ON POLYTOPAL UPPER BOUND SPHERES

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Abstract

Generalizing a result (the case $k = 1$) due to M. A. Perles, we show that any polytopal upper bound sphere of odd dimension $2k+1$ belongs to the generalized Walkup class $\mathcal{K}_k(2k+1)$, i.e., all its vertex links are k-stacked spheres. This is surprising since the k-stacked spheres minimize the face-vector (among all polytopal spheres with given $f_0, \dots, f_{k-1}$) while the upper bound spheres maximize the face vector (among spheres with a given f_0).

It has been conjectured that for $d \neq 2k+1$, all $(k+1)$-neighborly members of the class $\mathcal{K}_k(d)$ are tight. The result of this paper shows that, for every k, the case $d = 2k+1$ is a true exception to this conjecture.

We recall that a simplicial complex is said to be *l-neighborly* if each set of l vertices of the complex spans a face. As a well known consequence of the Dehn-Sommerville equations, any triangulated sphere of odd dimension $d = 2k+1$ can be at most $(k+1)$-neighborly (unless it is the boundary complex of a simplex). A $(2k+1)$-dimensional triangulated sphere is said to be an *upper bound sphere* if it is $(k+1)$-neighborly. This is because, by the celebrated Upper Bound Theorem, any such sphere maximizes the face vector componentwise among all $(2k+1)$-dimensional triangulated closed manifolds with a given number of vertices [9].

A simplicial complex is said to be a *polytopal sphere* if it is isomorphic to the boundary complex of a simplicial convex polytope. For $n \geq 2k+3$, the boundary complex of an n-vertex $(2k+2)$-dimensional cyclic polytope P (defined as the convex hull of any set of n points on the moment curve $t \mapsto (t, t^2, \dots, t^{2k+2})$) is an example of an n-vertex polytopal upper bound sphere of dimension $2k+1$.

We recall that a triangulated homology sphere S is said to be *k-stacked* if there is a triangulated homology ball B bounded by S all whose faces of codimension $k+1$ are in the boundary S. The generalized lower bound conjecture (GLBC) due to McMullen and Walkup [7] states that a k-stacked d-sphere S minimizes the face-vector componentwise among all triangulated d-spheres T such that $f_i(T) = f_i(S)$ for $0 \leq i < k$. (Here, as usual, the *face-vector* $(f_0(T), \dots, f_d(T))$ of a d-dimensional simplicial complex T is given by $f_i(T)$ = the number of i-dimensional faces of T). For polytopal spheres T, this conjecture was proved by Stanley [10] and McMullen [6]. Recently, Murai and Nevo [8] proved that a polytopal sphere (more generally, a triangulated homology sphere with the weak Lefschetz property) satisfies equality in GLBC only if it is k-stacked.

A triangulated homology ball B is said to be *k-stacked* if all its faces of codimension $k+1$ are in its boundary ∂B. Thus, a triangulated (homology) d-sphere S is k-stacked if

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and only if there is a k-stacked (homology) $(d+1)$-ball B such that $\partial B = S$. As an aside, we mention that in [8, Theorem 2.3 (ii)], Murai and Nevo prove:

Proposition 1. *If S is a k-stacked triangulated homology sphere of dimension $d \geq 2k$ then there is a unique k-stacked homology $(d+1)$-ball B such that $\partial B = S$. It is the largest simplicial complex (in the sense of set inclusion) whose k-skeleton agrees with that of S. That is, B is given by the formula*

$$B = \left\{ \alpha \subseteq V(S) \, : \, \binom{\alpha}{\leq k+1} \subseteq S \right\}.$$

(Here $V(S)$ is the vertex set of S and $\binom{\alpha}{\leq k+1}$ denotes the set of all subsets of α of size $\leq k+1$. Actually, Murai and Nevo give this formula with $d-k+1$ in place of $k+1$. But, their proof shows that it also holds with $k+1$ in place of $d-k+1$, and - of course - in view of the uniqueness statement the two formulae give the same $(d+1)$-ball.)

This theorem gives a common generalization of Propositions 2.10, 2.11 and Corollary 3.6 of [3] as well as a complete answer to Question 6.4 of that paper. Notice that if S is an upper bound sphere of dimension $2k-1$ then S is trivially k-stacked. Such a sphere S fails to satisfy the conclusion of Proposition 1 unless it is the boundary complex of a simplex. Thus, the hypothesis $d \geq 2k$ in Proposition 1 is best possible.

We also recall that, for a $(d+1)$-dimensional convex polytope $P \subseteq \mathbb{R}^{d+1}$ and a point $x \notin P$, a facet F of P is said to be *visible* from x if, for any $y \in F$, $[x,y] \cap P = \{y\}$. As a consequence of the Bruggesser-Mani construction of a shelling order on a simplicial polytope (cf. [11, Theorem 8.12]), we know that if P is a simplicial polytope then the facets of P visible from any given point outside P form a (shellable) ball. The same holds for the facets which are invisible from a point outside P.

The *generalized Walkup class* $\mathcal{K}_k(d)$ consists of the triangulated d-manifolds all whose vertex-links are k-stacked homology spheres. (We also note that the vertex-links of a polytopal sphere are polytopal spheres, hence actually triangulated spheres.) The case $k=1$ of the following result is due to M. A. Perles (cf. [1, Theorem 1]).

Theorem 1. *Let S be a polytopal upper bound sphere of dimension $2k+1$. Then S belongs to the generalized Walkup class $\mathcal{K}_k(2k+1)$.*

Proof. Let S be the boundary complex of the simplicial polytope P of dimension $2k+2$. Then P is a $(k+1)$-neighborly $(2k+2)$-polytope. Fix a vertex v of S, and let L be the link of v in S. We need to prove that L is k-stacked. This is trivial if P is a simplex. Otherwise, the convex hull Q, of the vertices of P excepting v, is again a $(2k+2)$-dimensional polytope. Clearly, Q is also $(k+1)$-neighborly and hence, by Radon's Theorem (cf. [5, Page 124]), Q is also simplicial. Let B be the pure $(2k+1)$-dimensional simplicial complex whose facets are the facets of Q visible from v. By Bruggesser-Mani, B is a shellable ball.

Claim: $\partial B = L = S \cap B$.

Let $\mathrm{ast}_S(v) := \{\alpha \in S \, : \, v \notin \alpha\}$ be the antistar of v in S. Then $\mathrm{ast}_S(v)$ is a triangulated $(2k+1)$-ball and $\mathrm{ast}_S(v) \cap \mathrm{star}_S(v) = \partial(\mathrm{ast}_S(v)) = \mathrm{lk}_S(v) = L$ (cf. [2, Lemma 4.1]).

Let A be the pure $(2k+1)$-dimensional simplicial complex whose facets are the facets of Q which are not in B (i.e., invisible from v). By the Bruggesser-Mani construction, A is also a shellable ball. Clearly, $\partial A = \partial B = A \cap B$.

We denote by $|A|$ the geometric carrier of A, i.e., the union of the facets in A. If $x \in \mathrm{int}(|A|)$ then $x \in |S|$ and $[v,x] \cap \mathrm{int}(Q)$ is a non-trivial interval. Therefore, $[v,x] \cap \mathrm{int}(P)$

is a non-trivial interval and hence $x \in \mathrm{int}(|\mathrm{ast}_S(v)|)$. Thus $\mathrm{int}(|A|) \subseteq |\mathrm{ast}_S(v)|$ and hence $|A| \subseteq |\mathrm{ast}_S(v)|$. Let $y \in \mathrm{int}(|\mathrm{ast}_S(v)|)$. Let $y \in \mathrm{int}(|\alpha|)$ for some $\alpha \in S$. Then α is a face of Q. So, $y \in \partial Q$. If $y \in \mathrm{int}(|B|)$ then the line l containing v and y intersect $\mathrm{int}(Q)$ in an interval (y, w), where $y \in (v, w)$. So, $(v, w) \subseteq \mathrm{int}(P)$ and $y \in (v, w)$. This is not possible since $y \in |\mathrm{ast}_S(v)|$. So, $y \in |A|$. Thus, $\mathrm{int}(|\mathrm{ast}_S(v)|) \subseteq |A|$ and hence $|\mathrm{ast}_S(v)| \subseteq |A|$. So, $|\mathrm{ast}_S(v)| = |A|$. Since both $\mathrm{ast}_S(v)$ and A are subcomplexes of S, $\mathrm{ast}_S(v) = A$. Then $\partial B = \partial A = \partial(\mathrm{ast}_S(v)) = L$. This proves the first equality of the claim.

Since $\partial B = L \subseteq S$, we have $\partial B \subseteq S \cap B$. Let $y \in |S \cap B|$. If $y \in \mathrm{int}(|B|)$ then, by the same argument as before, there exists $w \in P$ such that $y \in (v, w) \subseteq \mathrm{int}(P)$. This is not possible since $y \in |S|$. Therefore, $y \in |B| \setminus \mathrm{int}(|B|) = |\partial B|$. Thus $|S \cap B| \subseteq |\partial B|$. Since $\partial B \subseteq S$, this implies $S \cap B = \partial B$. This completes the proof of the claim.

If α is a k-face of B then $\alpha \in S$ since S is $(k+1)$-neighborly. Thus, by the claim, $\alpha \in S \cap B = \partial B$. So, B is k-stacked. Since, by the claim, $L = \partial B$, it follows that L is k-stacked. Since L is the link in S of an arbitrary vertex of S, it follows that $S \in \mathcal{K}_k(2k+1)$. $\square$

In [3], we defined a *k-stellated sphere* to be a triangulated sphere which may be obtained from the boundary complex of a simplex by a finite sequence of bistellar moves of index $< k$. We also defined $\mathcal{W}_k(d)$ as the class of all triangulated d-manifolds with k-stellated vertex-links. An easy induction on the number of bistellar moves used shows that (cf. [3, Proposition 2.9]) :

Proposition 2. *For $d \geq 2k-1$, a triangulated d-sphere S is k-stellated if and only if S is the boundary of a shellable k-stacked $(d+1)$-ball. In consequence, all k-stellated spheres of dimension $\geq 2k-1$ are k-stacked. Therefore, for $d \geq 2k$, $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$.*

Thus, the proof of the above theorem shows that the polytopal upper bound spheres of dimension $2k+1$ are ($(k+1)$-neighborly) members of the smaller class $\mathcal{W}_k(2k+1)$. In [4], we show that $(k+1)$-neighborly members of $\mathcal{W}_k(d)$ are tight for $d \neq 2k+1$. The theorem proved here shows that the case $d = 2k+1$ is a true exception to this tightness criterion (since, except for the boundary complex of simplices, no triangulated sphere can be tight).

In [1], the case $k = 1$ of Theorem 1 was used to classify the polytopal upper bound 3-spheres with 9 vertices. The case $k = 2$ of this theorem may be useful in similarly classifying polytopal upper bound spheres of dimension 5 with few vertices.

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