass finite-impulse response (FIR) filter is presented for flat Rayleigh and Rician fading channels. In [6], we derived an SNR estimation scheme for Nakagami-$m$ fading channels without diversity combining and used this estimate in the decoding of turbo codes.

In this letter, we propose an online SNR estimation scheme for Nakagami-$m$ fading with $L$ branch equal gain combining (EGC) diversity. The proposed SNR estimation scheme does not estimate the fade amplitudes and thus, does not require the transmission of known training symbols. The SNR estimate is derived using the statistical ratio of certain observables over a block of data. Our SNR estimator is valid for any value of $m \geq 0.5$. The SNR estimates in Rayleigh fading and AWGN channels can be obtained as special cases corresponding to $m = 1$ and $m = \infty$, respectively. As an example, we use the SNR estimates in the iterative decoding of turbo codes on Nakagami-$m$ fading channels with $L$ branch EGC diversity. We evaluate the turbo decoder performance using the SNR estimate under various fading and diversity scenarios ($m = 0.5, 1, 5$ and $L = 1, 2, 3$) and compare it with the performance using perfect knowledge of the SNR and the fade amplitudes.

## II. SNR Estimation

Let the encoded data symbols be binary phase-shift keying (BPSK) modulated and transmitted over a Nakagami fading channel. We assume $L$ antennas at the receiver with sufficient spacing between them so that these antennas receive signals through independent fading paths. We denote the $k$th symbol received at the $i$th antenna by $r_i^{(k)}$, and assume that the receiver performs EGC, after coherently demodulating the received symbols on these independent diversity paths. Then, the $k$th received symbol, $v_k$, at the output of the combiner, is given by

$$v_k = \sum_{i=1}^{L} r_i^{(k)} \tag{1}$$

where

$$r_i^{(k)} = \pm\alpha_i^{(k)}\sqrt{E_s} + n_i^{(k)}. \tag{2}$$

Here, $\alpha_i^{(k)}$ is the random fading amplitude experienced by the $k$th symbol on the $i$th antenna path, $E_s$ is the symbol energy, and $n_i^{(k)}$ is the AWGN component at the receiver front end having zero mean and variance $\sigma^2 = N_o/2$. We assume that the $\alpha$'s are Nakagami-$m$ distributed [7] and independent of the noise. Specifically, the probability density function (pdf) of $\alpha$, $p_\alpha(a)$, is given by

$$p_\alpha(a) = \frac{2m^m a^{2m-1}}{\Gamma(m)} e^{-ma^2} \tag{3}$$

Paper approved by W. E. Ryan, the Editor for Modulation, Coding, and Equalization of the IEEE Communications Society. Manuscript received December 15, 2000; revised November 4, 2001 and January 23, 2002. This work was supported in part by the Office of Naval Research under Grant N00014-98-1-0875. This paper was presented in part at the IEEE ICC'2001, Helsinki, Finland, June, 2001.

A. Ramesh was with the Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, Karnataka, India. He is now with the Department of Electrical and Computer Engineering, University of California at San Diego, La Jolla, CA 92093 USA (e-mail: ramesh@cwc.ucsd.edu).

A. Chockalingam is with the Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, India (e-mail: achockal@ece.iisc.ernet.in).

L. B. Milstein is with the Department of Electrical and Computer Engineering, University of California at San Diego, La Jolla, CA 92093 USA (e-mail: milstein@ece.ucsd.edu).

Digital Object Identifier 10.1109/TCOMM.2002.805276

where we have normalized the second moment of the fading amplitude, $E(\alpha^2)$, to unity.

We want to estimate the average received SNR, $\gamma = (E_s/2\sigma^2)E(\alpha^2) = E_s/2\sigma^2$. Our interest is to devise a blind algorithm which does not require the transmission of known training symbols to estimate the SNR. Accordingly, we formulate an estimator for the SNR based on a block observation of the $v_n$'s. We define a parameter $z_{\text{div}}$, to be the ratio of two statistical computations on the block observation of $v_n$'s, as

$$z_{\text{div}} = \frac{\left[E(v^2)\right]^2}{E(v^4)}. \tag{4}$$

We derive $z_{\text{div}}$ as a function $f(\cdot)$ of the received SNR, $\gamma$. Thus, the ratio of the two statistical computations and the known function $f(\gamma)$ provide a means to estimate $\gamma$. The parameter $z_{\text{div}}$ in (4) can be derived in closed form as (see Appendix)

$$z_{\text{div}} = \frac{\left[L + \left((L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2 + L\right)2\gamma\right]^2}{3L^2 + 4\Delta_m\gamma^2 + 12L\left(L + (L^2 - L)\left(\frac{\Gamma(m+\frac{1}{2})}{\sqrt{m}\Gamma(m)}\right)^2\right)\gamma} \tag{5}$$

where

$$\begin{aligned}\Delta_m =& LE(\alpha^4) + 4L(L-1)E(\alpha^3)E(\alpha)\\ &+ 3L(L-1)[E(\alpha^2)]^2\\ &+ 6L(L-1)(L-2)E(\alpha^2)[E(\alpha)]^2\\ &+ L(L-1)(L-2)(L-3)[E(\alpha)]^4\end{aligned} \tag{6}$$

with $E(\alpha^k) = (\Gamma(m + k/2)/\Gamma(m)m^{k/2})$. Note that (5) assumes the knowledge of Nakagami parameter $m$, which can be computed accurately using the method given in [11]. For the case when $m = 1$ (i.e., Rayleigh fading), (5) becomes

$$z_{\text{ray}} = \frac{\left[L + \frac{L(\pi L + 4 - \pi)}{2}\gamma\right]^2}{3L^2 + 4\Delta_1\gamma^2 + 3L^2(\pi L + 4 - \pi)\gamma}. \tag{7}$$

Now, for a given value of $z_{\text{div}}$ (computed from a block observation of the $v_n$'s), the corresponding estimate of $\gamma$ can be found by inverting (5). For easy implementation, an approximate relation between $z_{\text{div}}$ and $\gamma$ can be obtained through an exponential curve fitting for (5). We use the exponential fit of the form

$$\gamma = d_3 e^{\left(d_0 e^{(d_1 z_{\text{div}})} + d_2 z_{\text{div}}\right)} \tag{8}$$

where the values of the coefficients $d_0$, $d_1$, $d_2$, and $d_3$ for different values of $m$ and $L$ are computed and presented in Table I. The coefficients $d_0$, $d_1$, $d_2$, and $d_3$ are chosen in such a way that the mean-square error $\sum_{j=1}^{P}\left[z(j) - d_3 * \exp(d_0 * \exp(d_1 * z(j)) + d_2 * z(j))\right]^2$ is minimized, where $P$ is the number of points (taken to be 30) on the $z = f(\gamma)$ curve. Fig. 1 shows the $\gamma$ versus $z_{\text{div}}$ plots corresponding to Rayleigh fading $(m = 1)$ for $L = 1$ and 2 as per (8), along with the true value plots as per (7). The fits are made in the magnitude domain and plotted in dB. It is seen that the fits are very accurate over the SNR values of interest. In

TABLE I
COEFFICIENTS OF THE EXPONENTIAL FIT FOR DIFFERENT VALUES OF $m$ AND $L$

| Channel | Exponential fit, $\gamma = d_3 \exp(d_0 \exp(d_1 z) + d_2 z)$ | | | |
|---|---|---|---|---|
| | $d_0$ | $d_1$ | $d_2$ | $d_3$ |
| AWGN ($m = \infty$, $L = 1$) | 4.10 | 0.40 | 4.07 | $9.87 \times 10^{-4}$ |
| Rayleigh ($m = 1$, $L = 1$) | 12.30 | $1.53 \times 10^{-7}$ | 22.61 | $6.28 \times 10^{-10}$ |
| Nakagami ($m = 0.5$, $L = 1$) | 2.44 | -57.86 | -9.82 | 4.68 |
| Nakagami($m = 5$, $L = 1$) | -1.03 | 23.59 | -75.16 | 215.10 |
| Rayleigh ($m = 1$, $L = 2$) | $7.25 \times 10^{-8}$ | 25.94 | 10.44 | $6.72 \times 10^{-3}$ |
| Rayleigh ($m = 1$, $L = 3$) | $1.85 \times 10^{-2}$ | 7.19 | 1.82 | $1.31 \times 10^{-1}$ |

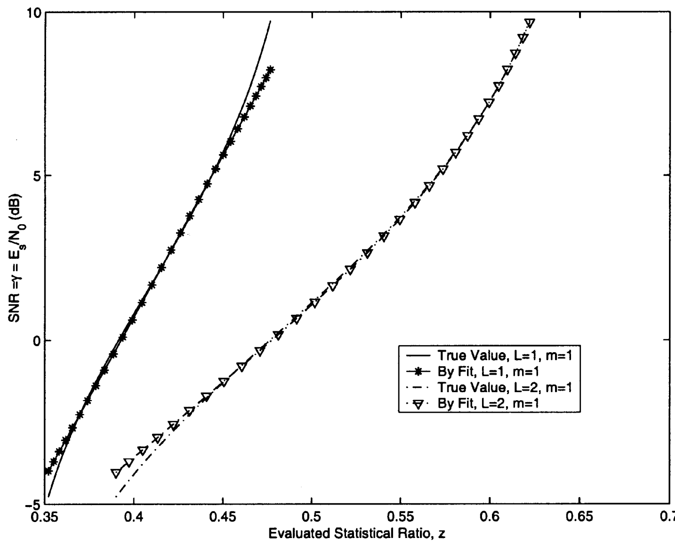


Fig. 1. $\gamma$ versus $z_{\text{div}}$ in Rayleigh fading $(m = 1)$ for $L = 1$ and 2.

order to obtain an estimate for $z_{\text{div}}$, we replace the expectations in (4) with the corresponding block averages, yielding

$$\hat{z}_{\text{div}} = \frac{[\overline{v^2}]^2}{\overline{v^4}}. \tag{9}$$

Substituting (9) into (8), we get the SNR estimates, $\widehat{\gamma}$. We tested the accuracy of the fit by evaluating the mean and standard deviation of the SNR estimates for $L = 2$ and $m = 1$, determined by over 20 000 blocks. The block sizes considered are 1000 and 5000 bits, and the code rate is 1/3 (i.e., 3000 and 15 000 code symbols per block). The range of $E_b/N_0$ values considered is from 0 dB to 8 dB in steps of 1 dB. For a code rate of 1/3, this corresponds to $E_s/N_0$ values from $-4.77$ to 3.23 dB, as shown in Table II. The mean and the standard deviation of the estimates are evaluated in the magnitude domain and converted to dB. From Table II, we observe that the SNR estimates $\widehat{\gamma}$ through the exponential fit in (8) are quite close to the true value of the SNR, $\gamma$, and the standard deviation of the estimate reduces as the block size is increased.

## III. LOG-MAXIMUM *A POSTERIORI* (MAP) DECODER WITH EGC

In this section, we modify the log-MAP decoder for the case of $L$ branch diversity with equal gain combining. To do so, we need to calculate the transition metric defined by $g_k(s, t, \underline{\alpha}_k) = \text{Prob}(\mathbf{y}_k, S_k = t|S_{k-1} = s, \underline{\alpha}_k)$, where $\mathbf{y}_k = (y_k^s, y_k^p)$ and $\underline{\alpha}_k = (\underline{\alpha}_k^s, \underline{\alpha}_k^p)$ [2], [9]. Here, $y_k^s$ is the received symbol corresponding to the transmitted information

TABLE II
MEAN AND STANDARD DEVIATION OF THE SNR ESTIMATE, $\widehat{\gamma}$, FOR DIFFERENT VALUES OF THE TRUE SNR, $\gamma$, FOR $m = 1$ AND $L = 2$

| True SNR, $\gamma$ (dB) | Block size=1000 bits | | Block Size=5000 bits | |
|---|---|---|---|---|
| | E[$\widehat{\gamma}$], dB | SD [$\widehat{\gamma}$], dB | E[$\widehat{\gamma}$], dB | SD[$\widehat{\gamma}$], dB |
| -4.77 | -4.01 | 0.397 | -4.03 | 0.181 |
| -3.77 | -3.32 | 0.395 | -3.34 | 0.180 |
| -2.77 | -2.54 | 0.396 | -2.57 | 0.178 |
| -1.77 | -1.69 | 0.395 | -1.71 | 0.176 |
| -0.77 | -0.77 | 0.394 | -0.78 | 0.176 |
| 0.23 | 0.18 | 0.395 | 0.17 | 0.178 |
| 1.23 | 1.19 | 0.407 | 1.66 | 0.182 |
| 2.23 | 2.18 | 0.421 | 2.16 | 0.189 |
| 3.23 | 3.19 | 0.443 | 3.17 | 0.200 |

symbol $x_k^s$, $y_k^p$ is the received symbol corresponding to the transmitted parity symbol $x_k^p$, $\underline{\alpha}_k^s = \left(\alpha_{k,1}^s, \alpha_{k,2}^s, \ldots, \alpha_{k,L}^s\right)$, and $\underline{\alpha}_k^p = \left(\alpha_{k,1}^p, \alpha_{k,2}^p, \ldots, \alpha_{k,L}^p\right)$. Also, $S_k$ and $S_{k-1}$ are the encoder states at time instants $k$, $k-1$, respectively [9]. When the symbol $x_k^\xi$ is transmitted, it will be received through $L$ independent paths and the output of the combiner will be

$$y_k^\xi = x_k^\xi \sum_{l=1}^{L} \alpha_{k,l}^\xi + \sum_{l=1}^{L} n_{k,l}^\xi \tag{10}$$

where $\xi \in \{s,p\}$, $\alpha_{k,l}^\xi$ is the random fading amplitude experienced by the $k$th symbol at the $l$th antenna path, and $n_{k,l}^\xi \sim \mathcal{N}(0,\sigma^2)$. Conditioning on $x_k^\xi$ and $\alpha_{k,1}^\xi, \alpha_{k,2}^\xi, \ldots, \alpha_{k,L}^\xi$, we have $y_k^\xi \sim \mathcal{N}\left(x_k^\xi \sum_{l=1}^{L} \alpha_{k,l}^\xi, L\sigma^2\right)$. Applying Bayes' theorem, we can write $g_k(s,t,\underline{\alpha}_k)$ as

$$\begin{aligned} g_k(s,t,\underline{\alpha}_k) =& \text{Prob}(\mathbf{y}_k, S_k = t | S_{k-1} = s, \underline{\alpha}_k) \\ =& \text{Prob}(\mathbf{y}_k | S_{k-1} = s, S_k = t, \underline{\alpha}_k) \\ & \times \text{Prob}(S_k = t | S_{k-1} = s) \\ =& p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k) \text{Prob}(S_k = t | S_{k-1} = s) \\ =& p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k) \text{Prob}\left(x_k^s\right). \end{aligned} \tag{11}$$

The last step in the above equation is due to the fact that the state transition between any given pair of states $s$ and $t$ uniquely determines the information bit $x_k^s$. Define

$$\begin{aligned} c_k(s,t,\underline{\alpha}_k) &= \log\left(g_k(s,t,\underline{\alpha}_k)\right) \\ &= \log(p\left(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k\right) \text{Prob}\left(x_k^s\right)) \\ &= \log(p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k)) + \log\left(\text{Prob}\left(x_k^s\right)\right). \end{aligned} \tag{12}$$

With perfect channel interleaving and knowledge of fade amplitudes, we get

$$p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k) = p\left(y_k^s | x_k^s, \underline{\alpha}_k^s\right) p\left(y_k^p | x_k^p, \underline{\alpha}_k^p\right). \tag{13}$$

Upon observing that $y_k^s \sim \mathcal{N}\left(x_k^s \sum_{l=1}^{L} \alpha_{k,l}^s, L\sigma^2\right)$ and $y_k^p \sim \mathcal{N}\left(x_k^p \sum_{l=1}^{L} \alpha_{k,l}^p, L\sigma^2\right)$ and, hence, upon substituting the expression for the Gaussian pdf into (13), we arrive at

$$\begin{aligned} p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k) = \frac{1}{2\pi L\sigma^2} & e^{-\left(y_k^s - x_k^s \sum_{l=1}^{L} \alpha_{k,l}^s\right)^2 / (2L\sigma^2)} \\ & \times e^{-\left(y_k^p - x_k^p \sum_{l=1}^{L} \alpha_{k,l}^p\right)^2 / (2L\sigma^2)}. \end{aligned} \tag{14}$$

Discarding all the constant terms and terms which do not depend on the code symbols $\{\mathbf{x}_k\}$, and taking logarithm of both sides of (14), we obtain

$$\begin{aligned} \log(p(\mathbf{y}_k | \mathbf{x}_k, \underline{\alpha}_k)) \\ = \frac{2E_s}{LN_0} \left( \sum_{l=1}^{L} y_k^s x_k^s \alpha_{k,l}^s + \sum_{l=1}^{L} y_k^p x_k^p \alpha_{k,l}^p \right). \end{aligned} \tag{15}$$

Defining the quantity $\hat{L}_k = \log(\text{Prob}(x_k^s = +1)/\text{Prob}(x_k^s = -1))$, and discarding all the terms independent of $x_k^s$, we can calculate $\log\left(\text{Prob}\left(x_k^s\right)\right)$ as [9]

$$\log\left(\text{Prob}\left(x_k^s\right)\right) = \frac{\hat{L}_k x_k^s}{2}. \tag{16}$$

Combining the results of (15) and (16) and substituting in (12), we obtain

$$\begin{aligned} c_k(s,t,\underline{\alpha}_k) = & \frac{\hat{L}_k x_k^s}{2} \\ & + \frac{2E_s}{LN_0} \left( \sum_{l=1}^{L} y_k^s x_k^s \alpha_{k,l}^s + \sum_{l=1}^{L} y_k^p x_k^p \alpha_{k,l}^p \right). \end{aligned} \tag{17}$$

The above quantity $c_k(s,t,\underline{\alpha}_k)$ can be used in the computation of the forward and backward recursion metrics in the log-MAP algorithm [10]. It is noted that the computation of the quantity $c_k(s,t,\underline{\alpha}_k)$ requires knowledge of the $E_s/N_0$ and the fade amplitudes. To obtain a metric in the absence of the fade amplitude knowledge, one needs to average $c_k(\cdot,\cdot,\underline{\alpha}_k)$ over the distribution of $\underline{\alpha}_k$, which is difficult even for the Rayleigh fading case [2]. Hence, in [2], an approximate, fade-independent metric was given for Rayleigh fading. Here, we use the following suboptimum metric in the absence of the knowledge of fade amplitudes $\underline{\alpha}$:

$$c_k^{\text{subopt}}(s,t) = \frac{\hat{L}_k x_k^s}{2} + 2\widehat{\gamma}\left(y_k^s x_k^s + y_k^p x_k^p\right). \tag{18}$$

The above metric is essentially the AWGN channel metric, which is equivalent to setting the fade amplitudes to unity.

## IV. TURBO DECODER PERFORMANCE RESULTS

We estimated $E_s/N_0$ using the SNR estimator derived in Section II and used this estimate in the decoding of turbo codes. We carried out simulations to evaluate the performance of a rate-1/3 turbo code with generator $(21/37)_8$ and random turbo interleaver for various values of $L$ $(= 1,2,3)$ and $m$ $(= 0.5,1,5)$. The number of information bits per block is 5000 b and all the 15 000 received symbols in the block are used to compute the SNR estimate. The trellis of the constituent encoder is terminated by appending five tail bits to the information bits, which makes the code rate to be fractionally less than 1/3. Different random interleavers are generated for different data blocks in the simulations. The number of iterations in the turbo decoding is eight. We evaluate the turbo decoder performance using our SNR estimate, and compare it with the performance using perfect knowledge of the $E_s/N_0$ and the fade amplitudes. In the ideal case, where perfect knowledge of the SNR as well as the symbol-by-symbol fade amplitudes are assumed, the

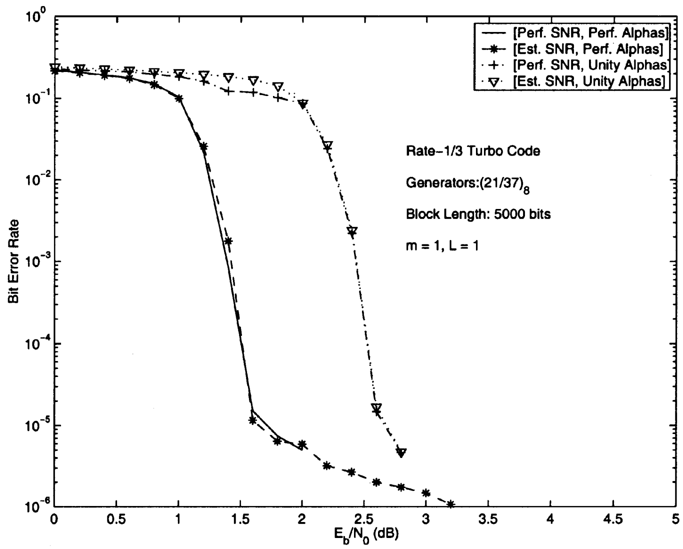


Fig. 2. Comparison of turbo decoder performance using the estimated SNR versus knowledge of channel information, for $L = 1$ and $m = 1$.

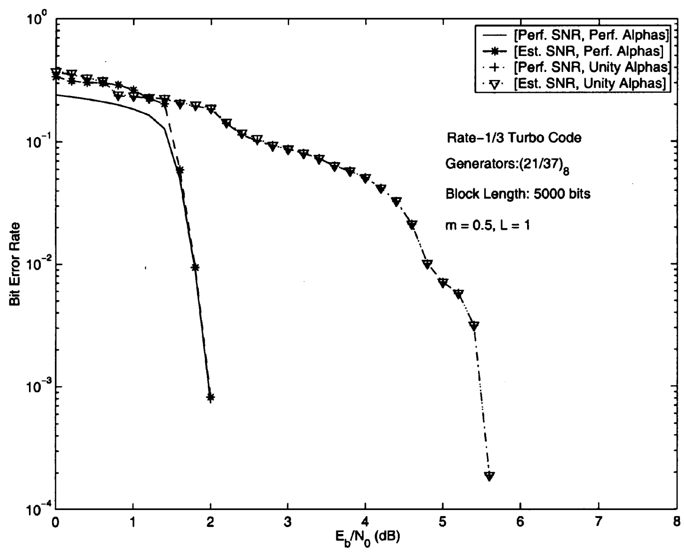


Fig. 3. Comparison of turbo decoder performance using the estimated SNR versus knowledge of channel information, for $L = 1$ and $m = 0.5$.

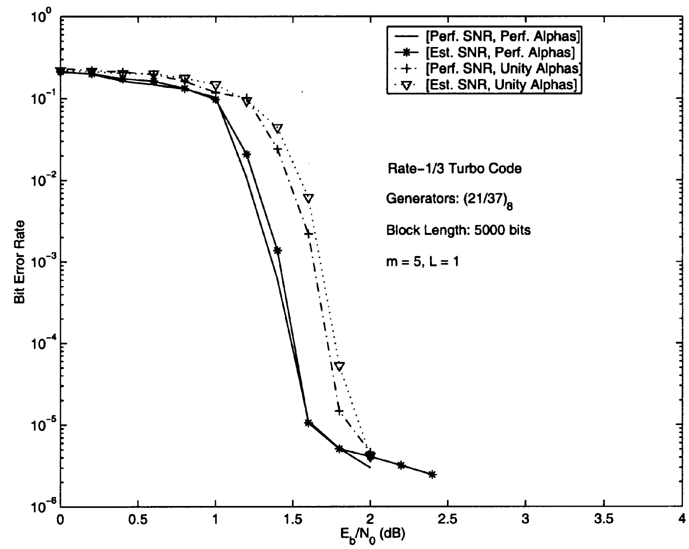


Fig. 4. Comparison of turbo decoder performance using the estimated SNR versus knowledge of channel information, for $L = 1$ and $m = 5$.

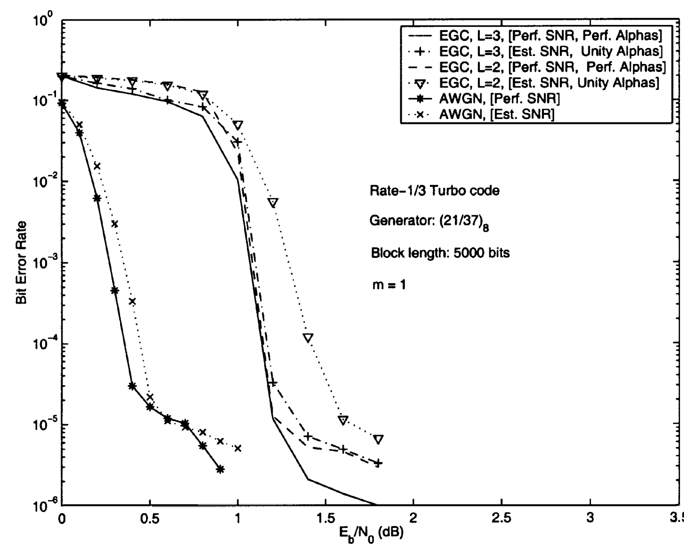


Fig. 5. Comparison of turbo decoder performance using the estimated SNR versus knowledge of the channel information, for $m = 1$ and $L = 2, 3$.

metric in (17) is used. In the nonideal case, however, since we are estimating only the SNR, the suboptimum metric in (18) is used.

Figs. 2–4 present the comparison of the turbo decoder performance using the estimated SNR versus perfect knowledge of channel information, for $m = 1, 0.5$, and $5$, respectively, and $L = 1$. The various combinations of the knowledge of the SNR and fade amplitudes considered are a) Perfect SNR, Perfect $\alpha$'s; b) Estimated SNR, Perfect $\alpha$'s; c) Perfect SNR, Unity $\alpha$'s; and d) Estimated SNR, Unity $\alpha$'s. The assumption of perfect fade amplitude knowledge here must be viewed from a comparison point of view only, i.e., it gives the best possible performance with EGC, although maximal ratio combining would give better performance. The difference in performance between case a) and case b) gives the degradation due to inaccuracy in the SNR estimate alone. The performance difference between cases b) and d) gives the effect due to nonavailability of fade amplitude information. From Figs. 2–4, it can be observed that, given the knowledge of the fade amplitudes at the receiver, the performance of the decoder using our SNR estimate is almost the same as the performance with perfect knowledge of the SNR. In other words, the proposed estimator provides adequately accurate SNR estimates for the purpose of turbo decoding in Nakagami-$m$ fading channels. It is also observed that the lack of knowledge of the fade amplitudes at the receiver results in noticeable degradation in performance, particularly when the fading is severe. For example, when $m = 5$ (light fading), the degradation due to lack of knowledge of fade amplitudes is about 0.4 dB for an error rate of $10^{-5}$, whereas the degradation increases to about 1.5 dB for $m = 1$ (Rayleigh fading) and 3.5 dB for $m = 0.5$ (severe fading). Fig. 5 illustrates the performance comparison for EGC diversity when $L = 2$ and $3$. Performance in AWGN is also shown. As $L$ is increased, the degradation due to lack of knowledge of fade amplitudes is reduced. For example, when $L = 2$, the degradation is 0.3 dB, whereas for $L = 3$, the degradation is just less than 0.1 dB. The turbo decoder performance is less sensitive to the lack of knowledge of the fade amplitudes in EGC diversity schemes compared with fading without diversity.

## APPENDIX

Here, we derive the expressions for the numerator and the denominator of (5). Removing the superscript for convenience, the denominator $E(v^4)$ is given by

$$E(v^4) = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(r_i r_j r_k r_l). \tag{19}$$

Substituting $r_i = \alpha_i X + n_i$, $r_j = \alpha_j X + n_j$, $r_k = \alpha_k X + n_k$, and $r_l = \alpha_l X + n_l$, we obtain

$$\begin{aligned} E(v^4) =& E_s^2 \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l) \\ &+ E_s \sum_{i=1}^{L}\sum_{j=1}^{L} E(\alpha_i\alpha_j) \sum_{k=1}^{L}\sum_{l=1}^{L} E(n_k n_l) \\ &+ E_s \sum_{i=1}^{L}\sum_{k=1}^{L} E(\alpha_i\alpha_k) \sum_{j=1}^{L}\sum_{l=1}^{L} E(n_j n_l) \\ &+ E_s \sum_{i=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_l) \sum_{j=1}^{L}\sum_{k=1}^{L} E(n_j n_k) \\ &+ E_s \sum_{j=1}^{L}\sum_{k=1}^{L} E(\alpha_j\alpha_k) \sum_{i=1}^{L}\sum_{l=1}^{L} E(n_i n_l) \\ &+ E_s \sum_{j=1}^{L}\sum_{l=1}^{L} E(\alpha_j\alpha_l) \sum_{i=1}^{L}\sum_{k=1}^{L} E(n_i n_k) \\ &+ E_s \sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_k\alpha_l) \sum_{i=1}^{L}\sum_{j=1}^{L} E(n_i n_j) \\ &+ \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l). \end{aligned} \tag{20}$$

Since $\alpha$ is Nakagami-$m$ distributed with $E(\alpha^2) = 1$, we have $E(\alpha) = (\Gamma(m+1/2)/\sqrt{m}\Gamma(m))$. Assuming the $\alpha$'s are independent and identically distributed (i.i.d.), the $n$'s are i.i.d, and assuming these groups are independent, and also independent of $X$, we obtain

$$\sum_{p=1}^{L}\sum_{q=1}^{L} E(\alpha_p\alpha_q) = L + (L^2 - L)\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{m}\Gamma(m)}\right)^2 \tag{21}$$

and

$$\sum_{p=1}^{L}\sum_{q=1}^{L} E(n_p n_q) = L\sigma^2. \tag{22}$$

Defining $\Delta_m = \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l)$, the expression for $\Delta_m$ can be obtained as

$$\begin{aligned} \Delta_m =& \sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(\alpha_i\alpha_j\alpha_k\alpha_l) \\ =& E(\alpha_1 + \alpha_2 + \cdots + \alpha_L)^4. \end{aligned} \tag{23}$$

Applying the multinomial theorem to the above equation, we get

$$\begin{aligned} \Delta_m =& \binom{L}{1} E(\alpha^4) + \binom{L}{2}\frac{4!}{1!3!}E(\alpha^3)E(\alpha) \\ &+ \binom{L}{2}\frac{4!}{2!2!}[E(\alpha^2)]^2 \\ &+ \binom{L}{3}\frac{4!}{1!1!2!}3E(\alpha^2)[E(\alpha)]^2 \\ &+ \binom{L}{4}\frac{4!}{1!1!1!1!}[E(\alpha)]^4. \end{aligned} \tag{24}$$

Simplifying the above equation, we arrive at

$$\begin{aligned} \Delta_m =& LE(\alpha^4) + 4L(L-1)E(\alpha^3)E(\alpha) \\ &+ 3L(L-1)[E(\alpha^2)]^2 \\ &+ 6L(L-1)(L-2)E(\alpha^2)[E(\alpha)]^2 \\ &+ L(L-1)(L-2)(L-3)[E(\alpha)]^4 \end{aligned} \tag{25}$$

where $E(\alpha^k) = (\Gamma(m+k/2)/m^{k/2}\Gamma(m))$. In the case of Rayleigh fading (i.e., $m = 1$), $E(\alpha) = \sqrt{\pi}/2$, $E(\alpha^2) = 1$, $E(\alpha^3) = 3\sqrt{\pi}/4$, and $E(\alpha^4) = 2$. Substituting these values of expectations in (25), we get $\Delta_1$ as

$$\begin{aligned} \Delta_1 =& 2L + L(L-1) \\ &\times\left\{3 + \frac{3(L-1)\pi}{2} + (L-2)(L-3)\frac{\pi^2}{16}\right\}. \end{aligned} \tag{26}$$

Next, to compute $\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l)$, we use the result of (25) with $\alpha$ replaced by $n$. Also, by recalling that the odd moments of Gaussian random variable $n$ with zero mean and variance $\sigma^2$ are all zero, $E(n^2) = \sigma^2$, and $E(n^4) = 3\sigma^4$, we arrive at

$$\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{k=1}^{L}\sum_{l=1}^{L} E(n_i n_j n_k n_l) = 3L^2\sigma^4. \tag{27}$$

Combining (21), (22), (25), and (27), we get $E(v^4)$ as

$$\begin{aligned} E(v^4) =& E_s^2\Delta_m + 3L^2\sigma^4 \\ &+ 6L\left[L + (L^2 - L)\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{m}\Gamma(m)}\right)^2\right]E_s\sigma^2 \\ =& \sigma^4\left\{3L^2 + \frac{E_s^2}{\sigma^4}\Delta_m + 6L\right. \\ &\times\left.\left[L + (L^2 - L)\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{m}\Gamma(m)}\right)^2\right]\frac{E_s}{\sigma^2}\right\}. \end{aligned} \tag{28}$$

Similarly, $E(v^2)$ can be calculated as

$$\begin{aligned} E(v^2) &= \sum_{i=1}^{L}\sum_{j=1}^{L} E(r_i r_j) \\ &= \sum_{i=1}^{L}\sum_{j=1}^{L} E\left(\alpha_i\alpha_j X^2 + \alpha_i X n_j + \alpha_j X n_i + n_i n_j\right) \\ &= \left[L + (L^2 - L)\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{m}\Gamma(m)}\right)^2\right] E_s + L\sigma^2 \\ &= \sigma^2\left\{L + \left[L + (L^2 - L)\left(\frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{m}\Gamma(m)}\right)^2\right]\frac{E_s}{\sigma^2}\right\}. \end{aligned} \tag{29}$$

Squaring (29) and dividing it by (28), and defining $\gamma = E_s/2\sigma^2$, we get (5). By substituting $m = 1$ in (5), we get (7).

## References


[1] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo codes," in *Proc. IEEE ICC*, 1993, pp. 1064–1070.

[2] E. K. Hall and S. G. Wilson, "Design and analysis of turbo codes on Rayleigh fading channels," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 160–174, Feb. 1998.

[3] T. A. Summers and S. G. Wilson, "SNR mismatch and online estimation in turbo decoding," *IEEE Trans. Commun.*, vol. 46, pp. 421–423, Apr. 1998.

[4] M. C. Valenti and B. D. Woerner, "Iterative channel estimation and decoding of pilot symbol assisted turbo codes over flat-fading channels," *IEEE J. Select. Areas Commun.*, vol. 19, pp. 1697–1705, Sept. 2001.

[5] ——, "Performance of turbo codes in interleaved flat fading channels with estimated channel state information," in *Proc. IEEE VTC*, 1998, pp. 66–70.

[6] A. Ramesh, A. Chockalingam, and L. B. Milstein, "SNR estimation in generalized fading channels and its application to turbo decoding," presented at the *IEEE ICC*, Helsinki, Finland, June 2001.

[7] M. Nakagami, "The $m$ distribution—A general formula of intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*. Oxford, U.K.: Pergamon, 1960, pp. 3–36.

[8] J. G. Proakis, *Digital Communications*. New York: McGraw-Hill, 1995.

[9] D. N. Rowitch, "Convolutional and turbo coded multicarrier direct sequence CDMA, and applications of turbo codes to hybrid ARQ communication systems," Ph.D. dissertation, Univ. Calif. San Diego, La Jolla, 1998.

[10] P. Robertson, P. Hoeher, and E. Villebrun, "Optimal and suboptimal maximum *a posteriori* algorithms suitable for turbo decoding," *Eur. Trans. Telecommun.*, vol. 8, pp. 119–125, Mar./Apr. 1997.

[11] A. Abdi and M. Kaveh, "Performance comparison of three different estimators for the Nakagami-$m$ parameter using Monte Carlo simulations," *IEEE Commun. Lett.*, vol. 14, pp. 119–121, Apr. 2000.