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An extension of Mangler transformation to a 3-D problem

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Abstract. Considering the linearized boundary layer equations for three-dimensional disturbances, a Mangler type transformation is used to reduce this case to an equivalent two-dimensional one.

Keywords. Boundary layer; three-dimensional; Mangler transformation.

1. Introduction

The Mangler transformation reduces an axisymmetric laminar boundary layer on a body of revolution to an equivalent planar boundary layer flow (Schlichting 1968). This transformation is also useful in turbulent boundary layer flow over a body of revolution (Cebeci & Bradshaw 1968). Another application of this transformation is in the reduction of a laterally strained boundary layer to the Blasius flow (Ramesh *et al* 1997). In this case the span-wise velocity is zero along a streamline but its non-zero span-wise gradient appears as a source/sink term in the contunuity equation (Schlichting 1968). In this paper, we show that a Mangler type transformation can reduce a specific three-dimensional flow considered here to an equivalent two-dimensional case.

2. Analysis

Let u^*, v^* and w^* denote the non-dimensional velocity components in the non-dimensional x, y and z directions, respectively. u_0 and v_0 will denote the Blasius velocity components. The governing equations considered here are the linearized boundary layer equations for two- and three-diemnsional disturbances of Libby & Fox (1964) and Luchini (1996). These authors perturbed the Blasius boundary layer as: $u^* = u_0(x, y) + u_1(x, y)exp(i\alpha z)$, $v^* = v_0(x, y) + v_1(x, y)exp(i\alpha z)$, $w^* = w_1(x, y)exp(i\alpha z)$; for 2-D flow ($z = 0$, $w = 0$), $u_1 = u$, $v_1 = v$. We first consider the two-dimensional case.

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2.1 *2-D Case*

In this case, the governing boundary layer equations are (Libby & Fox 1964),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u_o \frac{\partial u}{\partial x} + v_0 \frac{\partial u}{\partial y} + u \frac{\partial u_o}{\partial x} + v \frac{\partial u_o}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The boundary conditions are: $u(x, 0) = v(x, 0) = u(x, \infty) = (x, \infty) = 0$. The Blasius boundary layer equations are,

$$\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} = 0, \tag{3}$$

$$u_o \frac{\partial u_o}{\partial x} + v_o \frac{\partial u_o}{\partial y} = \frac{\partial^2 u_o}{\partial y^2}, \tag{4}$$

along with the boundary conditions, $u_0(y = 0) = v_0(y = 0) = 0$, $u_0(y \to \infty) \to 1$.

Adding and subtracting the quantity u/x in the continuity eq. (1), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{x} - \frac{u}{x} = 0. \tag{5}$$

Consider the Mangler transformation,

$$X = \frac{x^3}{3},\ Y = yx,\ u(x, y) \to U(X, Y),$$

$$V(X, Y) = \frac{1}{x}\left[\frac{yu}{x} + v\right],\ u_o(x, y) \to U_o(X, Y),\ V_o = \frac{1}{x}\left[\frac{yu_o}{x} + v_o\right]. \tag{6}$$

The usual Mangler variables are X, Y, U and V. The variables U_o and V_o are additional here. The boundary layer equations for an axi-symmetric body of radius r differ from those for two-dimensional flows by the term $(u/r)(dr/dx)$ in the continuity equation, $\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial y} = 0$; for $r = x$, the term $(u/r)(dr/dx)$ becomes u/x, which acts as a source term in the continuity equation.

In terms of the variables in (6), the governing equations (1)–(4) become,

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} - \frac{U}{3X} = 0, \tag{7}$$

$$U_o \frac{\partial U}{\partial X} + V_o \frac{\partial U}{\partial Y} + U \frac{\partial U_o}{\partial X} + V \frac{\partial U_o}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}, \tag{8}$$

$$\frac{\partial U_o}{\partial X} + \frac{\partial V_o}{\partial Y} - \frac{U_o}{3X} = 0, \tag{9}$$

$$U_o \frac{\partial U_o}{\partial X} + V_o \frac{\partial U_o}{\partial Y} = \frac{\partial^2 U_o}{\partial Y^2}, \tag{10}$$

respectively.

The mean flow continuity eq. (9) now has an artificial sink term $U_o/3X$. However, the similarity variables $\eta = Y/\sqrt{3X}$, $U_o = f'(\eta)$ reduce the mean flow to the Blasius one; here, a prime denotes the derivative with respect to η. (This may be an interesting application of the Mangler transformation to the Blasius flow.)

In terms of the variables,

$$U = 3XU_1(X,Y),\ V = 3XV_1(X,Y), \tag{11}$$

equations (7) and (8) become

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} + \frac{2U_1}{3X} = 0, \tag{12}$$

$$U_o\frac{\partial U_1}{\partial X} + V_o\frac{\partial U_1}{\partial Y} + \frac{U_oU_1}{X} + \left[U_1\frac{\partial U_o}{\partial X} + V_1\frac{\partial U_o}{\partial Y}\right] = \frac{\partial^2 U_1}{\partial Y^2}, \tag{13}$$

respectively.

2.2 *3-D Case*

The governing boundary layer equations in this case are the linearized disturbance equations of Luchini (1996; his equations 7a–c),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + w = 0, \tag{14}$$

$$u_o\frac{\partial u}{\partial x} + v_0\frac{\partial u}{\partial y} + u\frac{\partial u_o}{\partial x} + v\frac{\partial u_o}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \tag{15}$$

$$u_o\frac{\partial w}{\partial x} + v_o\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2}. \tag{16}$$

($i\alpha$ in Luchini's eq. (7a) is eliminated by taking $u_1 = i\alpha u,\ v_1 = i\alpha v,\ w = w_1$). The boudary conditions are: $u(x,0) = v(x,0) = w(x,0) = u(x,\infty) = w(x,\infty) = 0$. The third equation is the span-wise disturbance equation.

As in eq. (5), adding and subtracting the quantity u/x in the continuity equation (14), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{x} - \frac{u}{x} + w = 0. \tag{17}$$

We consider the Mangler type transformation (6), along with an additional variable W, below

$$X = \frac{x^3}{3},\ Y = yx, u(x,y) \to U(X,Y),\ V(X,Y) = \frac{1}{x}\left[\frac{yu}{x} + v\right]$$

$$W(X,Y) = \frac{1}{x^2}\left[w - \frac{u}{x}\right], u_o(x,y) \to U_o(X,Y),\ V_o = \frac{1}{x}\left[\frac{yu_o}{x} + v_o\right]. \tag{18}$$

This additional variable W is to relate the span-wise velocity component, w, to the stream-wise velocity component, u, as discussed below. In terms of these variables, the disturbance equations (14), (15) and (16) are,

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + W = 0, \tag{19}$$

$$U_o\frac{\partial U}{\partial X}+V_o\frac{\partial U}{\partial Y}+U\frac{\partial U_o}{\partial X}+V\frac{\partial U_o}{\partial Y}=\frac{\partial^2 U}{\partial Y^2}, \tag{20}$$

$$2U_oW+3X\left[U_o\frac{\partial W}{\partial X}+V_o\frac{\partial W}{\partial Y}\right]+U_o\frac{\partial U}{\partial X}-\frac{UU_o}{3X}+V_o\frac{\partial U}{\partial Y}=3X\frac{\partial^2 W}{\partial Y^2}+\frac{\partial^2 U}{\partial Y^2}, \tag{21}$$

respectively.

Following Squire (1933), we add (20) and (21) to obtain

$$2U_oW+3X\left[U_o\frac{\partial W}{\partial X}+V_o\frac{\partial W}{\partial Y}\right]+2\left[U_o\frac{\partial U}{\partial X}+V_o\frac{\partial U}{\partial Y}\right]$$

$$-\frac{UU_o}{3X}+U\frac{\partial U_o}{\partial X}+V\frac{\partial U_o}{\partial Y}=3X\frac{\partial^2 W}{\partial Y^2}+2\frac{\partial^2 U}{\partial Y^2}. \tag{22}$$

In terms of the variables in (11), the continuity equation (19) and the momentum equation (22) become,

$$\frac{\partial U_1}{\partial X}+\frac{\partial V_1}{\partial Y}+\frac{U_1}{X}+\frac{W}{3X}=0, \tag{23}$$

$$U_o\frac{\partial W}{\partial X}+V_o\frac{\partial W}{\partial Y}-\frac{U_1U_o}{3X}+\frac{2U_oW}{3X}+2\left[U_o\frac{\partial U_1}{\partial X}+\frac{U_oU_1}{X}+V_o\frac{\partial U_1}{\partial Y}\right]$$

$$+U_1\frac{\partial U_o}{\partial X}+V_1\frac{\partial U_o}{\partial Y}=\frac{\partial^2 W}{\partial Y^2}+2\frac{\partial^2 U_1}{\partial Y^2}, \tag{24}$$

respectively.

By letting $W=aU_1$ the disturbance equations (23) and (24) become,

$$\frac{\partial U_1}{\partial X}+\frac{\partial V_1}{\partial Y}+\frac{(3+a)U_1}{3X}=0, \tag{25}$$

$$U_o\frac{\partial U_1}{\partial X}+V_o\frac{\partial U_1}{\partial Y}+\frac{(2a+5)}{3(2+a)}\frac{U_oU_1}{X}+\frac{1}{(2+a)}\left[U_1\frac{\partial U_o}{\partial X}+V_1\frac{\partial U_o}{\partial Y}\right]=\frac{\partial^2 U_1}{\partial Y^2}, \tag{26}$$

respectively. Comparing these with the two-dimensional equations (12) and (13), we find them similar. We may note that the span-wise velocity component, w, is w=(1+a)u/x. For $a=-1$, w=0, equations (25) and (26) are exactly the same as (12) and (13), as it should be. Also, u/x ($\sim w$) acts as a source term in the continuity eq. (14). Thus enabling the use of the Mangler type transformation.

By letting $U_1=X^N g'(\eta)$, and satisfying the continuity equation (25), the similarity form of (26) is readily obtained as,

$$g'''+\frac{fg''}{2}+\frac{gf''}{2}+gf''\left[\frac{(7+a)}{2(2+a)}+\frac{3N}{(2+a)}\right]-f'g'\left[3N+\frac{(5+2a)}{(2+a)}\right]=0. \tag{27}$$

In terms of the similarity variables, the perturbed velocity components are: $u=3^{-N}x^{3(N+1)}g'$, $w=(1+a)u/x$.

For both (i) $a = -1$, $N = -4/3$ and (ii) $a = -8/3$, $N = -1/2$, eq. (27) reduces to the two-dimensional equation of Libby & Fox (1964)

$$g''' + \frac{fg''}{2} + \frac{gf''}{2} + f'g' - gf'' = 0. \tag{28}$$

The solution (Libby & Fox 1964) of this equation is $g = f - \eta f'$. The first case of $w = 0$ ($a = -1$) is obvious. In the second case, the perturbed velocities are: $u = 3^{-1/2}x^{3/2}g'$, and $w = -(5/3)(x/3)^{1/2}g'$. That is, the proposed Mangler type transformation (18) could reduce the three-dimensional problem considered here to a two-dimensional equivalent one.

3. Conclusion

A three-dimensional boundary layer flow is considered here. A Mangler type of transformation is proposed to reduce this flow to an equivalent two-dimensional one. This reduction has been possible by relating the span-wise velocity component to the stream-wise velocity component leading to an equivalent source term in the continuity equation.

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