DOMAINS IN COMPLEX SURFACES WITH A NONCOMPACT AUTOMORPHISM GROUP – II

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ABSTRACT. Let X be an arbitrary complex surface and $D \subset X$ a domain that has a noncompact group of holomorphic automorphisms. A characterization of those domains D that admit a smooth real analytic, finite type, boundary orbit accumulation point and whose closures are contained in a complete hyperbolic domain $D' \subset X$ is obtained.

1. INTRODUCTION

Let $D \subset \mathbb{C}^n$, $n \geq 1$ be a bounded domain and let $\mathrm{Aut}(D)$ be the group of holomorphic automorphisms of D. There is a natural action of $\mathrm{Aut}(D)$ on D given by

$$(f,z) \mapsto f(z)$$

where $f \in \mathrm{Aut}(D)$ and $z \in D$. Suppose the orbit of some point $p \in D$ under this action accumulates at $p_\infty \in \partial D$ – call such a point a boundary orbit accumulation point. In this situation, it has been shown that (see [1]–[4], [5], [11], [22], [26] and [33] among others) the nature of ∂D near p_∞ provides global information about D. The question of investigating this phenomenon when D is a domain in a complex manifold was raised in [7] and [14] and it was shown in the latter article that the Wong-Rosay theorem remains valid when D is a domain in an arbitrary complex manifold with $p_\infty \in \partial D$ a strongly pseudoconvex point. In short, such a domain D is biholomorphic to the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. Motivated by this result, it was shown in [28] that the analogues of [1] and [5] are also valid, with the same conclusion, when D is a domain in an arbitrary complex surface and p_∞ is a smooth weakly pseudoconvex point of finite type. The pseudoconvexity hypothesis near p_∞ was dropped in [4] and a local version of this result for bounded domains in $\mathbb{C}^2$ and with the boundary ∂D near p_∞ being smooth real analytic and of finite type can be found in [31]. The purpose of this article is to generalise the result in [31] by finding all possible model domains when D is a domain in an arbitrary complex surface X.

Theorem 1.1. *Let X be an arbitrary complex surface and $D \subset X$ a domain. Suppose that $\overline{D}$ is contained in a complete hyperbolic domain $D' \subset X$ and that there exists a point $p \in D$ and a sequence $\{\phi_j\} \in \mathrm{Aut}(D)$ such that $\{\phi_j(p)\}$ converges to $p_\infty \in \partial D$. Assume that the boundary of D is smooth real analytic and of finite type near p_∞. Then exactly one of the following alternatives holds:*

(i) If $\dim \mathrm{Aut}(D) = 2$ then either
- *$D \backsimeq \mathcal{D}_1 = \left\{(z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\right\}$ where P_1 is a polynomial that depends on $\Re z_1$, or*
- *$D \backsimeq \mathcal{D}_2 = \left\{(z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\right\}$ where P_2 is a polynomial that depends on $|z_1|^2$, or*

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- $D \backsimeq \mathcal{D}_3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0 \right\}$ *where* P_{2m} *is a homogeneous polynomial of degree* $2m$ *without harmonic terms.*

(ii) If $\dim \mathrm{Aut}(D) = 3$ *then* $D \backsimeq \mathcal{D}_4 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0 \right\}$ *for some integer* $m \geq 2$.

(iii) If $\dim \mathrm{Aut}(D) = 4$ *then* $D \backsimeq \mathcal{D}_5 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0 \right\}$ *for some integer* $m \geq 2$.

(vi) If $\dim \mathrm{Aut}(D) = 8$ *then* $D \backsimeq \mathcal{D}_6 = \mathbb{B}^2$ *the unit ball in* $\mathbb{C}^2$.

The dimensions $0, 1, 5, 6, 7$ *cannot occur with D as above.*

To clarify several points, first note that D is hyperbolic since it is contained in D' which is assumed to be complete hyperbolic in the sense of Kobayashi. Therefore $\mathrm{Aut}(D)$ is a real Lie group endowed with the topology of uniform convergence on compact subsets of D. Moreover, the family $\phi_j : D \to D \subset D'$ is normal since D' is complete. By theorem 2.7 in [30] (which generalises Cartan's theorem – see [25] pp. 78) we see that every possible limit map ϕ is either in $\mathrm{Aut}(D)$ or satisfies $\phi(D) \subset \partial D$. Since $\phi(p) = p_\infty \in \partial D$, it follows that $\phi(D) \subset \partial D$. Fix a neighbourhood U of p_∞ and a biholomorphism $\psi : U \to \psi(U) \subset \mathbb{C}^2$ such that $\psi(p_\infty) = 0$ and $\psi(U \cap \partial D)$ is a smooth real analytic hypersurface of finite type – note that the type is a biholomorphic invariant and hence it suffices to work with a fixed, sufficiently small neighbourhood of p_∞. Let W be a neighbourhood of p small enough so that $\phi(W) \subset U$. If possible, let $k > 0$ be the maximal rank of ϕ which is attained on the complement of an analytic set $A \subset D$. If $p \in W \setminus A$, then the image of a small neighbourhood of p that does not intersect A under ϕ is a germ of a positive dimensional complex manifold contained in $U \cap \partial D$ and this is a contradiction. On the other hand if $p \in A$, pick $q \in W \setminus A$ and repeat the above argument to see that $k = 0$ in this case as well. Thus $\phi(D) \equiv p_\infty$. Since this is true of any limit map, it follows that the entire sequence ϕ_j converges uniformly on compact subsets of D to the constant map $\phi(z) \equiv p_\infty$. It follows that D must be simply connected (see for example [24]) for any loop $\gamma \subset D$ is contractible if and only if $\phi_j(\gamma)$ is so for all j. However, for all large j the loop $\phi_j(\gamma) \subset U \cap D$ which is simply connected if U is small enough. Hence $\phi_j(\gamma)$ is a trivial loop for large j and hence so is γ.

Second, note that $\psi(p_\infty)$ cannot belong to the envelope of holomorphy of $\psi(U \cap D)$. Indeed, for if not, then on the one hand we see from the above reasoning that the Jacobian determinant $\det(\psi \circ \phi_j)'$ must tend to zero uniformly on compact subsets of D. On the other hand, all the maps $\phi_j^{-1} \circ \psi^{-1} : \psi(U \cap D) \to D \subset D'$ extend to a fixed, open neighbourhood of $\psi(p_\infty)$ by a theorem of Ivashkovich (see [20]) since D' is complete. Moreover, the extensions of these maps near $\psi(p_\infty)$ take values in D'. Hence there is an upper bound for $\det(\phi_j^{-1} \circ \psi^{-1})'$ near $\psi(p_\infty)$ and this is a contradiction. As a consequence, this observation of Greene-Krantz is also valid in the situation of the main theorem.

Third, recall the stratification of the smooth real analytic finite type hypersurface $U \cap \partial D$ that was used in [31]. There is a biholomorphically invariant decomposition of $U \cap \partial D$ as the union of two relatively open sets, namely ∂D^+ (for brevity, we drop the reference to ψ) which consists of points near which $U \cap \partial D$ is pseudoconvex and $\hat{D} \cap \partial D$ that has those points which are in the envelope of holomorphy of $U \cap D$, and their closed complement M_e which is a locally finite union of smooth real analytic arcs and points. Note that M_e is contained in the set of Levi flat points $\mathcal{L}$ which by the finite type assumption is a codimension one real analytic subset of $U \cap \partial D$. By the second remark above, $p_\infty \notin \hat{D} \cap \partial D$. If $p_\infty \in \partial D^+$ then by [28] it follows that

$$D \backsimeq \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0 \right\}$$

where $P_{2m}(z_1, \bar{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ (this being the 1-type of $U \cap \partial D$ near p_∞) without harmonic terms. In this case, the assumption that $D \subset D'$ plays no role for pseudoconvexity of $U \cap \partial D$ near p_∞ (an orbit accumulation point) is enough to guarantee that D is complete hyperbolic – see [5], [13] for example. In particular, in the situation of the main theorem, the Levi form of $U \cap \partial D$ changes sign in every neighbourhood of p_∞. Finally, a word about the assumption that $\overline{D}$ is contained in a complete hyperbolic domain $D' \subset X$. Perhaps the most natural assumption would be to *not* assume anything except finite type and smooth real analyticity of $U \cap \partial D$ near p_∞. In this situation, the first thing to do would be to show the normality of $\mathcal{O}(\Delta, D)$, the family of holomorphic mappings from the unit disc Δ into D. And as in [5] and [13] this should be a consequence of understanding the rate of blow up of the Kobayashi metric on D near p_∞. That the metric can even be localised near p_∞ near which the Levi form changes sign does not seem to be known. Therefore another possibility is to assume that D is locally taut near p_∞, i.e., $V \cap D$ is taut for some fixed neighbourhood V of p_∞. However, working with this also requires knowledge that an analytic disc $f : \Delta \to D$ with $f(0)$ close to p_∞ can be localised. Moreover, if we strengthen the hypothesis on D by assuming that it is complete hyperbolic, then D would be pseudoconvex near p_∞. The model domains in this case have been determined in [28]. With these observations a plausible hypothesis seemed that of requiring that $\overline{D} \subset D'$ where $D' \subset X$ is complete – and this, though being global in nature, seemed to complement well the assumption made in [31] that $D \subset \mathbb{C}^2$ is a bounded domain.

The general strategy is the same as in [31]. Note that since D is hyperbolic it follows from [21], [23] that $0 \leq \dim \mathrm{Aut}(D) \leq n^2 + 2n = 8$ as $n = 2$. Furthermore by [21] it is known that if $\dim \mathrm{Aut}(D) \geq 5$, then D is homogeneous and hence there is an orbit that clusters at strongly pseudoconvex points in $U \cap \partial D$. Such points form a non-empty open subset of $U \cap \partial D$ that contains p_∞ in its closure and this follows from the decompositon of $U \cap \partial D$ alluded to above. Consequently by [14], $D \backsimeq \mathbb{B}^2$. Therefore it suffices to treat the case when $0 \leq \dim \mathrm{Aut}(D) \leq 4$. An initial scaling of D using the orbit $\{\phi_j(p)\}$ as described below shows that D is biholomorphic to a model domain of the form

$$G = \left\{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P(z_1, \bar{z}_1) < 0 \right\}$$

where $P(z_1, \bar{z}_1)$ is a polynomial without harmonic terms. Let $g : G \to D$ be the biholomorphism. G is evidently invariant under the one parameter subgroup of translations in the imaginary z_2-direction, i.e., $T_t(z_1, z_2) = (z_1, z_2 + it)$ for $t \in \mathbb{R}$. This shows that $\dim \mathrm{Aut}(D) \geq 1$. If $\dim \mathrm{Aut}(D) = \dim \mathrm{Aut}(G) = 1$, it is possible to explicitly write down what can element of $\mathrm{Aut}(G)$ should look like and this description shows that the orbits in G stay uniformly away from the boundary and accumulate only at the point at infinity in ∂G. Using the assumption that D is contained in a taut domain, it can be seen that the Kobayashi metric in D blows up near p_∞. Let $\mathcal{X} = g_*(i\partial/\partial z_2)$; note that $i\partial/\partial z_2$ is a holomorphic vector field in G whose real part generates the translations T_t. Then p_∞ is seen to be an isolated zero of $\mathcal{X}$ on ∂D and the arguments of [4] show that $\mathcal{X}$ must be parabolic and this forces D to be equivalent to an ellipsoid whose automorphism group is four dimensional. This is a contradiction. When $\dim \mathrm{Aut}(D) = 2$, two cases arise depending on whether $\mathrm{Aut}(D)^c$, the connected component of the identity in $\mathrm{Aut}(D)$ is abelian or not. In the former case, $\mathrm{Aut}(D)^c$ must be isomorphic to either $\mathbb{R}^2$ or to $\mathbb{R} \times \mathbb{S}^1$. These lead to the conclusion that $D \backsimeq \mathcal{D}_1$ or $D \backsimeq \mathcal{D}_2$. In the non-abelian case $\mathrm{Aut}(D)^c$ is solvable and it can be shown that $D \backsimeq \mathcal{D}_3$. A case-by-case analysis is used when $\dim \mathrm{Aut}(D) = 3, 4$ to identify the relevant domain from the classification obtained by A. V. Isaev in [17], [18]. While the argument remains the same in some cases, we take this opportunity to streamline and provide alternate proofs in some instances – for example, ruling out the possibility that $\dim \mathrm{Aut}(D) = 1$ and identifying the right model domain when $\dim \mathrm{Aut}(D) = 3$. There are several possibilities in [17] and here we focus on three interesting classes from that list, as the proof for the others remains the same. Nothing changes when $\dim \mathrm{Aut}(D) = 2, 4$, i.e., the same proofs from [31] carry over to these cases and we have decided to be brief, the emphasis being not to merely repeat what carries over to this situation from [31], but to identify and focus on the differences instead.

2. THE DIMENSION OF $\mathrm{Aut}(D)$ IS AT LEAST TWO

To describe the scaling of D using the base point p and the sequence $\{\phi_j\} \in \mathrm{Aut}(D)$, first note that for j large, there is a unique point $\tilde{p}_j \in \psi(U \cap \partial D)$ such that

$$\mathrm{dist}(\psi \circ \phi_j(p), \psi(U \cap \partial D)) = |\tilde{p}_j - \psi \circ \phi_j(p)|. \tag{2.1}$$

By a rotation of coordinates, we may assume that the defining function $\rho(z)$ for $\psi(U \cap \partial D)$ is of the form

$$\rho(z) = 2\Re z_2 + \sum_{k,l} c_{kl}(y_2) z_1^k \overline{z}_1^l$$

where $c_{00}(y_2) = O(y_2^2)$ and $c_{10}(y_2) = \overline{c}_{01}(y_2) = O(y_2)$. Let m be the type of $\psi(U \cap \partial D)$ at the origin. By definition, there exist k, l both at least one and $k + l = m$ for which $c_{kl}(0) \neq 0$ and $c_{kl}(0) = 0$ for all other $k + l < m$. The pure terms, if any, up to order m in the defining function can be removed by a polynomial automorphism of the form

$$(z_1, z_2) \mapsto \big(z_1, z_2 + \sum_{k \leq m} (c_{k0}(0)/2) z_1^k\big). \tag{2.2}$$

These coordinate changes will be absorbed in ψ. Let $\psi^j_{p,1}(z) = z - \tilde{p}_j$ so that $\psi^j_{p,1}(\tilde{p}_j) = 0$. A unitary rotation $\psi^j_{p,2}(z)$ then ensures that the outer real normal to $\psi^j_{p,1} \circ \psi(U \cap \partial D)$ at the origin is the real z_2-axis. The defining function for $\psi^j_{p,2} \circ \psi^j_{p,1} \circ \psi(U \cap \partial D)$ near the origin is then of the form

$$\rho_j(z) = 2\Re z_2 + \sum_{k,l} c^j_{kl}(y_2) z_1^k \overline{z}_1^l \tag{2.3}$$

with the same normalisations on the coefficients $c^j_{00}(y_2)$ and $c^j_{10}(y_2)$ as described above. Since $\tilde{p}_j \to 0$ it follows that both $\psi^j_{p,1}$ and $\psi^j_{p,2}$ converge to the identity uniformly on compact subsets of $\mathbb{C}^2$. Note that the type of $\psi^j_{p,2} \circ \psi^j_{p,1} \circ \psi(U \cap \partial D)$ is at most m for all j and an automorphism of the form (2.2) can be used to remove all pure terms up to order m from $\rho_j(z)$. Denote this by $\psi^j_{p,3}$. Lastly, note that $\psi \circ \phi_j(p)$ is on the inner real normal to $\psi(U \cap \partial D)$ at $\tilde{p}_j$ and it follows that $\psi^j_{p,2} \circ \psi^j_{p,1} \circ \psi \circ \phi_j(p) = (0, -\delta_j)$ for some $\delta_j > 0$ and the explicit form of (2.2) shows that this is unchanged by $\psi^j_{p,3}$. Let

$$\psi^j_{p,4}(z_1, z_2) = (z_1/\varepsilon_j, z_2/\delta_j)$$

where $\varepsilon_j > 0$ will be chosen in the next step. The defining function for $\psi^j_p \circ \psi(U \cap \partial D)$ near the origin, where $\psi^j_p = \psi^j_{p,4} \circ \psi^j_{p,3} \circ \psi^j_{p,2} \circ \psi^j_{p,1}$, is given by

$$\rho_{j,p}(z) = \delta_j^{-1} \, \rho_j(\varepsilon_j z_1, \delta_j z_2) = 2\Re z_2 + \sum_{k,l} \varepsilon_j^{k+l} \delta_j^{-1} c^j_{kl}(\delta_j y_2) z_1^k \overline{z}_1^l.$$

Observe that $\psi_p^j \circ \psi \circ \phi_j(p) = (0,-1)$ for all j. Now choose $\varepsilon_j > 0$ by demanding that

$$\max\left\{|\varepsilon_j^{k+l}\delta_j^{-1}c_{kl}^j(0)| : k+l \leq m\right\} = 1$$

for all j. In particular, note that $\{\varepsilon_j^m \delta_j^{-1}\}$ is bounded and by passing to a subsequence it follows that

$$\rho_{j,p}(z) \to \rho_p = 2\Re z_2 + P(z_1, \bar{z}_1)$$

in the C^∞ topology on compact subsets of $\mathbb{C}^2$, where $P(z_1,\bar{z}_1)$ is a polynomial of degree at most m without any harmonic terms. Therefore the domains $G_{j,p} = \psi_p^j \circ \psi(U \cap D)$ converge to

$$G_p = \{(z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P(z_1,\bar{z}_1) < 0\}$$

in the Hausdorff sense. Let $K \subset G_p$ be a relatively compact domain containing the base point $(0,-1)$. Then $K \subset \psi_p^j \circ \psi(U \cap D)$ for all large j and therefore the mappings

$$g_p^j : (\psi_p^j \circ \psi \circ \phi_j)^{-1} : K \to D \subset D'$$

are well defined and satisfy $g_p^j(0,-1) = p$. The completeness of D' shows that the family $\{g_p^j\}$ is normal and hence there is a holomorphic limit $g_p : G_p \to \overline{D}$ with $g_p(0,-1) = p$. It remains to show that g_p is a biholomorphism from G_p onto D. For this, recall the observation made in [4] that since $P(z_1,\bar{z}_1)$ is not harmonic, the envelope of holomorphy of G_p is either all of $\mathbb{C}^2$ or ∂G_p contains a strongly pseudoconvex point. The former situation cannot hold – indeed, by [20] again, the map g_p will extend to $\mathbb{C}^2$ taking values in D' and since D' is complete, $g_p(z) \equiv p$. Let $W \subset \mathbb{C}^2$ be a bounded domain that intersects infinitely many of the boundaries $\psi_p^j \circ \psi(U \cap \partial D)$ – and hence also ∂G_p. Then for each j, note that the cluster set of $W \cap \psi_p^j \circ \psi(U \cap \partial D)$ under g_p^j is contained in ∂D since $\phi_j \in \mathrm{Aut}(D)$. Now, if the envelope of G_p were all of $\mathbb{C}^2$, it is possible to find a domain Ω with $\Omega \cap \partial G_p \neq \emptyset$ on which the family $\{g_p^j\}$ would converge uniformly. In this case, by passing to the limit, we see that $g_p(U \cap \partial G_p) \subset \partial D$ and thus g_p cannot be the constant map. Therefore there must be a strongly pseudoconvex point, say ζ on ∂G_p. Fix $r > 0$ so that all points on $\partial G_p \cap B(\zeta, r)$ are strongly pseudoconvex and since $\rho_{j,p} \to \rho_{\infty,p}$ in the C^∞ topology on $B(\zeta,r)$, it follows that each of the open pieces $\psi_p^j \circ \psi(U \cap \partial D) \cap B(\zeta,r)$ are themselves strongly pseudoconvex for $j \gg 1$. For a complex manifold M, let $F_M(z,v)$ denote the Kobayashi metric at $z \in M$ along a tangent vector v at z. By the stability of the Kobayashi metric under smooth strongly pseudoconvex perturbations, it follows that for all $q \in B(\zeta,r) \cap G_p$

$$F_{G_{j,p}}(q,v) \geq c|v|$$

for some uniform $c > 0$ and by the invariance of the Kobayashi metric we see that

$$F_{\phi_j^{-1}(U \cap D)}(g_p^j(q), dg_p^j(q)v) = F_{G_{j,p}}(q,v) \geq c|v|. \tag{2.4}$$

Since the automorphisms $\phi_j \to p_\infty$ uniformly on compact subsets of D, it can be seen that the domains $\phi_j^{-1}(U \cap D)$ form an exhaustion of D in the sense that for any compact $K \subset D$, there is an index j_0 for which $K \subset \phi_{j_0}^{-1}(U \cap D)$. Furthermore, as $g_p(0,-1) = p \in D$, it follows that $g_p^{-1}(\partial D)$ is closed and nowhere dense in G_p and therefore it is possible to choose a $q \in (B(\zeta,r) \cap G_p) \setminus g_p^{-1}(\partial D)$. This ensures that $g_p^j(q) \to g_p(q) \in D$. Now, the completeness of D'

implies that the Kobayashi metrics on $\phi_j^{-1}(U \cap D)$ converge to the corresponding metric on D and thus (2.4) shows that

$$F_D(g_p(q), dg_p(q)v) \gtrsim c|v|. \tag{2.5}$$

Thus $dg_p(q)$ has full rank. Thus the rank of dg_p can be smaller only on an analytic set $A \subset G_p$ of dimension at most one. Pick $\tilde{q} \in A$ and let N_1, N_2 be small neighbouroods of $\tilde{q}$ and $g_p(\tilde{q})$ respectively such that $g_p^j(N_1) \subset N_2$ for $j \gg 1$. By identifying N_2 with an open subset of $\mathbb{C}^2$, Hurwitz's theorem applied to the Jacobians $\det(dg_p^j)$ shows that either $\det(dg_p^j)$ never vanishes or is identically zero in N_1. Since A has strictly smaller dimension it follows that dg_p has full rank everywhere, i.e., A must be empty. Hence g_p is locally biholomorphic in G_p and therefore $g_p(G_p) \subset D$. Injectivity of g_p is now a consequence of the fact that g_p^j are all biholomorphic and they converge uniformly on compact subsets of G_p to g_p.

To conclude, we have to show that $D_p = g_p(G_p)$ is all of D. If not, pick $\tilde{p} \in \partial D_p \cap D$ and note that since $\phi_j(\tilde{p}) \to p_\infty$, the scaling argument above can be repeated to get a biholomorphism $g_{\tilde{p}} : G_{\tilde{p}} \to g_{\tilde{p}}(G_{\tilde{p}}) \subset D$. Here $G_{\tilde{p}}$ has the same form as G_p with possibly a different polynomial than $P(z_1, \overline{z}_1)$. Note that $V = D_p \cap D_{\tilde{p}}$ is then a nonempty open subset of D. Let $f_p^j = (g_p^j)^{-1}$, $f_p = g_p^{-1}$ and $f_{\tilde{p}}^j = (g_{\tilde{p}}^j)^{-1}$, $f_{\tilde{p}} = g_{\tilde{p}}^{-1}$. Observe that both $f_p, f_{\tilde{p}}$ are biholomorphic on V, and that both $f_p^j, f_{\tilde{p}}^j$ are defined on a given compact set in D for large j. We may write $f_p = A \circ f_{\tilde{p}}$ where $A = g_p^{-1} \circ g_{\tilde{p}}$ is biholomorphic on $f_{\tilde{p}}(V)$. But more can be said about A – indeed, by definition we have

$$g_p^j \circ \psi_p^j \circ (\psi_{\tilde{p}}^j)^{-1} = g_{\tilde{p}}^j$$

where $A_j = \psi_p^j \circ (\psi_{\tilde{p}}^j)^{-1}$ are polynomial automorphisms of $\mathbb{C}^2$ of bounded degree as their construction shows. Since g_p^j and $g_{\tilde{p}}^j$ converge to g_p and $g_{\tilde{p}}$ respectively, we may take A as the limit of A_j on $f_{\tilde{p}}(V)$ and conclude that A is also a polynomial automorphism of $\mathbb{C}^2$. Now the functional equation $f_p = A \circ f_{\tilde{p}}$ extends f_p as a biholomorphic mapping from a small neighbourhood W of $\tilde{p}$ onto W', a neighbourhood of $f_p(\tilde{p})$. On the other hand, note first that since $g_{\tilde{p}}$ is biholomorphic near $(0, -1)$ and maps it to $\tilde{p}$, it follows that $f_{\tilde{p}}^j$ form a normal family on W, after possibly shrinking it if necessary. As a consequence, the equality

$$f_p^j = A_j \circ f_{\tilde{p}}^j$$

which holds on W for j large, shows that f_p^j converges to f_p on W and hence in the limit we see that $f_p(W) \subset \overline{G}_p$. That is, $f_p(W)$ cannot contain a neighbourhood of $f_p(\tilde{p})$ which is a contradiction. Hence $g_p : G_p \to D$ is biholomorphic and since G_p is invariant under the translations T_t, it follows that $\dim \mathrm{Aut}(D) = \dim \mathrm{Aut}(G_p) \geq 1$. In the sequel, we will write g, G in place of g_p, G_p respectively.

Recall that p_∞ is not in the envelope of holomorphy of $U \cap D$ where (U, ϕ) is the coordinate chart around p_∞ that was fixed earlier. Let $\Delta \subset \mathbb{C}$ be the unit disc. The following estimate on the Kobayashi metric near p_∞ will be useful.

Lemma 2.1. *For every $r \in (0,1)$, there is a neighbourhood V of p_∞ compactly contained in U such that every analytic disc $f : \Delta \to D$ with $f(0) \in V$ satisfies $f(r\Delta) \subset U$. As a result, the Kobayashi metric can be localised near p_∞ – there is a constant $C > 0$ such that*

$$C \cdot F_{U \cap D}(p, v) \leq F_D(p, v) \leq F_{U \cap D}(p, v)$$

uniformly for all $p \in V \cap D$ and tangent vectors v at p. Moreover,

$$F_D(p,v)/|v| \to \infty$$

as $p \to p_\infty$. In particular, for any neighbourhood V of p_∞ and $R < \infty$, there exists another neighbourhood $W \subset V$ of p_∞ such that the Kobayashi ball $B^k_D(p,R) \subset V$ whenever $p \in W \cap D$.

Proof. Let $f_\nu : \Delta \to D \subset D'$ be a sequence of holomorphic disks with $f_\nu(0) = p_\nu \to p_\infty$. The completeness of D' implies that some subsequence of $\{f_\nu\}$ converges uniformly on compact subsets of Δ to a holomorphic limit $f : \Delta \to \overline{D}$ and $f(0) = p_\infty$. Suppose that $f(z) \not\equiv p_\infty$ on Δ. Let $\eta > 0$ be such that $f(\eta\Delta) \subset U$. Since $U \cap \partial D$ is of finite type, no open subset of $f(\eta\Delta)$ can be contained in it and hence $f(\eta\Delta) \cap D \neq \emptyset$. By the strong disk theorem ([32]) it follows that p_∞ belongs to the envelope of holomorphy of $U \cap D$ which is a contradiction. Therefore $f(z) \equiv p_\infty$ and this shows that all limit functions for the given family of holomorphic disks are constant. The first claim follows and the equivalence of the metrics on $U \cap D$ and D is then a consequence of the definition of the Kobayashi metric.

If there exists a sequence $p_\nu \to p_\infty$ and non-zero vectors v_ν at p_ν and a constant C such that $F_D(p_\nu, v_\nu) \le C|v_\nu|$, then there would exist a uniform $r > 0$ and holomorphic disks $f_\nu \in \mathcal{O}(r\Delta, D)$ with $f_\nu(0) = p_\nu$ and $df_\nu(0) = v_\nu$. By the homogeneity of the metric in the vector variable, we may assume that $|v_\nu| = 1$ for all ν. The argument above shows that every possible limit function f of the family $\{f_\nu\}$ is constant which contradicts $|df(0)| = 1$. Therefore $F_D(p,v)/|v|$ blows up as $p \to p_\infty$.

For the claim about the size of $B^k_D(p,R)$, let us work in local coordinates around $\phi(p_\infty) = 0$. For $a, b \in U \cap D$, let $d(a,b)$ denote the euclidean distance on $U \cap D$ induced by ϕ. For a given neighbourhood V of p_∞ and $R < \infty$, let $p_\infty \in N_2 \subset V$ be such that $F_D(p,v)/|v| \ge 2R$ for all $p \in N_2 \cap D$ and tangent vectors v at p. We may assume without loss of generality that $N_2 \subset U$ and $N_2 = \phi^{-1}(B(0,2))$. Let $N_1 = \phi^{-1}(B(0,1))$. Fix $p \in N_1 \cap D$ and $q \in D$ and let $\gamma(t)$ be a path in D parametrised by $[0,1]$ with $\gamma(0) = p$ and $\gamma(1) = q$ such that

$$\int_0^1 F_D(\gamma(t), \gamma'(t))\, dt \le d^k_D(p,q) + \varepsilon$$

where $\varepsilon > 0$ is given and $d^k_D(p,q)$ is the Kobayashi distance between p, q. Suppose that $q \in N_1 \cap D$; two cases now arise – first, if the entire path $\gamma \subset N_2 \cap D$, then

$$2R\, d(p,q) \le \int_0^1 F_D(\gamma(t), \gamma'(t))\, dt \le d^k_D(p,q) + \varepsilon.$$

Second, if γ does not entirely lie in $N_2 \cap D$, then there is a connected component of γ that contains p and a point $a \in \partial N_2 \cap D$. The length of this connected component is at least $2 \ge d(p,q)$. On the other hand, if $q \in D \setminus N_1$, then the length of this path can be bounded from below by simply $2R$. Thus we get

$$d^k_D(p,q) \ge 2R\, d(p,q) - \varepsilon$$

if $q \in N_1 \cap D$ and $d^k_D(p,q) \ge 2R$ otherwise. Now if $p \in N_1 \cap D$ and $q \in B^k_D(p,R)$, it follows from these comparisons that $q \in N_1 \cap D$ which completes the proof. $\square$

The holomorphic vector field $\mathcal{X} = g_*(i\partial/\partial z_2)$ on D is such that its real part $\Re\mathcal{X} = (\mathcal{X} + \overline{\mathcal{X}})/2$ generates the one parameter subgroup $L_t = g \circ T_t \circ g^{-1} = \exp(t\, \Re\mathcal{X}) \in \mathrm{Aut}(D)$. Two observations can be made about $\mathcal{X}$ at this stage – first, Proposition 2.3 of [31] shows that (L_t) induces a local one parameter group of holomorphic automorphisms of a neighbourhood of p_∞ when

$D \subset \mathbb{C}^2$ is a bounded domain. In particular, $\mathcal{X}$ extends as a holomorphic vector field near p_∞. The proof of this relies on a local parametrised version of the reflection principle from [10], the main tools being the use of Segre varieties and their invariance property under biholomorphisms to construct the desired extension of (L_t) near p_∞ for all $|t| < \eta$ for a fixed $\eta > 0$. The same arguments can be applied in the local coordinates induced on U by ϕ to get the same conclusion in the setting of the main theorem as well. Second, consider the pullback of the orbit $\{\phi_j(p)\} \in D$ under the equivalence $g : G \to D$, i.e., let $g^{-1} \circ \phi_j(p) = (a_j, b_j) \in G$ and

$$2\varepsilon_j = 2\Re b_j + P(a_j, \overline{a}_j).$$

Note that $\varepsilon_j < 0$ for all j. Proposition 2.5 of [31] shows that if $|\varepsilon_j| > c > 0$ for all large j, then $\mathcal{X}$ vanishes to finite order at p_∞. The proof of this uses the boundedness of $D \subset \mathbb{C}^2$ which in particular implies that a family of holomorphic maps into D is normal. The same argument can be applied in the situation of the main theorem since $D \subset D'$ and D' is assumed to be complete hyperbolic. Thus we have:

Proposition 2.2. *The group (L_t) induces a local one parameter group of holomorphic automorphisms of a neighbourhood of p_∞ in X. In particular, $\mathcal{X}$ extends as a holomorphic vector field near p_∞. Moreover, if $|\varepsilon_j| > c > 0$ for all large j, then $\mathcal{X}$ vanishes to finite order at p_∞.*

The next step is to describe what the elements of $\mathrm{Aut}(G)$ look like under the assumption that $\dim \mathrm{Aut}(G) = 1$. This calculation was done in Propositions 2.6 and 2.7 of [31] and they remain valid here since they do not involve any features of D. The conclusion is that if $g \in \mathrm{Aut}(G)$ then

$$g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2)) = (\alpha z_1 + \beta, \phi(z_1) + a z_2)$$

where $|\alpha| = 1, a = \pm 1, \beta \in \mathbb{C}$ and $\phi(z_1)$ is a holomorphic polynomial. Moreover, if $q = (q_1, q_2) \in G$, $g \in \mathrm{Aut}(G)$ and

$$E = 2\Re(g_2(q_1, q_2)) + P(g_1(q_1, q_2), \overline{g_1(q_1, q_2)})$$

then $|E| = |2\Re q_2 + P(q_1, \overline{q}_1)|$ as Lemma 2.8 of [31] shows. Hence $|E|$ is independent of g.

Proposition 2.3. *The dimension of* $\mathrm{Aut}(D)$ *is at least two.*

Proof. Suppose that $\dim \mathrm{Aut}(D) = \dim \mathrm{Aut}(G) = 1$. Write

$$(a_j, b_j) = g^{-1} \circ \phi_j(p) = g^{-1} \circ \phi_j \circ g(g(p))$$

and note that $g^{-1} \circ \phi_j \circ g \in \mathrm{Aut}(G)$ for all $j \geq 1$. Let $g^{-1}(p) = q = (q_1, q_2) \in G$. By the arguments summarized above, it follows that

$$|2\Re b_j + P(a_j, \overline{a}_j)| = |2\Re q_2 + P(q_1, \overline{q}_1)| > 0 \tag{2.6}$$

for all $j \geq 1$. This shows that the orbit $\{g^{-1} \circ \phi_j(p)\} \in G$ can only cluster at the point at infinity in ∂G. Let

$$\eta = |2\Re q_2 + P(q_1, \overline{q}_1)| > 0$$

and for $r > 0$ define

$$G_r = \left\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P(z_1, \bar{z}_1) < -r\right\} \subset G.$$

Observe that the boundaries of G and G_r intersect only at the point at infinity for all $r > 0$. Furthermore, the entire orbit (a_j, b_j) and q are contained in $G_{\eta/2}$ by (2.6). By Proposition 2.2 above it follows that $\mathcal{X}(p_\infty) = 0$ and by Lemma 3.5 of [4] the intersection of the zero set of $\mathcal{X}$ with ∂D contains p_∞ as an isolated point. Now regard g as a holomorphic mapping from $G_{\eta/2}$ into D. The sequence $(a_j, b_j) \in G_{\eta/2}$ converges to the point at infinity in $\partial G_{\eta/2}$ and its image under g, namely $\phi_j(p)$, converges to p_∞. Proposition 2.2 also shows that if the cluster set of the point at infinity in $\partial G_{\eta/2}$ intersects ∂D near p_∞, then the vector field $\mathcal{X}$ vanishes at all such points. Since the cluster set of the point at infinity in $\partial G_{\eta/2}$ under g is connected and contains p_∞ as an isolated point, it must equal p_∞.

Thus for a given small neighbourhood U of p_∞ there exists a neighbourhood of the point at infinity in $\partial G_{\eta/2}$ which is mapped by g into $U \cap D$. However, a neighbourhood of infinity in $G_{\eta/2}$ contains G_M for some large $M > 0$. Fix a point $s \in g(G_M) \subset D$ and let $\tilde{s} = (\tilde{s}_1, \tilde{s}_2) \in G_M$ be such that $s = g(\tilde{s})$. Note that

$$L_t(s) = L_t \circ g(\tilde{s}) = g \circ T_t(\tilde{s}_1, \tilde{s}_2) = g(\tilde{s}_1, \tilde{s}_2 + it)$$

which gives $L_t(s) \to p_\infty$ as $|t| \to \infty$. For any compact $K \subset D$ there exists $R > 0$ such that K is contained in the Kobayashi ball $B^k_D(s,R)$. Hence $L_t(K) \subset B^k_D(L_t(s),R)$ for any $t \in \mathbb{R}$. By Lemma 2.1 it follows that L_t moves any point in D in both forward and backward time to p_∞, i.e., the action of L_t on D is parabolic. The arguments of [4] can now be applied to show that

$$D \backsimeq \left\{ (z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1 \right\}$$

for some integer $m \geq 1$. Thus $\dim \mathrm{Aut}(D) = 4$ which is a contradiction. $\square$

In case $\dim \mathrm{Aut}(D) = 2$, note that the calculations done in section 3 of [31] deal with only the defining function of G and hence they apply in this situation as well. Indeed, the following dichotomy holds – here $\mathrm{Aut}(D)^c$ is the connected component of the identity.

(i) If $\mathrm{Aut}(D)^c$ is abelian, then D is biholomorphic to either

$$\mathcal{D}_1 = \left\{ (z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0 \right\}$$

or

$$\mathcal{D}_2 = \left\{ (z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0 \right\}$$

for some polynomials P_1, P_2 that depend only on $\Re z_1$ or $|z_1|^2$ respectively.

(ii) If $\mathrm{Aut}(D)^c$ is non-abelian then D is biholomorphic to

$$\mathcal{D}_3 = \left\{ (z_1,z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0 \right\}$$

where $P_{2m}(z_1, \bar{z}_1)$ is a homogeneous polynomial of degree $2m$ without harmonic terms.

3. Model domains when $\mathrm{Aut}(D)$ is three dimensional

3.1. **A tube domain and its finite and infinite sheeted covers.** For $0 \leq s < t < \infty$ define

$$\mathfrak{S}_{s,t} = \left\{ (z_1,z_2) \in \mathbb{C}^2 : s < (\Re z_1)^2 + (\Re z_2)^2 < t \right\}$$

which is a non-simply connected tube domain over a nonconvex base. Evidently D cannot be biholomorphic to $\mathfrak{S}_{s,t}$ since D is simply connected as observed earlier. It is possible to consider finite and infinite sheeted covers of $\mathfrak{S}_{s,t}$. To obtain a finite sheeted cover, consider the n-sheeted covering self map

$$\Phi^{(n)}_\chi : \mathbb{C}^2 \setminus \left\{ \Re z_1 = \Re z_2 = 0 \right\} \to \mathbb{C}^2 \setminus \left\{ \Re z_1 = \Re z_2 = 0 \right\}$$

whose components are given by

$$\tilde{z}_1 = \Re\left((\Re z_1 + i\Re z_2)^n \right) + i\Im z_1,$$
$$\tilde{z}_2 = \Im\left((\Re z_1 + i\Re z_2)^n \right) + i\Im z_2.$$

Equip $\mathbb{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\}$ with the pull-back complex structure using $\Phi_\chi^{(n)}$ and call the resulting complex surface $M_\chi^{(n)}$. For $0 \le s < t < \infty$ and $n \ge 2$ define

$$\mathfrak{S}_{s,t}^{(n)} = \big\{(z_1, z_2) \in M_\chi^{(n)} : s^{1/n} < (\Re z_1)^2 + (\Re z_2)^2 < t^{1/n}\big\}.$$

Then $\Phi_\chi^{(n)}$ is an n-sheeted holomorphic covering map from $\mathfrak{S}_{s,t}^{(n)}$ onto $\mathfrak{S}_{s,t}$. It is clear that the domains $\mathfrak{S}_{s,t}^{(n)}$ are not simply connected and hence D cannot be equivalent to any of them. Proposition 4.7 in [31] provides a different proof of this fact which uses ideas that are applicable for other classes of domains as well. This can be adapted in the setting of theorem 1.1 as follows:

Proposition 3.1. *There cannot exist a proper holomorphic mapping from D onto $\mathfrak{S}_{s,t}$ for all $0 \le s < t < \infty$. In particular, D cannot be equivalent to $\mathfrak{S}_{s,t}^{(n)}$ for any $n \ge 2$ and $0 \le s < t < \infty$.*

Proof. Let $\pi : D \to \mathfrak{S}_{s,t}$ be a proper holomorphic mapping. The case when $0 < s < t < \infty$ will be considered first. The boundary of $\mathfrak{S}_{s,t}$ has two components, namely

$$\partial\mathfrak{S}_{s,t}^{+} = \big\{(z_1, z_2) \in \mathbb{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = t\big\}, \text{ and}$$
$$\partial\mathfrak{S}_{s,t}^{-} = \big\{(z_1, z_2) \in \mathbb{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = s\big\}.$$

The orientation induced on these pieces by $\mathfrak{S}_{s,t}$ makes them strongly pseudoconvex and strongly pseudoconcave respectively. Lemma 2.1 of [31] shows that there is a two dimensional stratum $S \subset \mathcal{L} \cap \hat{D}$ that clusters at p_∞ – this is a purely local assertion and hence it remains valid here as well. Pick $a \in S$ near p_∞ and let W be a small neighbourhood of a so that π extends holomorphically to W. Note that $(W \cap \partial D) \setminus S$ consists of points that are either strongly pseudoconvex or strongly pseudoconcave. Let $V_\pi \subset W$ be the branching locus of $\pi : W \to \mathbb{C}^2$. Since ∂D is finite type, it follows that $V_\pi \cap \partial D$ has real dimension at most one. There are two possibilities now – first, if $\pi(a) \in \mathfrak{S}_{s,t}^{+}$, then choose a strongly pseudoconcave point $a' \in (W \cap \partial D) \setminus V_\pi$. Thus π maps a neighbourhood of a', which is strongly pseudoconcave, locally biholomorphically onto a neighbourhood of $\pi(a') \in \partial\mathfrak{S}_{s,t}^{+}$ and this is a contradiction. A similar argument can be given when $\pi(a') \in \mathfrak{S}_{s,t}^{-}$. The only possibility then is that there are no pseudoconcave points near p_∞, i.e., ∂D is weakly pseudoconvex near p_∞. In this case, [28] shows that

$$D \backsimeq \tilde{D} = \big\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + P_{2m}(z_1, \bar{z}_1) < 0\big\} \tag{3.1}$$

where $P_{2m}(z_1, \bar{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ – this being the 1-type of ∂D at p_∞, without harmonic terms. In particular D is globally pseudoconvex and as π is proper, it follows that $\mathfrak{S}_{s,t}$ is also pseudoconvex. However, this is not the case.

When $0 = s < t < \infty$, the two components of $\partial\mathfrak{S}_{0,t}$ are

$$\partial\mathfrak{S}_{0,t}^{+} = \big\{(z_1, z_2) \in \mathbb{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = t\big\}, \text{ and}$$
$$i\mathbb{R}^2 = \big\{\Re z_1 = \Re z_2 = 0\big\}.$$

Choose $a \in S$ as above and let W, W be small neighbourhoods of a and $\pi(a)$ so that $\pi : W \to W'$ is a well defined holomorphic mapping. Suppose that $\pi(a) \in i\mathbb{R}^2$. Since $V_\pi \cap \partial D$ has real dimension at most one, it follows that there is an open piece of $W \cap \partial D$ near a that is mapped locally biholomorphically onto an open piece in $i\mathbb{R}^2$ and this is a contradiction. A similar argument shows that $\pi(a) \notin \partial\mathfrak{S}_{0,t}^{+}$ and therefore the only possibility is that ∂D is weakly pseudoconvex near p_∞. By [28] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Let π still denote the proper mapping

$$\pi : \tilde{D} \to \mathfrak{S}_{0,t}.$$

Let ϕ be a holomorphic function on $\tilde{D}$ that peaks at the point at infinity in $\partial\tilde{D}$. Then $\psi = \log|\phi - 1|$ is a plurisubharmonic function that is bounded above on $\tilde{D}$ and has the property that $\psi \to -\infty$ at the point at infinity in $\partial\tilde{D}$. If $\pi_1^{-1}, \pi_2^{-1}, \ldots, \pi_m^{-1}$ are the local branches of π^{-1}, then it is known that

$$\tilde{\psi} = \max\{\psi \circ \pi_j^{-1} : 1 \le j \le m\}$$

extends to a plurisubharmonic function on $\mathfrak{S}_{0,t}$. If there is an open piece of $\partial\mathfrak{S}_{0,t}^+$ on which $\tilde{\psi} \to \infty$, then the uniqueness theorem shows that $\tilde{\psi} \equiv -\infty$ and this is a contradiction. Thus there is a point, say $p \in \partial\tilde{D}$ whose cluster set under π intersects $\partial\mathfrak{S}_{0,t}^+$. Then π extends continuously up to $\partial\tilde{D}$ near p and this extension is even locally biholomorphic across strongly pseudoconvex points which are known to be dense on $\partial\tilde{D}$. By Webster's theorem, π is algebraic. Away from a codimension one algebraic variety Z, the inverse π^{-1} defines a correspondence that is locally given by finitely many holomorphic maps. Since $Z \cap i\mathbb{R}^2$ has real dimension at most one, it is possible to pick $p' \in i\mathbb{R}^2 \setminus Z$. The branches of π^{-1} will now map an open piece of $i\mathbb{R}^2$ near p' locally biholomorphically (shift p' if necessary to achieve this) to an open piece on $\partial\tilde{D}$. This cannot happen as $\partial\tilde{D}$ is not totally real.

To conclude, let $f : D \to \mathfrak{S}_{s,t}^{(n)}$ be biholomorphic. Since $\mathfrak{S}_{s,t}^{(n)}$ inherits the complex structure from $\mathfrak{S}_{s,t}$ via $\Phi_\chi^{(n)}$, it follows that

$$\pi = \Phi_\chi^{(n)} \circ f : D \to \mathfrak{S}_{s,t}$$

is an unbranched, proper holomorphic mapping between domains with the standard complex structure. Such a map cannot exist as shown above. $\square$

To construct an infinite sheeted cover of $\mathfrak{S}_{s,t}$, consider the infinite sheeted covering map

$$\Phi_\chi^{(\infty)} : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\}$$

whose components are given by

$$\tilde{z}_1 = \exp(\Re z_1)\cos(\Im z_1) + i\Re z_2, \text{ and}$$
$$\tilde{z}_2 = \exp(\Re z_1)\sin(\Im z_1) + i\Im z_2.$$

Equip $\mathbb{C}^2$ with the pull-back complex structure using $\Phi_\chi^{(\infty)}$ and denote the resulting complex manifold by $M_\chi^{(\infty)}$. For $0 \le s < t < \infty$ define

$$\mathfrak{S}_{s,t}^{(\infty)} = \{(z_1, z_2) \in M_\chi^{(\infty)} : (\ln s)/2 < \Re z_1 < (\ln t)/2\}.$$

This is seen to be an infinite sheeted covering of $\mathfrak{S}_{s,t}$, the holomorphic covering map being $\Phi_\chi^{(\infty)}$.

Proposition 3.2. *D is not biholomorphic to $\mathfrak{S}_{s,t}^{(\infty)}$ for $0 \le s < t < \infty$.*

Proof. Let $f : D \to \mathfrak{S}_{s,t}^{(\infty)}$ be a biholomorphism. Then

$$\pi = \Phi_\chi^{(\infty)} \circ f : D \to \mathfrak{S}_{s,t}$$

is a holomorphic infinite sheeted covering map between domains equipped with the standard complex structure. Using the explicit description of $\Phi_\chi^{(\infty)}$, we see that it maps the boundary of $\mathfrak{S}_{s,t}^{(\infty)}$ into the boundary of $\mathfrak{S}_{s,t}$. Hence the cluster set of ∂D under π is contained in $\partial\mathfrak{S}_{s,t}$. Now if $0 < s < t < \infty$, then by choosing an appropriate point on the two dimensional stratum $S \subset \mathcal{L}$ as in the previous proposition, it follows that ∂D must be weakly pseudoconvex near p_∞. By [28], $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Hence $\tilde{D}$ covers $\mathfrak{S}_{s,t}$ and since the Kobayashi metric on $\tilde{D}$ is complete, it follows that the same must hold for $\mathfrak{S}_{s,t}$. Completeness then forces $\mathfrak{S}_{s,t}$ to be pseudoconvex which it is not. Contradiction.

If $0 = s < t < \infty$, then first note that the conclusion that $\tilde{D}$ covers $\mathfrak{S}_{0,t}$ still holds and let π still denote this infinite sheeted covering map. By [8], there exists a point on $\partial\tilde{D}$ whose cluster set under π intersects $\partial\mathfrak{S}_{0,t}^+$. By standard arguments involving the Kobayashi metric, π extends continuously up to $\partial\tilde{D}$ near this point. This extension is even locally biholomorphic near strongly pseudoconvex points that are known to be dense in $\partial\tilde{D}$. By Webster's theorem, π is algebraic and therefore the cardinality of a generic fibre of π is finite. This contradicts the fact that π is an infinite sheeted cover. $\square$

3.2. **A domain in $\mathbb{P}^2$.** Let $\mathcal{Q}_+ \subset \mathbb{C}^3$ be the smooth complex analytic set given by

$$z_0^2 + z_1^2 + z_2^2 = 1.$$

For $1 \le s < t < \infty$ define

$$E_{s,t}^{(2)} = \left\{ (z_0, z_1, z_2) \in \mathbb{C}^3 : s < |z_0|^2 + |z_1|^2 + |z_2|^2 < t \right\} \cap \mathcal{Q}_+.$$

This is a two sheeted covering of

$$E_{s,t} = \left\{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : s|z_0^2 + z_1^2 + z_2^2| < |z_0|^2 + |z_1|^2 + |z_2|^2 < t|z_0^2 + z_1^2 + z_2^2| \right\},$$

the covering map being $\psi(z_0, z_1, z_2) = [z_0 : z_1 : z_2]$. Similarly, for $1 < t < \infty$, the map

$$\psi : E_t^{(2)} \to E_t$$

is a two sheeted covering, where

$$E_t^{(2)} = \left\{ (z_0, z_1, z_2) \in \mathbb{C}^3 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t \right\} \cap \mathcal{Q}_+$$

and

$$E_t = \left\{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t|z_0^2 + z_1^2 + z_2^2| \right\}.$$

To construct a four sheeted cover of $E_{s,t}$, consider the map $\Phi_\mu : \mathbb{C}^2 \setminus \{0\} \to \mathcal{Q}_+$ whose components are given by

$$\begin{aligned}
\tilde{z}_1 &= -i(z_1^2 + z_2^2) + i(z_1\bar{z}_2 - \bar{z}_1 z_2)/(|z_1|^2 + |z_2|^2), \\
\tilde{z}_2 &= z_1^2 - z_2^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)/(|z_1|^2 + |z_2|^2), \text{ and} \\
\tilde{z}_3 &= 2z_1 z_2 + (|z_1|^2 - |z_2|^2)/(|z_1|^2 + |z_2|^2).
\end{aligned}$$

Note that Φ_μ is a two sheeted cover onto $\mathcal{Q}_+ \setminus \mathbb{R}^3$. Therefore we may equip the domain of Φ_μ, i.e., $\mathbb{C}^2 \setminus \{0\}$ with the pull back complex structure using Φ_μ and denote the resulting complex surface by $M_\mu^{(4)}$. For $1 \le s < t < \infty$, the domain

$$E_{s,t}^{(4)} = \left\{ (z_1, z_2) \in M_\mu^{(4)} : ((s-1)/2)^{1/2} < |z_1|^2 + |z_2|^2 < ((t-1)/2)^{1/2} \right\}$$

is a four sheeted cover of $E_{s,t}$, the holomorphic covering map being $\psi \circ \Phi_\mu$.

Proposition 3.3. *There cannot exist a proper holomorphic mapping from D onto $E_{s,t}$ for all $1 \leq s < t < \infty$. In particular, D is not equivalent to either $E^{(2)}_{s,t}$ or $E^{(4)}_{s,t}$.*

Proof. Let $f : D \to E_{s,t}$ be a proper holomorphic mapping. Consider the case when $1 < s < t < \infty$. The boundary $\partial E_{s,t}$ has two components, namely

$$\partial E^{+}_{s,t} = \left\{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = t|z_0^2 + z_1^2 + z_2^2| \right\}, \text{ and}$$
$$\partial E^{-}_{s,t} = \left\{ [z_0 : z_1 : z_2] \in \mathbb{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = s|z_0^2 + z_1^2 + z_2^2| \right\},$$

which are strongly pseudoconvex and strongly pseudoconcave hypersurfaces respectively. The argument used in proposition 3.1 can be applied here to conclude that $p_\infty \in \partial D$ must be a weakly pseudoconvex point. By [28] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Thus we have a proper mapping from $\tilde{D}$ onto $E_{s,t}$ which implies that $E_{s,t}$ must be holommorphically convex and this is a contradiction.

Now suppose that $1 = s < t < \infty$. Then the boundary $\partial E_{1,t}$ consists of a strongly pseudoconvex piece, namely $\partial E^{+}_{1,t}$ and a maximally totally real piece given by $\psi(\partial \mathbb{B}^3 \cap Q_+)$. The argument in the preceeding paragraph applies again to show that $D \backsimeq \tilde{D}$ with $\tilde{D}$ as in (3.1). Let f still denote the proper map from $\tilde{D}$ onto $E_{1,t}$. Let ϕ be a holomorphic function on $\tilde{D}$ that peaks at the point at infinity in $\partial \tilde{D}$ and denote by $f_1^{-1}, f_2^{-1}, \ldots, f_l^{-1}$ the locally defined branches of f^{-1} that exist away from a closed codimension one analytic set in $E_{1,t}$. Then

$$\tilde{\phi} = (\phi \circ f_1^{-1}) \cdot (\phi \circ f_2^{-1}) \cdots (\phi \circ f_l^{-1})$$

is a well defined holomorphic function on $E_{1,t}$ and satisfies $|\tilde{\phi}| < 1$ there. Now $\tilde{\phi}$ extends across $\psi(\partial \mathbb{B}^3 \cap Q_+)$, which has real codimension two and is totally real strata, as well. Thus $\tilde{\phi} \in \mathcal{O}(E_t)$ and $|\tilde{\phi}| \leq 1$. If $|\tilde{\phi}(a')| = 1$ for some $a' \in \psi(\partial \mathbb{B}^3 \cap Q_+)$, the maximum principle implies that $|\tilde{\phi}| \equiv 1$ on $E_{1,t} \subset E_t$ and this is a contradiction. This argument shows that for every $a' \in \psi(\partial \mathbb{B}^3 \cap Q_+)$, there is a point $a \in \partial \tilde{D}$ such that the cluster set of a under f contains a'. On the other hand, by [8], there are points b, b' on $\partial \tilde{D}, \partial E^{+}_{1,t}$ respectively such that the cluster set of b contains b'. Thus f will be algebraic by Webster's theorem as before. Away from an algebraic variety $Z \subset \mathbb{P}^2$, f^{-1} defines a holomorphic correspondence that locally splits into finitely many holomorphic mappings. Since $Z \cap \psi(\partial \mathbb{B}^3 \cap Q_+)$ has real dimension at most one, it is possible to choose $a' \in \psi(\partial \mathbb{B}^3 \cap Q_+) \setminus Z$. Now one of the branches of f^{-1} will map a' into $\partial \tilde{D}$ and therefore an open piece of the totally real component $\psi(\partial \mathbb{B}^3 \cap Q_+)$ will be mapped locally biholomorphically onto an open piece of $\partial \tilde{D}$. Contradiction.

To conclude, if $D \backsimeq E^{(2)}_{s,t}$ or $E^{(4)}_{s,t}$, then this would imply the existence of an unbranched proper holomorphic mapping from D onto $E_{s,t}$ and this cannot happen by the arguments given above. ☐

Proposition 3.4. *There cannot exist a proper holomorphic mapping from D onto E_t for all $1 < t < \infty$. In particular, D cannot be equivalent to $E^{(2)}_t$ for all $1 < t < \infty$.*

Proof. By working in local coordinates it can be seen that E_t is described as a sub-level set of a strongly plurisubharmonic function. Hence E_t must be holomorphically convex and therefore D is pseudoconvex if there were to exist a proper map $f : D \to E_t$. By standard arguments involving the Kobayashi metric, this map f will be continuous up to ∂D near p_∞. By [9] it follows that p_∞ is a weakly spherical point on ∂D, i.e., there is a defining function for ∂D near $p_\infty = 0$ of the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \dots .$$

Since p_∞ is an orbit accumulation point, [28] shows that D is equivalent to the model domain at p_∞, i.e.,

$$D \backsimeq \big\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\big\}.$$

This shows that $\dim \mathrm{Aut}(D) = 4$ which is a contradiction. To conclude, if $D \backsimeq E_t^{(2)}$, then there would exist an unbranched proper mapping from D onto E_t which is not possible. $\square$

3.3. **Domains constructed by using an analogue of Rossi's map.** For $-1 \le s < t \le 1$ let

$$\Omega_{s,t} = \big\{(z_1, z_2) \in \mathbb{C}^2 : s|z_1^2 + z_2^2 - 1| < |z_1|^2 + |z_2|^2 - 1 < t|z_1^2 + z_2^2 - 1|\big\}$$

and for $-1 < t < 1$ let

$$\Omega_t = \big\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 - 1 < t|z_1^2 + \zeta_2^2 - 1|\big\}.$$

It was shown in [17] that Ω_t has a unique maximally totally real $\mathrm{Aut}(\Omega_t)^c$-orbit, namely

$$\mathcal{O}_5 = \big\{(\Re z_1, \Re z_2) \in \mathbb{R}^2 : (\Re z_1)^2 + (\Re z_2)^2 < t\big\}$$

for all $t \in (-1, 1)$. Moreover $\Omega_t = \Omega_{-1,t} \cup \mathcal{O}_5$ for all $t \in (-1, 1)$.

For $1 \le s < t \le \infty$ let

$$D_{s,t} = \big\{(z_1, z_2) \in \mathbb{C}^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2 < t|1 + z_1^2 - z_2^2|, \Im(z_1(1 + \bar{z}_2)) > 0\big\}$$

where it is assumed that the domain $D_{s,\infty}$ does not contain the complex curve

$$\mathcal{O} = \big\{(z_1, z_2) \in \mathbb{C}^2 : 1 + z_1^2 - z_2^2 = 0, \Im(z_1(1 + \bar{z}_2)) > 0\big\}.$$

For $1 \le s < \infty$ let

$$D_s = \big\{(z_1, z_2) \in \mathbb{C}^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2, \Im(z_1(1 + \bar{z}_2)) > 0\big\}$$

and note that $D_s = D_{s,\infty} \cup \mathcal{O}$.

Observe that D cannot be equivalent to $\Omega_{s,t}$ or $D_{s,t}$ as neither is simply connected. It remains to consider whether D can be equivalent to Ω_t or D_s.

Proposition 3.5. *There cannot exist a proper holomorphic mapping from D onto Ω_t for $-1 < t < 1$ or to D_s for $1 \le s < \infty$.*

Proof. We first consider Ω_t. Let $z_1 = x + iy, z_2 = u + iv$ so that

$$\mathcal{O}_5 = \big\{(x, u) \in \mathbb{R}^2 : x^2 + u^2 < 1\big\}$$

and its boundary

$$\partial\mathcal{O}_5 = \big\{(x, u) \in \mathbb{R}^2 : x^2 + u^2 = 1\big\} \subset \partial\Omega_t$$

for all $t \in (-1, 1)$. Note that $\partial\Omega_t \setminus \partial\mathcal{O}_5$ is a smooth strongly pseudoconvex hypersurface. Suppose that $f : D \to \Omega_t$ is proper. As in proposition 3.1, it is possible to choose $a \in S \subset \mathcal{L}$ such that f extends holomorphically to a neighbourhood of a. By shifting $a \in S$ if necessary we may assume that f is in fact locally biholomorphic near a. Note that $f(a) \notin \partial\Omega_t \setminus \partial\mathcal{O}_5$, as otherwise there are strongly pseudoconcave points near a that will be mapped to strongly pseudoconvex points. The remaining possibility is that $f(a) \in \partial\mathcal{O}_5$ which is totally real. Since f is locally biholomorphic near a, f cannot map an open piece of ∂D near a into $\partial\mathcal{O}_5$. Again, there are strongly pseudoconcave points near a that are mapped by f to $\partial\Omega_t \setminus \partial\mathcal{O}_5$ which is strongly pseudoconvex and this is a contradiction.

Hence the boundary ∂D is weakly pseudoconvex near p_∞ and thus $D \backsimeq \tilde{D}$ by [28] where $\tilde{D}$ is as in (3.1). Let $f : \tilde{D} \to \Omega_t$ still denote the biholomorphism. Observe that the automorphism group of $\tilde{D}$ is at least two dimensional; apart from the translations T_t, it is also invariant under the one parameter subgroup

$$S_s(z_1, z_2) = (\exp(s/2m)z_1, \exp(s)z_2),$$

$s \in \mathbb{R}$. The corresponding real vector fields $X = \Re(i\partial/\partial z_2)$ and $Y = \Re((z_1/2m)\partial/\partial z_1 + z_2\partial/\partial z_2)$ satisfy $[X,Y] = X$. By the arguments in the last part of the proof of proposition 4.1 in [31], it follows that $D \backsimeq \mathcal{D}_4$ where

$$\mathcal{D}_4 = \big\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\big\}.$$

Let $f : \mathcal{D}_4 \to \Omega_t$ still denote the proper map. Choose an arbitrary strongly pseudoconvex point $b' \in \partial\Omega_t \setminus \partial\mathcal{O}_5$. By [8] there exists $b \in \partial\mathcal{D}_4$ such that the cluster set of b under f contains b'. Then by well known arguments involving the Kobayashi metric on $\mathcal{D}_4$ and Ω_t near b and b' respectively, it follows that f is continuous up to $\partial\mathcal{D}_4$ near b and $f(b) = b'$. By [9], it follows that $b \in \partial\mathcal{D}_4$ must be a weakly spherical point, i.e., there exists a coordinate system near b in which the defining equation for $\partial\mathcal{D}_4$ is of the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \dots,$$

the dots indicating terms of higher order. However, the explicit form of ∂D_4 shows that no point on it is weakly spherical.

It remains to show that no proper map $f : D \to D_s$ can exist for $1 \le s < \infty$. Suppose the contrary. Observe that if $s > 1$ then ∂D_s is the disjoint union of three components, namely

$$\mathcal{C}^1 = \big\{1 + |z_1|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1+\bar{z}_2)) > 0\big\},$$
$$\mathcal{C}^2 = \big\{1 + |z_1|^2 - |z_2|^2 > s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1+\bar{z}_2)) = 0\big\},$$
$$\mathcal{C}^3 = \big\{1 + |z_1|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1+\bar{z}_2)) = 0\big\}.$$

Note that $\mathcal{C}^1$ is a strongly pseudoconvex hypersurface and that $\Im(z_1(1+\bar{z}_2)) = 0$ has an isolated singularity at $(z_1, z_2) = (0, -1)$ away from which it is smooth Levi flat. Also, $(0, -1) \notin \mathcal{C}^2$ as $s > 1$. As above, choose $a \in S \subset \mathcal{L}$ near which f extends locally biholomorphically. Since $\mathcal{C}^1$ is strongly pseudoconvex, it follows that $f(a) \notin \mathcal{C}^1$. Further if $f(a) \in \mathcal{C}^2$, then a small open piece of ∂D near a will be mapped locally biholomorphically into the Levi flat piece $\big\{\Im(z_1(1+z_2)) = 0\big\}$ and this is a contradiction as points on $\partial D \setminus S$ near a are Levi non-degenerate. The remaining possibility is that $f(p) \in \mathcal{C}^3$. However, an open piece of ∂D near a cannot be mapped by f into $\mathcal{C}^3$ as it has real dimension at most 2 near each of its points. Thus there is an open dense set of points near a that are mapped locally biholomorphically into either $\mathcal{C}^1$ or $\mathcal{C}^2$. Both cannot occur for reasons mentioned above. Thus ∂D must be weakly pseudoconvex near p_∞ and we may now argue as before to get a contradiction.

When $s = 1$, it was noted in [17] that there is a proper mapping g from the bidisc Δ^2 onto D_1. If $f : D \to D_1$ is proper, then $F : f^{-1} \circ g : \Delta^2 \to D$ is a proper holomorphic correspondence. Thus D is pseudoconvex and by [28], it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (3.1). Let $F : \Delta^2 \to \tilde{D}$ still denote the proper correspondence. Using the holomorphic function on $\tilde{D}$ that peaks at the point at infinity in $\partial\tilde{D}$ it can be seen that there is an open dense subset of $\partial\Delta^2$ whose cluster set under

F intersects the finite part of $\partial\Delta^2$ – call this subset Γ. Fix $\zeta_0 \in \Gamma$ and a small neighbourhood W containing it such that $W \cap \partial\Delta^2$ is smooth. Note that $W \cap \partial\Delta^2$ is defined as the zero locus of either $|z_1|^2 - 1$ or $|z_2|^2 - 1$ both of which are plurisubharmonic. Now well known arguments using the branches of F^{-1}, these plurisubharmonic defining equations and a suitable version of the Hopf lemma show that

$$\text{dist}(F(z), \partial\tilde{D}) \lesssim \text{dist}(z, \partial\Delta^2)$$

whenever $z \in W \cap \Delta^2$ – here $F(z)$ denotes any one of the finitely many branches of F. By [6] it follows that F extends continuously up to $W \cap \partial\Delta^2$ as a correspondence. The branching locus of F in Δ^2 is therefore defined by a holomorphic function in Δ^2 that extends continuously up to $W \cap \partial\Delta^2$. Let $h \in \mathcal{O}(\Delta^2)$ define the branching locus. If $h \equiv 0$ on $W \cap \partial\Delta^2$, the uniqueness theorem shows that $h \equiv 0$ in Δ^2 which cannot happen. By shifting ζ_0 we may assume that $h(\zeta_0) \neq 0$. Therefore near ζ_0 the correspondence F splits into well defined holomorphic functions, say $F_1, F_2, \ldots, F_k$ each of which is holomorphic on W (shrink W if needed) and continuous up to $W \cap \partial\Delta^2$. Since $W \cap \Delta^2$ is a product domain and each point of $\partial\tilde{D}$ supports a holomorphic peak function, arguments from [27] show that these branches $F_1, F_2, \ldots, F_k$ must be independent of either z_1 or z_2. This contradicts the assumption that F is proper. □

It is also possible to construct finite and infinite sheeted covers of $D_{s,t}, \Omega_{s,t}$ as explained in [17]. That D cannot be equivalent to any of them follows by similar arguments and we omit the details.

Finally proposition 4.1 of [31] shows that a bounded domain $D \subset \mathbb{C}^2$ that satisfies the hypotheses of the main theorem and admits a Levi flat $\text{Aut}(D)^c$-orbit must be equivalent to

$$\mathcal{D}_4 = \left\{(z_1, z_2) \in \mathbb{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\right\}.$$

The proof is purely local and can be applied here as well to conclude that a domain $D \subset X$ as in the main theorem with a Levi flat $\text{Aut}(D)^c$-orbit must be equivalent to $\mathcal{D}_4$. This is the only possibility that remains after eliminating all others and the conclusion is that if $\dim \text{Aut}(D) = 3$ then $D \backsimeq \mathcal{D}_4$.

4. Model domains when $\text{Aut}(D)$ is four dimensional

Of the 7 isomorphism classes listed in [18] of hyperbolic surfaces with four dimensional automorphism group, the following cannot be equivalent to D for topological reasons.

- The spherical shell $S_r = \left\{z \in \mathbb{C}^2 : r < |z| < 1\right\}$ for $0 \leq r < 1$ – the automorphism group here is the unitary group U_2 which is compact, or the quotient $S_r/\mathbb{Z}_m$ for some $m \in \mathbb{N}$, none of which are simply connected.
- $\mathcal{E}_{r,\theta} = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, r(1 - |z_1|^2)^\theta < |z_2| < (1 - |z_1|^2)^\theta\right\}$, where $\theta \geq 0, 0 < r < 1$ or $\theta < 0, r = 0$. This is not simply connected.
- $D_{r,\theta} = \left\{(z_1, z_2) \in \mathbb{C}^2 : r\exp(\theta|z_1|^2) < |z_2| < \exp(\theta|z_1|^2)\right\}$, where $\theta = 1, 0 < r < 1$ or $\theta = -1, r = 0$. This is again not simply connected.

The remaining four classes listed below have a common feature that a large part of their boundary, if not the whole, is spherical.

- $\Omega_{r,\theta} = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, r(1 - |z_1|^2)^\theta < \exp(\Re z_2) < (1 - |z_1|^2)^\theta\right\}$, where $\theta = 1, 0 \leq r < 1$ or $\theta = -1, r = 0$
- $\mathfrak{S} = \left\{(z_1, z_2) \in \mathbb{C}^2 : -1 + |z_1|^2 < \Re z_2 < |z_1|^2\right\}$.

- $\mathcal{E}_\theta = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < (1-|z_1|^2)^\theta\}$, for $\theta < 0$. Here the boundary $\partial\mathcal{E}_\theta$ contains a Levi flat piece $L = \{|z_1| = 1\} \times \mathbb{C}_{z_2}$. Away from L, $\partial\mathcal{E}_\theta$ is spherical and strongly pseudoconcave as seen from $\mathcal{E}_\theta$
- $E_\theta = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^\theta < 1\}$, where $\theta > 0$ and $\theta \neq 2$.

To see that D cannot be equivalent to $\Omega_{r,\theta}, \mathfrak{S}$ or to $\mathcal{E}_\theta$, suppose the contrary. Let $f : D \to \mathcal{E}_\theta$ be biholomorphic. Let $p \in \partial D$ be a strongly pseudoconcave point near p_∞ across which f extends locally biholomorphically. Note that $f(p) \notin L$ as ∂D is of finite type near p_∞. Then $f(p) \in \partial\mathcal{E}_\theta$. Let g be a local biholomorphism defined on a open neighbourhood W of $f(p)$ that takes $W \cap \partial\mathcal{E}_\theta$ into $\partial\mathbb{B}^2$. Then $g \circ f$ is a biholomorphic germ at p that maps an open piece of ∂D into $\partial\mathbb{B}^2$. By [29], this germ can be analytically continued along all paths in $U \cap \partial D$ that start at p. Thus p_∞ must be a weakly pseudoconvex point and by [9], it must be weakly spherical as well. By [28], it follows that $D \backsimeq E_{2m}$ and so $\mathcal{E}_\theta \backsimeq E_{2m}$ which is a contradiction.

To conclude, it remains to show that if $D \backsimeq E_\theta$, then $\theta = 2m$ for some integer $m \geq 2$. Proposition 5.1 in [31] remains valid here too and we omit the details. The conclusion is that if $\dim \mathrm{Aut}(D) = 4$ then $D \backsimeq E_{2m} \backsimeq \mathcal{D}_5$.

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