Vandermonde-Cocyclic Codes and a suitable DFT

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Abstract — A new transform is defined for the class of cocyclic group ring codes over a Galois ring, where the underlying group is Abelian and the cocycle is the Vandermonde-cocycle on the Abelian group. The class of Vandermonde-cocyclic (VC) group ring codes are characterized using this new transform by means of a structure theorem for VC group rings.

For a prime p , let $GR(p^a, l)$ denote the Galois ring with characteristic p^a [1]. Let $\mathcal{T}_{a,l}^*$ denote the unique cyclic subgroup of order $p^l - 1$ in the group of units $GR(p^a, l)^*$. Let G , a finite Abelian group (with $|G| = n$), be a direct product of r cyclic subgroups denoted by $C_{r-1}, \dots, C_0$ of orders $m_{r-1}, \dots, m_0$ respectively. Any element $g \in G$ can be denoted as g_i or $g_{[i]}$, where $[i] = [i_{r-1}, i_{r-2}, \dots, i_0]$ is the mixed-radix representation of $i \in I_n = \{0, \dots, n-1\}$ using $m_{r-1}, m_{r-2}, \dots, m_0$ as the mixed-radixes and the group operation of G can also be specified using the mixed-radix addition [2]. We consider linear codes of length $n = m_{r-1} \dots m_1 m_0$ over the Galois ring $GR(p^a, l)$ such that $\gcd(p, n) = 1$. We use q for p^l for notational simplicity.

A function $\psi : G \times G \rightarrow \mathcal{T}_{a,l}^*$ is called a cocycle on G if $\psi(x, y)\psi(xy, z) = \psi(y, z)\psi(x, yz)$ for all $x, y, z \in G$ [3]. Let $\beta = (\beta_{r-1}, \beta_{r-2}, \dots, \beta_0)$, where each β_λ is an element of $\mathcal{T}_{a,l}^*$, such that d_λ (the order of β_λ) divides m_λ for all $\lambda = 0, 1, \dots, r-1$. The map on $G \times G$ defined as $\psi_\beta(g_{[i]}, g_{[j]}) = \prod_{\lambda=0}^{r-1} \beta_\lambda^{i_\lambda j_\lambda}$, for $[i] = [i_{r-1}, i_{r-2}, \dots, i_0]$ and $[j] = [j_{r-1}, j_{r-2}, \dots, j_0]$, is a cocycle. We call this cocycle as *Vandermonde-cocycle on an Abelian group*. For any cocycle ψ on G , let $GR(p^a, l)^\psi[G]$ be the twisted group ring which has the same module structure as the group ring $GR(p^a, l)[G]$, but the multiplication is twisted, which is a linear extension of $g.h = \psi(g, h)gh$ for all $g, h \in G$. A *cocyclic group ring code* is an ideal in $GR(p^a, l)^\psi[G]$. Vandermonde-cocyclic (VC) group ring codes are ideals in the cocyclic group ring $GR^{\psi_\beta}(p^a, l)[G]$. Let m be the smallest positive integer such that $m_\lambda \mid (q^m - 1)$ and also $2d_\lambda \mid (q^m - 1)$ for all $\lambda = 0, 1, \dots, r-1$. Let $\alpha_\lambda = \zeta^{s_\lambda}$ be a primitive root of unity of order m_λ and $\gamma_\lambda = \zeta^{h_\lambda}$ be a square-root of β_λ , for all λ , belonging to $\mathcal{T}_{a,lm}^* = \langle \zeta \rangle \subset GR(p^a, lm)$.

Definition 1 (VC-DFT) Let $\vec{a} = (a_0, a_1, \dots, a_{n-1}) \in GR(p^a, l)^n$ and let α_λ and γ_λ be as defined above. The VC-DFT vector $\vec{A} = (A_0, A_1, \dots, A_{n-1}) \in GR(p^a, lm)^n$ of $\vec{a}$ is defined as $A_j = \sum_{i=0}^{n-1} \left(\prod_{\lambda=0}^{r-1} \gamma_\lambda^{-i_\lambda^2} \alpha_\lambda^{i_\lambda j_\lambda} \right) a_i$ for all $j \in I_n$, where $i = [i_{r-1}, i_{r-2}, \dots, i_0]$ and $j = [j_{r-1}, j_{r-2}, \dots, j_0]$ are mixed-radix representations of i and j .

Lemma 1 (Properties of VC-DFT)

$$\left(\prod_{\lambda=0}^{r-1} \beta_\lambda^{i_\lambda} \right) a_{i \oplus [1, 1, \dots, 1]} \xleftrightarrow{vc-dft} \left(\prod_{\lambda=0}^{r-1} \gamma_\lambda \alpha_\lambda^{j_\lambda} \right) A_j,$$

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$$c_i = \sum_{k=0}^{n-1} a_{i \ominus k} b_k \psi_\beta(g_{i \ominus k}, g_k) \xleftrightarrow{vc-dft} C_j = B_j A_j,$$

$$\left(\prod_{\lambda=0}^{r-1} \alpha_\lambda^{i_\lambda} \right) a_i \xleftrightarrow{vc-dft} A_{j \oplus [1, 1, \dots, 1]},$$

$$c_i = a_i b_i \xleftrightarrow{vc-dft} \frac{1}{n} \sum_{k=0}^{n-1} B_k A_{j \oplus k}^{(u)}$$

where $A_k^{(u)} = \sum_{i=0}^{n-1} \left(\prod_{\lambda=0}^{r-1} \alpha_\lambda^{i_\lambda k_\lambda} \right) a_i$ is the Generalized DFT for Abelian codes [2] and $\oplus, \ominus$ denote the mixed-radix addition and subtraction respectively.

Let σ_0 denote the Frobenius automorphism of $GR(p^a, lm)$ and let $\sigma = \sigma_0^l$. A vector $\vec{A} \in GR(p^a, lm)^n$ is the VC-DFT vector of $\vec{a} \in GR(p^a, l)^n$, if and only if $\sigma^k(A_j) = A_i$ where $i = [i_{r-1}, i_{r-2}, \dots, i_0]$ is such that, $i_\lambda = \phi_\lambda^k(j_\lambda) = \left(\frac{-(q^k - 1)h_\lambda}{s_\lambda} + q^k j_\lambda \right)$ modulo m_λ for all λ . This is the *conjugate symmetry property*. Let Φ^k denote the mapping from I_n to I_n , which maps $i = [i_{r-1}, \dots, i_0]$ to $\Phi^k(i) = [\phi_{r-1}^k(i_{r-1}), \dots, \phi_0^k(i_0)]$. For every $j \in I_n$, the set $\widehat{j} = \{j, \Phi^1(j), \Phi^2(j), \dots, \Phi^{e_j-1}(j)\}$ where e_j is the smallest integer such that $\Phi^{e_j}(j) = j$, is called the *cyclotomic coset* containing $[j]$ and, e_j is called the exponent of $[j]$. Let $\mathbf{L} \subseteq I_n$ be the set containing one element from each of the cyclotomic cosets. The set of transform components $\widehat{A}_{[i]} = \{A_{[j]} \mid [j] \in \widehat{i}\}$ will be called the *conjugacy class* containing $A_{[i]}$. For a code $\mathcal{C}$ over $GR(p^a, l)$, let $\mathbf{C}_j = \{A_j \mid \forall \vec{a} \in \mathcal{C}\}$ denote the set of distinct values taken by the j -th transform component of all the codewords in $\mathcal{C}$ and let $\mathbf{C}_{i,j} = \{(A_i, A_j) \mid \forall \vec{a} \in \mathcal{C}\}$.

Theorem 1 (Structure theorem) For any integer $n = m_{r-1} m_{r-2} \dots m_0$ and $\gcd(p, n) = 1$, $GR(p^a, l)^n$ is isomorphic to $\bigoplus_{i \in \mathbf{L}} GR(p^a, l_{e_i})$. Further, a β -VC group ring code $\mathcal{C}$ over $GR(p^a, l)$ is isomorphic to an ideal $\bigoplus_{i \in \mathbf{L}} \mathbf{C}_i$, where $\mathbf{C}_i = p^{\eta_i} GR(p^a, l_{e_i})$ for some fixed value of η_i , $0 \leq \eta_i \leq a$ and transform components belonging to different conjugacy classes take values independently.

Here, A_i and A_j take values independently implies $\mathbf{C}_{i,j} = \mathbf{C}_i \times \mathbf{C}_j$.

Transform Domain Characterization: A code over Galois ring $GR(p^a, l)$ is β -VC iff the VC-DFT vector of each codeword satisfies the conjugate symmetry property and for any $i \in I_n$, $\mathbf{C}_i = p^{\eta_i} GR(p^a, l_{e_i})$ for some fixed value of η_i , $0 \leq \eta_i \leq a$.

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