

Wavelet Denoising for Plane Wave DOA Estimation By MUSIC

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Abstract—MUSIC is a widely used technique of plane wave direction-of-arrival (DOA) estimation, which is a problem of great interest in several applications. The performance of MUSIC degrades under low Signal-to-Noise Ratio (SNR) conditions due to errors in estimating the data covariance matrix from finite data. This paper explores the possibility of employing the wavelet denoising technique to arrest the degradation in the finite-data performance of MUSIC under low SNR. We propose the application of wavelet denoising to the noisy signal at each sensor to boost the SNR before performing DOA estimation by MUSIC. A comparative study of the finite data performance of MUSIC is presented for the undenoised and denoised data, and it is shown that denoising leads to a significant reduction in the bias and Mean Square Errors (MSE) of the DOA estimates.

1. INTRODUCTION

Multiple signal classification (MUSIC) is a popular high-resolution technique for estimating the directions-of-arrival (DOA) of multiple plane waves in a noisy environment, using an array of M sensors [1]. The method involves eigen decomposition of the spectral covariance matrix \mathbf{R} of the M -dimensional data vector to determine the noise subspace. The matrix \mathbf{R} is estimated from a finite number of samples of the data vector. For a given data size N , reduction of the signal-to-noise ratio (SNR) at the sensor array output causes an increase in the covariance matrix estimation error and a corresponding increase in the DOA estimation error [2]. The estimation errors may be reduced by increasing N , but requirements of temporal coherence and speed impose an upper limit on the permissible value of N . Inevitably, the performance of the MUSIC estimator suffers a progressive degradation as the SNR is reduced. In this work, we explore the possibility of using a wavelet denoising technique to improve the performance of MUSIC in a low SNR environment. The wavelet denoising algorithm of Donoho and Johnstone [3] is used to enhance the SNR at the output of each sensor. MUSIC is employed on the denoised data vector for DOA estimation. The effect of denoising on the performance of MUSIC is analysed by numerically evaluating and comparing: (1) the MSE in the spectral covariance estimates obtained using finite data, and (2) the bias and MSE of the DOA estimates, for both undenoised and denoised data. It is shown that denoising leads to a significant improvement in the performance of the MUSIC estimator.

2. DATA MODEL AND MUSIC ALGORITHM

Consider a uniform linear array of M sensors with intersensor spacing d . If plane waves from different narrowband sources arrive at the array at angles $\theta_1, \theta_2, \dots, \theta_n$, with respect to the array normal, the output of the m^{th} sensor can be written as

$$y_m(t) = \sum_{p=1}^n b_p(t) e^{j\omega t + j(m-1)k d \sin(\theta_p)} + n_m(t), \quad (1)$$

where $b_p(t)$ is the slowly varying zero-mean complex random amplitude of the signal from the p^{th} source, ω is the center frequency of the signal, k is the wavenumber and $n_m(t)$ is zero-mean additive white complex Gaussian noise of variance σ^2 . If we write $\mathbf{y} = [y_1(t) \ y_2(t) \ \dots \ y_M(t)]^T$, then

$$\mathbf{y} = \mathbf{A}\mathbf{s} + \mathbf{n}, \quad (2)$$

where $\mathbf{n} = [n_1(t) \ n_2(t) \ \dots \ n_M(t)]^T$ is the noise vector that is also spatially white, and

$$\mathbf{s} = [b_1(t)e^{j\omega t} \ b_2(t)e^{j\omega t} \ \dots \ b_n(t)e^{j\omega t}]^T \quad (3)$$

is the vector of source signals at time t .

The *steering-vector* matrix \mathbf{A} is given by,

$$\mathbf{A} = [\mathbf{a}(w_1) \ \mathbf{a}(w_2) \ \dots \ \mathbf{a}(w_n)], \quad (4)$$

where $\mathbf{a}(w_l) = [1 \ e^{jw_l} \ e^{j2w_l} \ \dots \ e^{j(M-1)w_l}]^T$ and $w_l = kd \sin(\theta_l)$. It is assumed that the sources are mutually uncorrelated and that the auto-correlation of each source decays exponentially, i.e.,

$$E[b_p(t)b_i^H(u)] = \sigma_p^2 \delta_{p,i} e^{-\alpha|t-u|}. \quad (5)$$

It is also assumed that the signal and noise are uncorrelated. The covariance matrix of the data vector would then be

$$\begin{aligned} \mathbf{R} &= E[\mathbf{y}\mathbf{y}^H] \\ &= \mathbf{A} E[\mathbf{s}\mathbf{s}^H] \mathbf{A}^H + E[\mathbf{n}\mathbf{n}^H] \\ &= \mathbf{A}\mathbf{P}\mathbf{A}^H + \sigma^2\mathbf{I}, \end{aligned} \quad (6)$$

where $\mathbf{P} = E[\mathbf{s}\mathbf{s}^H] = \text{diag}[\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2]$, and $E[\mathbf{n}\mathbf{n}^H] = \sigma^2\mathbf{I}$.

Let \mathbf{R} be eigen decomposed as

$$\mathbf{R} = [\mathbf{U} \ \mathbf{V}] \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}^H \\ \mathbf{V}^H \end{bmatrix}, \quad (7)$$

where \mathbf{U} and \mathbf{V} are the signal subspace and noise subspace eigen vector matrices, respectively. It can be shown that $\mathbf{A}^H \mathbf{V} = \mathbf{0}$ [1]. Or equivalently,

$$\mathbf{a}^H(w) \mathbf{V} \mathbf{V}^H \mathbf{a}(w) = 0 \quad (8)$$

at the true DOAs.

An estimate of \mathbf{R} is obtained from N samples of the data vectors as,

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=1}^N \mathbf{y}(t) \mathbf{y}^H(t). \quad (9)$$

If $\hat{\mathbf{R}}$ is eigen decomposed as in Eqn (7), one would arrive at the estimate of the noise subspace eigen vector matrix as $\hat{\mathbf{V}}$. Since $\hat{\mathbf{V}}$ is only an estimate, the left-hand side of Eqn (8) would be minimum, and not zero, at the true DOAs if \mathbf{V} is replaced by $\hat{\mathbf{V}}$. Spectral MUSIC utilises this fact, so that the ambiguity function,

$$B(w) = \left(\frac{1}{\mathbf{a}^H(w) \hat{\mathbf{V}} \hat{\mathbf{V}}^H \mathbf{a}(w)} \right) \quad (10)$$

peaks at the true DOA, whereas Root MUSIC simply roots the polynomial $\mathbf{a}^H(w) \hat{\mathbf{V}} \hat{\mathbf{V}}^H \mathbf{a}(w)$ to find the DOA [1].

3. WAVELET DENOISING

Wavelet denoising attempts to recover the discrete-time signal $f(i)$, $i = 1, 2, \dots, N$ from the noise-corrupted observations

$$x(i) = f(i) + z(i), \quad i = 1, 2, \dots, N \quad (11)$$

where $z(i)$ is zero-mean white Gaussian noise of variance σ^2 . If \mathbf{x} , \mathbf{f} and \mathbf{z} represent $N \times 1$ column vectors containing the samples $x(i)$, $f(i)$ and $z(i)$ and \mathbf{W} represents a $N \times N$ Discrete Wavelet Transform (DWT) matrix, then in the wavelet domain Eqn (11) becomes

$$\mathbf{x}_w = \mathbf{f}_w + \mathbf{z}_w, \quad (12)$$

where $\mathbf{x}_w = \mathbf{W}\mathbf{x}$, $\mathbf{f}_w = \mathbf{W}\mathbf{f}$ and $\mathbf{z}_w = \mathbf{W}\mathbf{z}$. The DWT is an orthonormal transform that compacts the signal into a few large coefficients in \mathbf{f}_w , while \mathbf{z} is mapped on to \mathbf{z}_w which likewise is zero-mean white Gaussian noise with variance σ^2 . The process of wavelet denoising is to threshold the coefficients \mathbf{x}_w to discard small values most likely due to the additive noise [3], [4]. The thresholding process can be thought of as a filtering operation with the filter being,

$$\mathbf{H} = \text{diag}[h(1) \ h(2) \ \dots \ h(N)]. \quad (13)$$

The threshold-filter coefficients are decided based on the type of threshold under consideration: *hard* or *soft* thresholding. If $\mathbf{x}_w = [x_w(1) \ x_w(2) \ \dots \ x_w(N)]^T$ are the wavelet coefficients of the noisy signal, the threshold-filter coefficients are given by

Hard Thresholding

$$h(i) = \begin{cases} 1 & \text{if } |x_w(i)| > \tau \\ 0 & \text{otherwise} \end{cases}$$

Soft Thresholding

$$h(i) = \begin{cases} \left(1 - \frac{\tau}{|x_w(i)|}\right) & \text{if } |x_w(i)| > \tau \\ 0 & \text{otherwise} \end{cases}$$

The signal estimate is obtained by inverting the thresholded coefficients, i.e.,

$$\hat{\mathbf{f}} = \mathbf{W}^{-1} \mathbf{H} \mathbf{W} \mathbf{x}. \quad (14)$$

The threshold developed by Donoho [3][4] is

$$\tau = \sigma \sqrt{2 \log(N)}, \quad (15)$$

where N is the total number of samples. For very large N the standard deviation of the noise can be estimated from the finest scale wavelet coefficients.

Comparison with Wiener Filtering

Optimum linear estimation of signal in the sense of minimizing MSE is achieved by the Wiener filter [5], given the knowledge of the statistics of the signal and noise. For the signal model in Eqn (11), if the covariance matrices of the vectors \mathbf{x} , \mathbf{f} and \mathbf{z} are \mathbf{R}_x , \mathbf{R}_f and $\sigma^2 \mathbf{I}$ respectively, then the optimum linear estimator of \mathbf{f} is

$$\hat{\mathbf{f}}_{\text{opt}} = [\mathbf{R}_f (\mathbf{R}_f + \sigma^2 \mathbf{I})^{-1}] \mathbf{x}. \quad (16)$$

If $\mathbf{R}_f = \sum_{k=1}^{N_f} \lambda_k \mathbf{u}_k \mathbf{u}_k^H = \mathbf{U}_f \mathbf{\Lambda} \mathbf{U}_f^H$ is the eigen decomposition of \mathbf{R}_f , then

$$\hat{\mathbf{f}}_{\text{opt}} = [\mathbf{U}_f \tilde{\mathbf{\Lambda}} \mathbf{U}_f^H] \mathbf{x}, \quad (17)$$

$$\tilde{\mathbf{\Lambda}} = \text{diag} \left\{ \frac{\lambda_1}{\lambda_1 + \sigma^2}, \dots, \frac{\lambda_{N_f}}{\lambda_{N_f} + \sigma^2} \right\}. \quad (18)$$

In Eqn (17), $\mathbf{U}_f^H \mathbf{x}$ denotes the KL transform of \mathbf{x} , so that the operator \mathbf{U}_f^H decorrelates the signal embedded in the data vector \mathbf{x} . The operator $\tilde{\mathbf{\Lambda}}$ in Eqn (17) performs optimal filtering of the decorrelated data vector, and the operator \mathbf{U}_f performs recorelation to yield the optimal estimate $\hat{\mathbf{f}}_{\text{opt}}$ of the signal \mathbf{f} . But in the absence of the knowledge of the signal statistics the optimum linear estimator cannot be used. A key property of DWT is that it approximates the KL transform for a large class of signals [3], and consequently, it tends to concentrate the signal energy into a relatively small number of large coefficients. Hence, by replacing \mathbf{U}_f^H by \mathbf{W} and the optimal weighting in $\tilde{\mathbf{\Lambda}}$ by hard thresholding \mathbf{H} , one can arrive at an estimate $\hat{\mathbf{f}}$ of \mathbf{f} as in Eqn (14).

The "closeness" of $\hat{\mathbf{f}}$ to the optimum estimate $\hat{\mathbf{f}}_{\text{opt}}$ depends on how small a number of coefficients the wavelet transform concentrates the signal energy into, the limit being the KLT of \mathbf{f} . The advantage of wavelet denoising over Wiener filtering is that, it is totally independent of the signal statistics and hence can be applied to signals of any kind. Secondly, wavelet transforms can be efficiently implemented using quadrature mirror filter bank structures.

Filter Bank Implementation

The wavelet coefficients of a signal measured at a finite resolution can be computed by a fast filter bank algorithm. Given the samples of a signal $f[i]$, $i = 1, 2, \dots, N$, the approximation and detail coefficients of the wavelet transform of \mathbf{f} at a level m are given [6], respectively, by

$$c_{m+1}[i] = \sum_n g[2i - n]c_m[n], \quad (19)$$

$$d_{m+1}[i] = \sum_n h[2i - n]c_m[n], \quad (20)$$

starting with $c_0[i] = f[i]$. The filters $g[n]$ and $h[n]$ are low pass and high pass filters, respectively. At level m , the coefficients c_m contain the low pass information of the signal and hence form an approximation of the original signal at that level. The detail coefficients d_m , however, are the result of the action of a high pass filter and at level m they represent the high frequency components of the signal c_{m-1} at level $m - 1$.

The regularity of the wavelets used forms an important factor in the process of wavelet denoising. The scaling filter $g[n]$ associated with a wavelet transform is said to be $K - regular$ if its Z-transform has K zeros at $z = e^{j\pi}$. Consequently, any polynomial of degree upto $K - 1$ can be expressed as a linear combination of shifted scaling functions at any scale [6], meaning that all the detail coefficients of the wavelet transform of the polynomial would be zero. If the original signal is band limited and can be approximated by a polynomial of a certain degree, then most of the detail coefficients at the initial levels would be negligible in magnitude, which is not the case with noise. Hence, the process of thresholding the detail coefficients eliminates noise at those levels. *Daubechies* wavelets of order K are specifically constructed to ensure that the scaling filter is $K - regular$ and hence well approximate signals that can be expressed as a linear combination of polynomials.

4. SIMULATION RESULTS

The signal model considered in this work relates to sinusoids with random amplitude modulation. As seen from Eqs (1) and (5) the sinusoid is multiplied by a random amplitude whose auto-correlation decays exponentially with the lag. This kind of random amplitude modulation occurs in media that introduce fading. The parameter α is a measure of the signal bandwidth. A small value of α represents a random narrowband signal.

In this work, all the sources were assumed to have the same center frequency and a narrow band width. The parameter α was chosen to be $\frac{1}{200}$. The signal was sampled at the rate of 32 times the source frequency so as to keep the narrowband assumption valid. Temporally and spatially white Gaussian noise was added to the signal at each sensor with appropriate variance so as to give an average SNR of 0 dB over the array. Figure 1a shows a realisation of a random amplitude modulated sinusoid for the parameters described above. Figure 1b is the noisy version of the signal in Figure 1a.

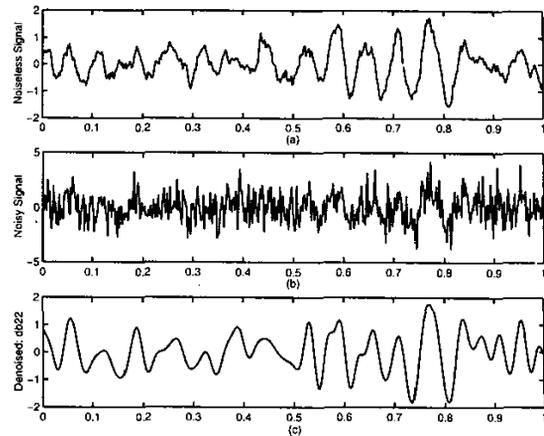


Figure 1. (a) Noiseless random amplitude modulated sinusoid. (b) Noisy signal SNR = 0 dB. (c) *db22* wavelet denoised signal.

We considered a uniform linear array of 14 sensors with intersensor spacing $d = \lambda/2$, where λ is the wavelength of the plane waves. Wavelet denoising was applied on each sensor output independently. The wavelet coefficients were hard-thresholded with the threshold mentioned in Eqn (15). Figure 1c shows the denoised version of the noisy signal in Figure 1b, denoised with *db22* wavelet. The SNR of the denoised signal was calculated as in Eqn (21) and was used as a performance measure of denoising by different *Daubechies* wavelets.

$$SNR_{out} = 10 \log_{10} \left(\frac{E[|\mathbf{f}|^2]}{E[|\mathbf{f} - \hat{\mathbf{f}}|^2]} \right), \quad (21)$$

where \mathbf{f} is the original noiseless signal and $\hat{\mathbf{f}}$ is the denoised signal. Figure 2 shows SNR_{out} for different *Daubechies* wavelets. The expectation in Eqn (21) was calculated by averaging over 100 Monte-Carlo simulations. It can be seen from Figure 2 that *db18 - 32* wavelets perform better than the other wavelets. *Daubechies* wavelets with larger number of filter coefficients have a larger number of vanishing moments and hence can approximate polynomials of higher degree. This is a desirable property for signal compaction. However, wavelets with a very large number of vanishing moments may prove to be disadvantageous because a given realisation of noise may be well approximated by a high degree polynomial resulting in much of the noise energy also being compacted into the approximation coefficients. On the other hand, wavelets with very few filter coefficients have very few vanishing moments and hence do not compact the signal energy efficiently.

In our simulations, the wavelets *db18* and *db22* have been chosen to denoise sensor outputs before performing the DOA estimation. Figure 3 depicts the gain in SNR due to denoising using *db22* wavelet, as a function of the input SNR. It is seen that denoising is effective for input SNR less than 4 dB. At

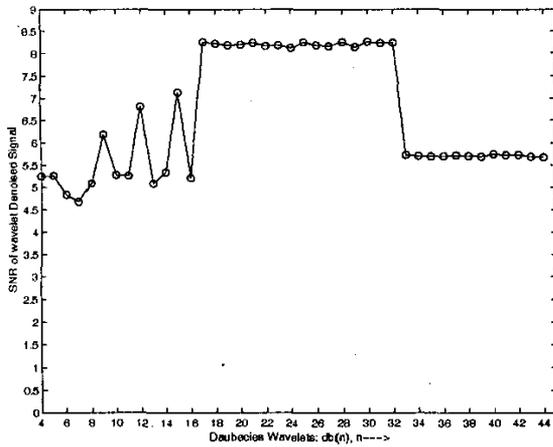


Figure 2. SNR_{out} for different Daubechies Wavelets for input SNR of 0dB.

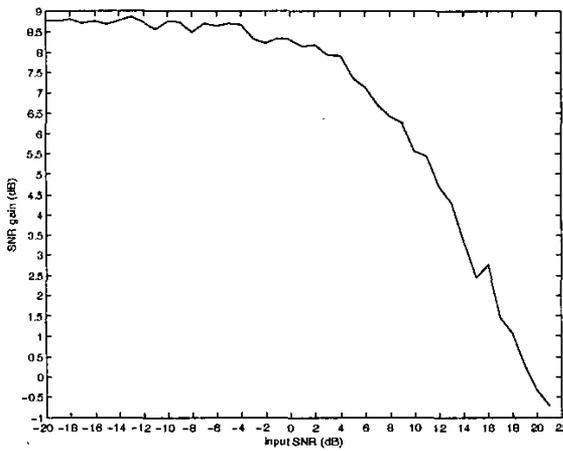


Figure 3. SNR gain as a function of input SNR for db22 wavelet.

higher SNR the gain decreases sharply. This may be due to the fact that at higher input SNR the threshold given in Eqn (15) is too high, resulting in removal of some of the signal coefficients leading to signal distortion.

The data covariance matrix was computed as in Eqn (9) for both the denoised and undenoised sensor outputs with $N = 512$ snapshots. Spectral MUSIC was then used to estimate the DOA. Figures 4, 5 and 6 show plots of spectral MUSIC ambiguity function for the undenoised, db18 and db22 denoised sensor data, respectively. The wavelet denoised data resolves the three DOA, viz., 10, 15 and 20 degrees, clearly, unlike the undenoised data.

The reason for the improved performance of the denoised data is that denoising reduces the MSE of the estimated covariance. Let r_{ml} denote the true covariance value of the noisy signal at the m^{th} and the l^{th} sensors, and let \hat{r}_{ml} de-

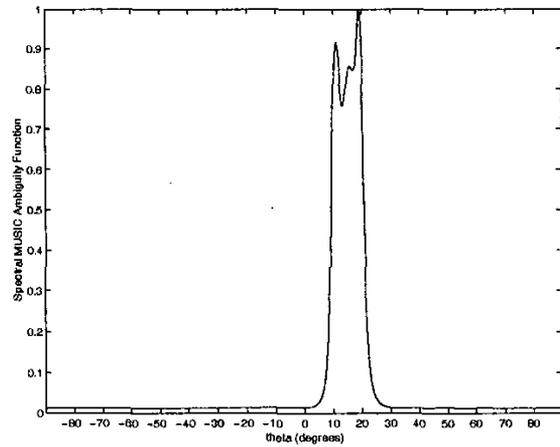


Figure 4. Spectral MUSIC for undenoised signal.

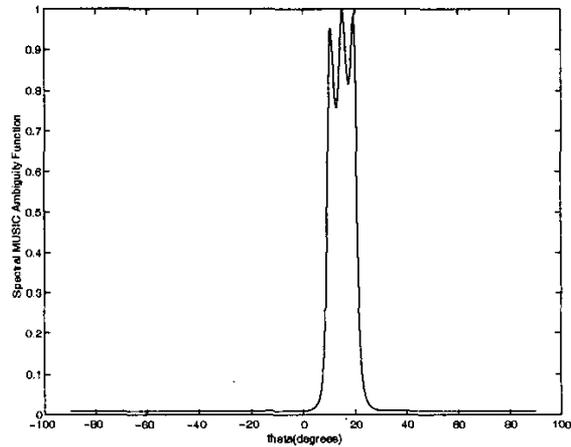


Figure 5. Spectral MUSIC for db18 wavelet denoised signal.

note the covariance estimate computed as

$$\hat{r}_{ml} = \frac{1}{N} \sum_{t=1}^N y_m(t)y_l^H(t), \quad m, l = 1, 2, \dots, M \quad (22)$$

It can be shown that the corresponding MSE is

$$E[|r_{ml} - \hat{r}_{ml}|^2] = \frac{1}{N^2} \left(\sum_{p=1}^n \sigma_p^2 \right) \sum_{l=1}^N \sum_{u=1}^N e^{-2\alpha|l-u|} \quad (23) \\ + \frac{2\sigma^2}{N} \sum_{p=1}^n \sigma_p^2 + \frac{\sigma^4}{N}$$

From the above equation one can infer that the MSE of the estimate \hat{r}_{ml} would reduce if N is increased or if the noise variance is reduced, implying higher SNR. Wavelet denoising can be used to effectively increase the SNR and thereby reduce the covariance estimation error. This is established in

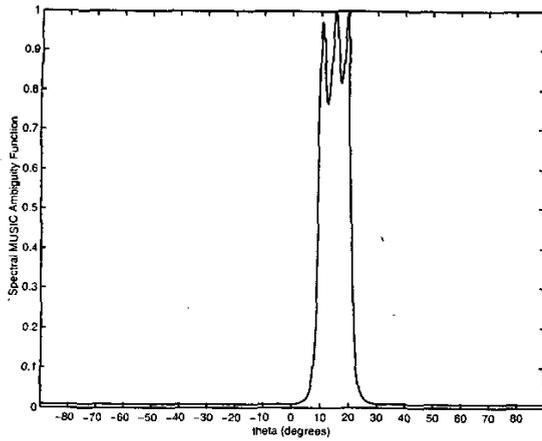


Figure 6. Spectral MUSIC for db22 wavelet denoised signal.

Table 1, which presents MSE of the covariance estimates \hat{r}_{12} , \hat{r}_{13} , ..., \hat{r}_{19} computed by averaging over 100 Monte-Carlo simulations.

Table 1. MSE of auto-correlation function

\hat{r}_{ml}	Undenoised	Denoised db18	Denoised db22
\hat{r}_{12}	1.9091	1.8900	1.8895
\hat{r}_{13}	1.9106	1.8898	1.8890
\hat{r}_{14}	1.9102	1.8899	1.8895
\hat{r}_{15}	1.9056	1.8867	1.8860
\hat{r}_{16}	1.9027	1.8820	1.8812
\hat{r}_{17}	1.8973	1.8764	1.8756
\hat{r}_{18}	1.8932	1.8719	1.8714
\hat{r}_{19}	1.8868	1.8674	1.8667

Root MUSIC was used to estimate the DOA and the associated bias and MSE were computed by averaging over 100 Monte-Carlo simulations. Tables 2, 3 list the bias and MSE of Root MUSIC DOA estimates. It is seen that wavelet denoising reduces the bias and MSE of the DOA estimates significantly.

Table 2. Bias of Root MUSIC DOA Estimates

	DOA = 10	DOA = 15	DOA = 20
Undenoised	6.3741	0.1439	3.0452
Denoised: db18	1.1902	0.0632	0.6037
Denoised: db22	1.0128	0.2678	1.0115

Table 3. MSE of Root MUSIC DOA estimates

	DOA = 10	DOA = 15	DOA = 20
Undenoised	269.2509	7.0742	79.9316
Denoised: db18	58.2647	2.4694	11.3223
Denoised: db22	54.7046	2.7200	17.0340

5. SUMMARY AND CONCLUSIONS

In this work, the use of wavelet denoising for plane wave DOA estimation has been investigated. Based on the idea that wavelet denoising improves the SNR of a noisy signal, we proceeded to perform wavelet denoising of the signal from each sensor of the array independently, prior to estimating the DOA. Wavelet denoising helps to reduce the MSE of the array data covariances estimated from finite data and thus reduce the bias and MSE of the DOA estimates.

6. ACKNOWLEDGEMENTS

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