Consta-Abelian codes over Galois rings in the DFT domain

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Abstract — Using Twisted-DFT, we characterize Consta-Abelian codes over Galois rings that are closed under two kinds of monomials.

I. INTRODUCTION AND PRELIMINARIES

For a prime p , Galois ring $R_{p^a,l} = Z_{p^a}[x]/\phi(x)$, where $\phi(x)$ is a monic irreducible polynomial of degree l over $Z_p[x]$ [1]. If $G = C_{r-1} \otimes \dots \otimes C_0$, is an Abelian group of order $n = \prod_{\lambda=0}^{r-1} m_\lambda$, the twisted group ring $R_{p^a,l}^\psi G$ [2] is defined by imposing in the $R_{p^a,l}$ -module $R_{p^a,l}G$ the following multiplication on each of the generators $g_{(m_k)}$ of the cyclic subgroup C_k : $g_{(m_k)}^i \cdot g_{(m_k)}^j = \psi_k(g_{(m_k)}^i, g_{(m_k)}^j) g_{(m_k)}^{i+j}$, where $g_{(m_k)}^{i+j} \in G$ and

$$\psi_k(g_{(m_k)}^i, g_{(m_k)}^j) = \begin{cases} 1 & \text{if } i+j < m_k \\ \beta_k & \text{if } i+j \geq m_k \end{cases} \quad (1)$$

where β_k , an element of order r_k belongs to the cyclic subgroup of size $p^l - 1$ in $R_{p^a,l}^*$. For n permutations $\pi_0, \dots, \pi_{n-1}$ of $R_{p^a,l}$ and a permutation τ of $I_n = \{0, 1, \dots, n-1\}$ a code C over $R_{p^a,l}$ is said to be $\pi\tau$ -invariant if $\pi\tau(\vec{c}) = (\pi_0(c_{\tau(0)}), \pi_1(c_{\tau(1)}), \dots, \pi_{n-1}(c_{\tau(n-1)})) \in C$ for all $\vec{c} = (c_0, c_1, \dots, c_{n-1}) \in C$. Consta-Abelian codes are ideals in the twisted group ring $R_{p^a,l}^\psi G$. For $\vec{\beta} = (\beta_{r-1}, \dots, \beta_0)$ these are also called $\vec{\beta}$ -constacyclic codes. For every codeword $\vec{c} = (c_0, \dots, c_{n-1})$ in the $\vec{\beta}$ -constacyclic code, the monomial-permuted vector $\pi\tau(\vec{c}) = (\pi_0(c_{0 \oplus j}), \dots, \pi_{n-1}(c_{(n-1) \oplus j}))$ also belongs to the code for all values of j , where $\pi_i(c_{i \oplus j}) = (\prod_{\lambda=0}^{r-1} \beta_\lambda^{k_\lambda}) c_{i \oplus j}$ and k_λ is such that $i_\lambda + j_\lambda = m_\lambda k_\lambda + (i_\lambda \oplus j_\lambda)$ where $i \oplus j$ denotes the mixed-radix addition with m_λ as radixes [3]. Let ζ_λ be a primitive $m_\lambda r_\lambda$ -th root of unity in the cyclic group of order $p^{lm} - 1$ in the extension ring $R_{p^a,lm}$ and $\gamma_\lambda = \zeta_\lambda^{h_\lambda}$, $\alpha_\lambda = \zeta_\lambda^{r_\lambda}$ be m_λ -th root of β_λ and m_λ -th root of unity respectively for all $\lambda = 0, 1, \dots, r-1$. The TDFT vector $\vec{A} = (A_0, \dots, A_{n-1}) \in R_{p^a,lm}^n$ of $\vec{a} = (a_0, \dots, a_{n-1}) \in R_{p^a,l}^n$ is defined as $A_j = \sum_{i=0}^{n-1} (\prod_{\lambda=0}^{r-1} \gamma_\lambda^{i_\lambda} \alpha_\lambda^{i_\lambda j_\lambda}) a_i$ where $i = [i_{r-1}, \dots, i_0]$ and $j = [j_{r-1}, \dots, j_0]$ are mixed-radix representations of i, j . The TDFT satisfies all the properties mentioned in [4] for a constacyclic DFT. Let σ_0 be the Frobenius automorphism of $R_{p^a,lm}$ then $\sigma = \sigma_0^l$ is an automorphism of $R_{p^a,lm}$ that fixes $R_{p^a,l} \subseteq R_{p^a,lm}$. Moreover, for $j \in I_n$, $\sigma^k(A_j) = A_i$, where $i = [i_{r-1}, i_{r-2}, \dots, i_0]$ with $i_\lambda = \left(\frac{(q^k-1)h_\lambda}{r_\lambda} + q^k j_\lambda \right)$ modulo m_λ for all $\lambda = 0, 1, \dots, r-1$ (Conjugate symmetry property). Let ϕ_λ^k denote a mapping from I_{m_λ} to I_{m_λ} which maps j_λ to $\left(\frac{(q^k-1)h_\lambda}{r_\lambda} + q^k j_\lambda \right)$. Let Φ^k denote the mapping $i = [i_{r-1}, i_{r-2}, \dots, i_0]$ to $\Phi^k(i) = [\phi_{r-1}^k(i_{r-1}), \phi_{r-2}^k(i_{r-2}), \dots, \phi_0^k(i_0)]$. For every $j \in I_n$, the set $\widehat{j} = \{\Phi^1(j), \Phi^2(j), \dots, \Phi^{e_j-1}(j)\}$ where e_j is the smallest integer such that $\Phi^{e_j}(j) = j$, is called the *cyclotomic coset*

modulo n containing $[j]$. A linear code C over $R_{p^a,l}$ is consta-Abelian iff it satisfies the conjugate-symmetry property and the set $C_j = \{A_j \mid \vec{a} \in C\} = p^{\eta_j} R_{p^a,le_j}$, an ideal of the subring R_{p^a,le_j} where $0 \leq \eta_j \leq a$.

In this paper, we study $\vec{\beta}$ -constacyclic codes which are also invariant under the monomial-permutations $\pi^{(b)} U_b$ and $\pi^{<s>} Q_{n_s}$ where: (i) U_b is defined by, $[i] = [i_{r-1}, \dots, i_0] \rightarrow [b_{r-1}i_{r-1}, \dots, b_0i_0]$ for $b = [b_{r-1}, \dots, b_0] \in I_n$ such that $\gcd(b_\lambda, m_\lambda) = 1$ and $r_\lambda \mid (b_\lambda^{-1} - 1)$ for all λ . The associated permutation $\pi^{(b)} = (\pi_0^{(b)}, \dots, \pi_{r-1}^{(b)})$ is given by $\pi_i^{(b)}(a_i) = \prod_{k=0}^{r-1} \psi_k \left(g_{m_k}^{b_k^{-1} i_k}, a_i \right)$ and (ii) for $n_s = \prod_{\lambda=0}^{s-1} m_\lambda$, Q_{n_s} takes $i \rightarrow (i + n_s)$ modulo n and the associated permutation $\pi^{<s>} = (\pi_0^{<s>}, \dots, \pi_{r-1}^{<s>})$ is given by $\pi_i^{<s>}(a_i) = a_i$ for $0 \leq i < n_{s+1} - n_s$ and for $n_{s+1} - n_s \leq i < n$, $\pi_i^{<s>}(a_i) = \prod_{k=s}^{r-1} \psi_k \left(g_{m_k}^{i_k + \delta_k^{(s)}}, a_i \right)$, where $\delta_k^{(s)}$ denotes the “carry value” in the k -th radix-component due to addition $[i + n_s]$.

II. TDFT DOMAIN CHARACTERIZATION

Definition 1 (i) For any $j \in I_n$ such that $[j] = [0, 0, \dots, 0, j_\mu, \dots, j_0]$ and $\mu \geq h > s \geq 0$, let the set $J^{(h,s)}(j)$ be $\{[0, \dots, 0, j_\mu, \dots, j_h, x_{h-1}, x_{h-2}, \dots, x_s, j_{s-1}, \dots, j_0]\}$ for all $x_\lambda \in I_{m_\lambda}$; $\lambda = h-1, h-2, \dots, s$. (ii) For every $\lambda = 0, 1, \dots, r-1$, let $\phi_\lambda^{k,b}$ denote the mapping from I_{m_λ} to I_{m_λ} that maps j_λ to $\left(\frac{(b_\lambda^{-1}-1)h_\lambda}{r_\lambda} + b_\lambda^{-k} j_\lambda \right)$ and let $\Phi^{k,b}$ be the map which maps $[j]$ to $\Phi^{k,b}([j]) = [\phi_{r-1}^{k,b_{r-1}}(j_{r-1}), \phi_{r-2}^{k,b_{r-2}}(j_{r-2}), \dots, \phi_0^{k,b_0}(j_0)]$.

Theorem 1 A length $n = m_{r-1}m_{r-2} \dots m_0$ $\vec{\beta}$ -constacyclic code with $C_j = p^{\eta_j} R_{p^a,le_j}$ for any $j \in I_n$ is (i) $\pi^{(b)} U_b$ -invariant iff $C_i = p^{\eta_i} R_{p^a,le_i}$ when some element of $\widehat{i}$ is of the form $\Phi^{k,b}([j])$ for some k . (ii) $\pi^{<s>} Q_{n_s}$ -invariant iff $C_i = p^{\eta_i} R_{p^a,le_i}$ when $j \geq n_{s+1}$ and some element of $\widehat{i}$ belongs to $J^{(r-1,s)}(j)$.

For the special case of $\beta_\lambda = 1$ for all λ , Theorem 1 characterizes (i) Quasicyclic-Abelian codes and (ii) U_b -invariant Abelian codes. Given the TDFT characterization of a $\vec{\beta}$ -constacyclic code, we obtain the characterization of its dual code (w.r.t normal inner product).

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