Hybrid Reconstruction in Bayesian Domain

Partha P. Mondal*, K. Rajan* and L. M. Patnaik**

*Department of Physics, ** Department of Computer Science and Automation
Indian Institute of Science, Bangalore 560012, India
partha@physics.iisc.ernet.in

Abstract— Image reconstruction in Bayesian framework is far more advantageous over other reconstruction methods like convolution back projection, weighted least square method and maximum likelihood estimation. The power of Bayesian estimation lies in its ability to incorporate the prior distribution knowledge, enabling better reconstruction. Proper specification of clique potentials in Bayesian estimation plays a crucial role in the reconstruction process by favors the presence of desired characteristics in the image lattice like nearest neighbor interactions and homogeneity. Homogenous Markov random fields have been successfully used for modeling such interactions. Though, reconstruction produced by such models are far more efficient, they often require large iterations for producing an approximate reconstruction. To deal with this problem, we have extended the Bayesian estimation in order to support sharp reconstruction. We propose to use sharp potential in Bayesian estimation once an approximate reconstruction is available using homogenous potentials in Bayesian domain. The advantage of the proposed potential is its ability to recognize correlated nearest neighbors. The proposed reconstruction is a hybrid of both smooth and sharp potential in Bayesian framework and hence it is termed as hybrid reconstruction.

Simulated experiments have shown that the proposed hybrid estimation method produces superior and sharp reconstruction as compared to the reconstruction produced by other Bayesian estimation methods.

1. INTRODUCTION

Emission Tomography is an imaging technique for mapping brain functioning during different physiological activities performed by cells. Depending upon the physiological activity of the region, cells concentrate these inhaled or injected radio-pharmaceuticals in different proportions. Being unstable these radio-pharmaceuticals emit subatomic particles. In positron emission tomography (PET) the radio-compound contains elements of lower atomic number which stabilize themselves by emitting positrons. Within a narrow neighborhood of approximately 5mm, the positrons annihilates with a nearby electron, resulting in the production of two γ -rays which run opposite to each other.

These γ -rays are then detected by the circular detector placed around the region of study as shown in Fig.2. Once the detector counts are registered and stored, reconstruction process requires correct modeling of the physical phenomena.

Though stochastic algorithms perform better than deterministic algorithm for emission tomography [3], often the reconstructed image is deteriorated by the noisy artifacts that appear in the reconstruction due to the application of unconstrained maximum likelihood estimation of density functions based on point process [7]. Broadly, two approaches have been suggested in literature for tackling this problem :

- (1) To stopping the ML-reconstruction procedure, before it starts to turn noisier [8].
- (2) Utilizing prior distribution knowledge in the reconstruction process [2,5,6].

Maximum a posterior (MAP) estimation produces improved reconstruction by utilizing both the measurement data information as well the available prior distribution knowledge in the reconstruction process [2,5,6]. MAP algorithms takes into advantage the physically relevant information like, nearest neighbor correlation and homogeneity in the local neighborhood. Utilizing these local properties has resulted in noise free smooth reconstruction. However, the assumption of homogeneity breaks down near the edges of an image, resulting in over-penalization of the edges, thus effectively producing over-smoothened reconstruction. A new prior is proposed which is capable of restoring the edges by taking into account the inhomogeneity in the image, once a close approximate image is available. So, a hybrid of two priors is used in the simulated experiments.

In section II, a brief introduction to Bayesian estimation approach to emission tomography is given. Section III, is devoted to the proposed potential and algorithm. Simulated experimental results are shown in section IV. Section V concludes the paper.

2. RECONSTRUCTION IN BAYESIAN-GIBBS

FRAMEWORK

Modeling prior distribution in Bayesian domain

In Bayesian domain prior distribution knowledge plays a crucial role for improved image reconstruction. Modeling a prior which has physical relevance is of utmost importance in image reconstruction. Uniform prior in MAP-estimation leads to maximum likelihood (ML) estimation. Only when prior having significant physical meaning is incorporated in the reconstruction procedure, then only one can achieve improved reconstruction over ML-estimation. Local correlation is an important property of an image, because of strong interaction between the nearest neighbor pixels.

Markov Random field (MRF) has been used to model such interactions in an image lattice. On a lattice, MRF is defined as the collection of random variables λ_i , corresponding to the lattice site λ_i such that,

$$P(\lambda_j | \lambda_i : i \neq j) = P(\lambda_j | \lambda_i : i \in N_j) \quad (1)$$

where, N_j are the subset of lattice sites in the neighborhood of i^{th} lattice site.

According to Hammersly-Clifford theorem, image λ is characterized by Gibbs distribution,

$$P(\lambda) = \frac{1}{Z} e^{-\frac{1}{\beta} U(\lambda)} \quad (2)$$

where, $U(\lambda)$ is the Gibbs energy, β is a parameter and Z is the partition function of the distribution.

Assuming homogenous MRF, one can write the energy function as :

$$U(\lambda) = \sum_i \sum_{j \in N_i} w_{ij} V(\lambda_i, \lambda_j) \quad (3)$$

where, N_i represents the nearest neighbors of lattice site i , and $V(\lambda_i, \lambda_j)$ is the potential at site i , w_{ij} is unity for nearest neighbors and $(1/\sqrt{2})$ for the diagonal neighbors.

The following are the commonly used potential functions suggested in literature [2,3,4],

$$V_1(\lambda_i, \lambda_j) = (\lambda_i - \lambda_j)^2 \quad (4)$$

$$V_2(\lambda_i, \lambda_j) = \frac{(\lambda_i - \lambda_j)^2}{\mu^2 + (\lambda_i - \lambda_j)^2} \quad (5)$$

$$V_3(\lambda_i, \lambda_j) = \ln \left[1 + \left(\frac{\lambda_i - \lambda_j}{\mu} \right)^2 \right] \quad (6)$$

Function $V_1(\bullet)$, increasingly penalizes the separation between neighboring pixels in the nearest neighborhood which accounts for the fact that nearest neighbor pixels possess similar value. On the other hand, $V_2(\bullet)$, penalizes the neighboring pixels to be of similar value until they are separated by a threshold μ , effectively encouraging the existence of genuine edges. $V_3(\bullet)$ is a compromise between $V_1(\bullet)$ and $V_2(\bullet)$ which penalizes the pixels at a slower rate.

General Methodology

Seeking a MAP-estimate is equivalent of determining that estimate λ which maximizes the posterior density function,

$$\max_{\lambda} \log(P(\lambda / y)) = \log P(y / \lambda) + \log P(\lambda) \quad (7)$$

where, $P(\lambda / y)$ is the posterior density function, $P(y / \lambda)$ is the likelihood function and $P(\lambda)$ is the prior probability distribution of the object image.

Since emission process is modeled as a Poisson process, the joint probability distribution function commonly termed as likelihood function $P(y / \lambda)$ is given by,

$$P(y / \lambda) = \prod_j \left(\frac{e^{-\sum_i \lambda_i p_{ij}} (\sum_i \lambda_i p_{ij})^{y_j}}{y_j!} \right) \quad (8)$$

Incorporating Gibbs prior as defined by (1), (2), (3) and the likelihood function (8) into (7) produces,

$$\max_{\lambda} P(\lambda / y) = \log P(y / \lambda) - \frac{1}{\beta} \sum_i \sum_{j \in N_i} w_{ij} V(\lambda_i, \lambda_j) \quad (9)$$

Due to the complicated nature of the prior, the solution for the above equation is very difficult. Green [6] has proposed one-step-late (OSL) for a recursive update of the estimates,

$$\lambda_i^{k+1} = \left(\frac{\lambda_i^k}{\left(\sum_j p_{ij} + \frac{1}{\beta} \sum_{j \in N_i} \left(\frac{\partial V(\lambda_i, \lambda_j)}{\partial \lambda_i} \right)_{\lambda=\lambda^k} \right)} \sum_{j=1}^M \frac{y_j p_{ij}}{\sum_i \lambda_i p_{ij}} \right) \quad (10)$$

One can write eqn.(10) in additive vector notation as,

$$\lambda^{k+1} = \lambda^k + \Delta(\lambda^k) \quad (11)$$

where,

$$\Delta(\lambda_i^k) = \frac{\lambda_i^k}{\left(\sum_j p_{ij} + \frac{1}{\beta} \sum_{i' \in N_i} \left(\frac{\partial V(\lambda_i, \lambda_{i'})}{\partial \lambda_i} \right)_{\lambda=\lambda^k} \right)} \left[\sum_{j=1}^M \left(\frac{(y_j - \sum_{i'} \lambda_{i'}^k p_{i'j}) p_{ij}}{\sum_{i'} \lambda_{i'}^k p_{i'j}} \right) + \frac{1}{\beta} \sum_{i' \in N_i} \left(\frac{\partial V(\lambda_i, \lambda_{i'})}{\partial \lambda_i} \right)_{\lambda=\lambda^k} \right] \quad (11a)$$

$\Delta(\lambda^k)$ is the correction vector which when added to the current estimate λ^k produces the new MAP-estimate λ^{k+1} .

In a MAP-MRF approach, prior distribution function takes into account the neighborhood of an pixel for correct determination of its value. The simplest case of a MAP-MRF estimation is ML-estimation in the absence of any prior distribution. Role of prior in MAP-MRF estimation is the correction of estimates by considering local correlation and homogeneity. It is the prior distribution which decides the amount of contribution from neighboring pixels.

3. PROPOSED POTENTIAL AND ALGORITHM

As, defined in section II, the homogenous MRF is considered for modeling nearest neighbor interaction. It is this homogenization that results in the over-smoothing of the edges in the reconstructed image. One can think of non-homogeneous MRF to model the prior distribution. This will enable us to separate two different uniform regions in the image without unduly penalizing the large separation at the boundary of two different regions of the image.

A MRF is said to be homogenous if Markovianity ((eqn.(1)) is retained regardless of the relative position of the lattice site i in an image lattice. It is clear that if edges have to be preserved in an image, one cannot assume homogeneity at all the lattice sites. Homogeneity assumption is valid only inside the uniform region of an image and should be restricted within the uniform region. But, for lattice sites at the edges separating two different uniform regions demand more correlation from neighboring sites inside the uniform region, and negligible contribution from outside, in order to preserve edges. This observation definitely distinguishes between pixels at the edge and pixels inside a uniform region, which simultaneously shows the presence of homogeneity inside an uniform region and in-homogeneity at the edges of the uniform region.

Non-homogenization can be invoked in the reconstruction by modifying the potential function so that the potential penalizes the non-smoothness in the image and at the same time encourages formation of the edges. The following

potential function is capable of doing this,

$$V(\lambda_i, \lambda_j) = \frac{c}{(\lambda_i - \lambda_j)} \quad (12)$$

So, the total potential in the neighborhood of the lattice site i is given by,

$$V_{Total} = \sum_{j \in N_i} w_{ij} \frac{c}{(\lambda_i - \lambda_j)} \quad (13)$$

where, c is a constant and N_i defines the subset of values in the neighborhood of lattice site i , and w_{ij} is defined in eqn.(3).

What we want from MAP-estimate is to increasingly penalize pixels inside the uniform region i.e, pixels marked k , and negligible penalization for pixels outside the uniform region i.e, pixel marked j , in order to preserve the boundary. On the other hand the same pixel should be equally penalized if the pixel is not at the edge of uniform region. To understand how this potential function can preserve the edges, consider the image lattice as shown in fig.1.

Let us consider the i^{th} lattice site which is at the edge of two different uniform regions of the image shown in Fig.1. Neighboring pixels are tagged by the letters k and j respectively. The potential function increasingly penalize the difference between the pixels i with k and negligibly penalizes the difference between the pixels i with j , results in the preservation of edges. On the other hand, if a pixel happens to be inside an uniform region, then it receives equal contribution from all the neighboring pixels. Hence, this potential distinguishing the edge pixels with the rest.

Fig.1. Schematic diagram of Shepp-Logan Phantom on an image lattice showing site i inside an uniform region.

Once an over-smoothed reconstruction is obtained using MAP-estimation employing homogenous prior, one can use the proposed in-homogenous prior in MAP-estimation for improved sharper reconstruction.

Proposed Algorithm :

Initialization : $\hat{\lambda}_i^0$ = very small positive initial estimate ,
 $n+m$ = total number of iteration

1. For iteration = 1 to n

generate $\hat{\lambda}_i^n$, using MAP-algorithm as given by eqn.(10), taking $\hat{\lambda}_i^0$ as the initial estimate and either of the given potential (eqn.(4), (5), (6)).

2. Initialize for the proposed MAP-estimate

$$\hat{\lambda}_i^0 = \hat{\lambda}_i^n$$

3. For iteration h = n to m

generate $\hat{\lambda}_i^m$, using MAP-algorithm as given by eqn.(10), with potential given by eqn.(12).

4. SIMULATED EXPERIMENTAL RESULTS

Simulated PET System

Proposed algorithm was implemented in a simulated positron emission tomography (PET) system having single detector ring as shown in Fig.2. The PET system consists of 64 detectors in a ring geometry and the object space is decomposed into 64 x 64 square pixels. The object is assumed to be confined within the circle of the square region in which the object is supported, to exploit the eight-fold symmetry available in the circular region of the system [1].

Fig.2. Schematic diagram of Positron Emission tomography System, showing that plane to be reconstruction, which is coplanar with the detector plane.

An annihilation event occurring inside a pixel is governed by a number of physical phenomena such as scattering,

absorption and detector characteristics. For simplicity, we have neglected the contribution of physical phenomena like attenuation and scattering. Incorporation of all the physical phenomena such as scattering, attenuation, detector characteristics are essential for the correct determination of the elements of probability matrix $P=[p_{ij}]$. Since modeling of these physical phenomena is very complex, hence we have to replace it by geometric probability as suggested by Shepp and Vardi [3]. Geometric probability p_{ij} is taken as the angle seen by the center of *pixel i* into the detector tube *j*.

Due to multiplicative nature of MAP-algorithm, one has to start the iterative process with a non-zero positive value. During the reconstruction process if any pixel becomes negative, it should be re-initialized to a small positive number. This constraint is due to the physical property of the positivity of pixels.

Reconstructed Results

Reconstructed images using the proposed algorithm are shown in Fig.3. It is clear that with a constant β -value, the reconstructed image is either blurred (for small β) or noisy (for large β). In the present simulation experiment, stable β -values are used in the range $2.5 \times 10^4 \leq \beta \leq 2.5 \times 10^7$. MAP-reconstruction after 65 iterations with potential V_1 and hyper-parameter $\beta=2.5 \times 10^4$, is shown in Fig.3(b). Experimentally, 50 iterations are found to be enough for getting smooth reconstruction. An additional 15 iterations using sharp potential V with $\beta=2.5 \times 10^4$ has produced better quality reconstructed image as shown in Fig.3(c).

Fig.3. (a) Original Test Shepp-Logan Phantom, (b) MAP-reconstruction (c) Hybrid Reconstruction

5. CONCLUSIONS

New potential function is proposed in the Bayesian image reconstruction framework. It is general nature of a prior to produce smooth reconstruction by incorporating local correlation in the reconstruction procedure. Identification of an approximate prior is important. It may end up with either over-smoothed reconstruction or noisy EM-reconstruction. This proposed hybrid approach is capable of preserving the minute details by enhancing the edges in the reconstructed image. This is possible only by invoking local in-homogeneity at the edges in an image.

Simulated experiments have shown that this hybrid approach produces better reconstructed images compared to those realized by MAP-reconstruction algorithms.

ACKNOWLEDGEMENT

The first author would like to thank Mr. Hemanth, research scholar at Indian Institute of Science for helping during the simulation experiment.

Also, he wish thanks to Council of Scientific and Industrial research (CSIR), Government of India, for providing Junior Research fellowship. He declares that this work is partially supported by CSIR, New Delhi, India.

REFERENCES

- [1] L. Kaufman, " Implementing and accelerating the EM algorithm for Positron Emission tomography ", *IEEE Trans. On Med. Img.*, Vol. MI-6, No.1, pp. 37-51, March , 1987.
- [2] T. Hebert and R. Leahy, " A generalized EM-algorithm for 3-D Bayesian Reconstruction from Poisson Data using Gibbs priors ", *IEEE Trans. on Med. Img.*, vol. 8, No.2, pp.194-202, 1989
- [3] L. A. Sheep and Y. Vardi, "Maximum Likelihood Reconstruction for Emission Tomography", *IEEE Trans. on Med. Img.*, Vol. MI-1, No.2, Oct. 1982
- [4] S. Geman and D.Geman, " Stochastic relaxation, Gibbs distribution and the Bayesian restoration of images ", *IEEE Trans. Pattern Anal. Machine Intell.*, vol. PAMI-6, pp. 721-741, Nov., 1984
- [5] K. Lange, M. Bahm, and R. Little, " A theoretical study of some maximum likelihood algorithm for emission and transmission tomography ", *IEEE Trans. on Med. Img.*, MI-6(2), pp.106-114, 1987
- [6] P.J. Green, "Bayesian reconstruction from emission tomography data using a modified EM algorithm", *IEEE Trans. on Med. Img.*, vol.9, No.1, March, 1990.
- [7] D.L. Snyder and M. I. Miller, "The use of sieves to stabilize images produces with the EM algorithm for emission tomography ", *IEEE Trans on Nuclear Science*, NS-32(5), pp. 3864-3872, 1985
- [8] E. Veklerov and J.Llacer, " Stopping rule for MLE algorithm based on statistical hypothesis testing ", *IEEE Trans. on Med. Img.*, MI-6 : pp. 313-319, 1987.