On Bandlimited Signals with Minimal Space/Time-Bandwidth Product

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Abstract—We deal with the problem of determining bandlimited signals which have the lowest space/time-bandwidth product. We propose two approaches to the problem: the first is based on a modification of the Gaussian function; and the second, on the solution of a set of nonlinear algebraic equation obtained from the samples of the signal.

I. INTRODUCTION

We deal with real signals, $f \in L_2$, which are treated as functions of the real variable $x \in \mathbb{R}$ (i.e., the class of *square integrable* functions) having unit energy and centered at the origin. The independent variable x can denote either time (for dealing with time-dependent phenomena) or space (for describing space-dependent functions, like scanlines of images). In what follows, we use the terms ‘space’ and ‘time’ interchangeably.

Let $F(\omega)$ denote the Fourier transform of $f(x)$. Based on the assumptions made above, we can establish the following identities:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega, \quad (1)$$

It is the practice to denote the space localization of a signal by its “effective spatial-width”, (Δ_x) :

$$(\Delta_x)^2 = \int_{-\infty}^{\infty} x^2 f^2(x) dx \quad (2)$$

Similarly, the frequency localization of a signal is described by its “effective spectral-width”:

$$(\Delta_\omega)^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega \quad (3)$$

It is well known that $f(x)$ and $F(\omega)$ **cannot both be of short duration**. And this is made explicit (a) *qualitatively* by the scaling theorem,

$$af(ax) \Leftrightarrow F\left(\frac{\omega}{a}\right),$$

and (b) quantitatively by the **uncertainty principle** which places a lower bound on the product of effective spatial and spectral widths of continuous signals (also called the **space-bandwidth product (SBP)**) [1]:

$$(\Delta_x)^2 (\Delta_\omega)^2 \geq \frac{1}{4} \quad (4)$$

The equality in (4) is satisfied *only* by the Gaussian function, $\exp(-\frac{x^2}{2})$. (For a survey on the uncertainty inequality (4), see [2].)

Let $\mathcal{B}$ denote the class of continuous and finite energy signals which are *bandlimited*, i.e.,

$$|F(\omega)| = 0, \quad \forall |\omega| > W \quad (5)$$

For signals belonging to $\mathcal{B}$, Ishii et al. [3], [4] obtain a strict uncertainty inequality based on the *further* assumption that $F(\omega) = 0$ for $|\omega| = W$. However, the following problem is still open:

Problem 1: What is the function belonging to $\mathcal{B}$ which attains the lowest uncertainty product? It appears that an answer to this problem does not exist. In this paper, we provide an answer to a slightly modified problem:

Problem 2: How close can the uncertainty product of functions belonging to $\mathcal{B}$ get to the lower bound obtained in (4)?

The rest of the paper is organized as follows. In Sec.II, we present the first approach to **Problem 2**, based on a modification of the Gaussian function. In Sec.III, we propose a general approach to the *same problem* using our earlier results [5] on the sampled signal. We conclude the paper in Sec. IV.

II. MAIN RESULTS: SOLUTION I

Using (2) and the properties of Fourier transform, it can be shown that the effective spatial width is finite only if $|\frac{dF(\omega)}{d\omega}|$ is square integrable. Therefore, for $f \in \mathcal{B}$, we require that $F(\omega)$ have finite derivatives at $\omega = W$ and $\omega = -W$. (Similarly, the effective spectral width is finite only if $|\frac{df(x)}{dx}|$ is square integrable.)

Let

$$F(\omega) = G(\omega)H(\omega) \quad (6)$$

where

$$G(\omega) = C \exp\left(\frac{-\omega^2}{2\sigma^2}\right) \quad (7)$$

and

$$H(\omega) = \begin{cases} \frac{X}{2} \left[\cos \left(\frac{\pi(\omega + W)}{\epsilon} \right) + 1 \right] & : -W - \epsilon \leq \omega < -W \\ X & : -W \leq \omega \leq W \\ \frac{X}{2} \left[\cos \left(\frac{\pi(\omega - W)}{\epsilon} \right) + 1 \right] & : W < \omega \leq W + \epsilon \\ 0 & : \text{elsewhere} \end{cases} \quad (8)$$

Fourier Transform of the interpolating function

Fig. 1. Fourier transform of the interpolating function

where $X = 1.0$, and $\epsilon > 0$ is a small positive quantity. See Fig. 1 for the plot of $H(\omega)$ which satisfies the following conditions:

$$\begin{aligned} H(W) &= X, \quad H(W + \epsilon) = 0, \\ H(-W) &= X, \quad H(-W - \epsilon) = 0 \end{aligned} \quad (9)$$

$$\begin{aligned} \left. \frac{dH(\omega)}{d\omega} \right|_{W+\epsilon} &= 0, \quad \left. \frac{dH(\omega)}{d\omega} \right|_W = 0, \\ \left. \frac{dH(\omega)}{d\omega} \right|_{-W-\epsilon} &= 0, \quad \left. \frac{dH(\omega)}{d\omega} \right|_{-W} = 0 \end{aligned} \quad (10)$$

From (8) and (6), we can conclude that $F(\omega)$ is a bandlimited function with bandlimit $W + \epsilon$. Also $F(\omega) = 0$, $\forall |\omega| = W + \epsilon$, and is differentiable everywhere. In the next section, we evaluate (2) and (3) for any given function f .

See Tables 1 and 2 for the (squared) SBP of $F(\omega)$ for various of values σ (in $G(\omega)$). The important point to be noted is that for any small value of $\epsilon > 0$, there are bandlimited functions of the type $F(\omega)$ whose uncertainty product is very close to the lower bound in (4). More precisely, there is a sequence of bandlimited functions whose (squared) SBP approaches the lower bound in (4) for a given bandwidth W and ϵ .

In the next section, we propose a new approach (based on our earlier results [5]) to find the optimal bandlimited signal whose SBP is the lowest.

III. MAIN RESULTS: SOLUTION 2

We now exploit the fact that a bandlimited signal is completely represented by a discrete set of signal samples if the

Sl.No	σ	Product
1	2.0	1.19870
2	1.8	0.74649
3	1.7	0.57802
4	1.5	0.36008
5	1.3	0.27172
6	1.0	0.25026
7	0.9	0.25002
8	0.8	0.25000
9	0.7	0.25000
10	0.5	0.25000

TABLE I

CONVERGENCE OF UNCERTAINTY PRODUCT USING COSINE INTERPOLATION FUNCTION WITH $\epsilon = 0.1$ AND $W = \pi$

Sl.No	σ	Product
1	2.0	10.42400
2	1.8	5.66300
3	1.7	3.86440
4	1.5	1.49740
5	1.3	0.50648
6	1.0	0.25349
7	0.9	0.25031
8	0.8	0.25001
9	0.7	0.25000
10	0.5	0.25000

TABLE II

CONVERGENCE OF UNCERTAINTY PRODUCT USING COSINE INTERPOLATION FUNCTION WITH $\epsilon = 0.01$ AND $W = \pi$

sampling is done at or above the Nyquist rate. To this end, let $\{f[n]\}_{n \in \mathbb{Z}}$ denote the samples obtained by sampling uniformly (with the sampling interval $= X$) a bandlimited signal $f(x)$. In [5], we have derived an expression for the SBP of $f(x)$ in terms of its samples.

The expressions for the squares of effective spectral and spatial widths, Δ_ω and Δ_x , of $f(x)$ in terms of its samples are as follows:

$$(\Delta_\omega)^2 = \int_{-\pi}^{\pi} |\omega F(\omega)|^2 d\omega \quad (11)$$

$$= \frac{1}{2\pi X} \left[\frac{2\pi^3}{3} \sum_{n \in \mathbb{Z}} f^2[n] + \right.$$

$$\left. \sum_{m \in \mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} \\ m \neq n}} \frac{4\pi(-1)^{(m-n)}}{(m-n)^2} f[m]f[n] \right]$$

$$(\Delta_x)^2 = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(mX)f(nX) \left(\frac{m^2 + n^2}{2} X^2 A + B \right) \quad (12)$$

where

$$A = \frac{2WX^2\delta_{mn}}{2} + \frac{X^2(-1)^{(m-n)}}{2} \left(\epsilon \frac{\sin((m-n)X\epsilon)}{(m-n)X\epsilon} + \frac{\sin((m-n)X\epsilon)}{2} \left[\frac{2(m-n)X}{\left(\frac{\pi}{\epsilon}\right)^2 - (m-n)^2X^2} \right] \right) + \frac{X^2(-1)^{(m-n)}}{2} \left(\epsilon \frac{\sin((m-n)X\epsilon)}{(m-n)X\epsilon} \left\{ 1 - \frac{\frac{1}{2}}{\left[1 - \left(\frac{(m-n)X\epsilon}{2\pi} \right)^2 \right]} \right\} \right) \right)$$

$$B = \frac{1}{4\epsilon} \frac{\sin((m-n)X\epsilon)}{(m-n)X\epsilon} \frac{X^2\pi^2(-1)^{(m-n)}}{\left[1 - \left(\frac{(m-n)X\epsilon}{2\pi} \right)^2 \right]}, \quad (13)$$

and ϵ is a (positive) parameter of the interpolating function appearing in (8), and is less than the difference between W and the actual bandwidth of $f(x)$. When ϵ tends to 0, it can be shown, by examining (13), that

$$A = 2WX^2\delta_{mn} \quad (14)$$

and, from (12),

$$(\Delta_x)^2 = \frac{2WX^4}{2\pi} \sum_{n \in \mathbb{Z}} n^2 f^2(nX) + \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(mX) f(nX) B \quad (15)$$

And the second term in the above expression involving B , when $\epsilon \rightarrow 0$, simplifies to

$$\lim_{\epsilon \rightarrow 0} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(mX) f(nX) B = \lim_{\epsilon \rightarrow 0} \frac{1}{4\epsilon} X^2 \left| \sum_{n \in \mathbb{Z}} f(nX) \exp(j\pi n) \right|^2 \quad (16)$$

Note that if $F(\pm W) \neq 0$, the above expression becomes infinite, when $\epsilon \rightarrow 0$. Assuming that $F(\pm W) = 0$, we get

$$(\Delta_x)^2 = \frac{2WX^4}{2\pi} \sum_{n \in \mathbb{Z}} n^2 f^2(nX) \quad (17)$$

In this special limiting case, the above expression is similar to the one obtained by Ishii [3]. But our procedure is believed to be more general.

In order to make our mathematical analysis more tractable, we analyze finite length discrete time signals. Let $\mathbf{f} = (f_1, f_2, f_3, \dots, f_N)^T$ denote a N -dimension column vector representing a discrete sequence. We assume that this finite

length discrete sequence is obtained by sampling a bandlimited signal using a sampling interval X . Using the samples $\{f_i\}$ of the bandlimited signal, and invoking (1) and (12), we get

$$(\Delta_x)^2 = \mathbf{f}^T \mathbf{S} \mathbf{f} \quad (18)$$

$$(\Delta_\omega)^2 = \mathbf{f}^T \mathbf{B} \mathbf{f} \quad (19)$$

where $\mathbf{S}$ and $\mathbf{B}$ are positive definite symmetric, real matrices.

We now find the $\{f_i\}$ which minimizes SBP, $P = (\Delta_x)^2 (\Delta_\omega)^2$, subject to the following two constraints:

- 1) $\sum_i (f_i)^2 = \frac{1}{X}$; and
- 2) $F(\pm W) = 0$ or, equivalently, $\mathbf{f}^T \mathbf{c} = 0$ where $c[n] = \exp(j\pi n)$.

Invoking the Lagrange multiplier technique, the above problem reduces to the minimization of modified function, $\mathcal{P}$:

$$\mathcal{P} = (\mathbf{f}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + \lambda (\mathbf{f}^T \mathbf{f} - 1) + \mu \mathbf{f}^T \mathbf{c} \quad (20)$$

where λ and μ are Lagrange multipliers. For minimizing the above function, we compute the derivative of P_L with respect to $\{f_i\}$, λ and μ , and equate them separately to 0:

$$\frac{\partial P_L}{\partial \mathbf{f}} = (2\mathbf{S}\mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (2\mathbf{B}\mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) + 2\lambda \mathbf{f} + \mu \mathbf{c} = 0 \quad (21)$$

$$\frac{\partial P_L}{\partial \lambda} = \mathbf{f}^T \mathbf{f} - \frac{1}{X} = 0 \quad (22)$$

$$\frac{\partial P_L}{\partial \mu} = \mathbf{f}^T \mathbf{c} = 0 \quad (23)$$

Now we find the values of the Lagrange multipliers λ and μ in terms of $\mathbf{f}$. To this end, we rewrite (21) as

$$(\mathbf{S}\mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{B}\mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) + \lambda \mathbf{f} + \frac{\mu}{2} \mathbf{c} = 0 \quad (24)$$

Multiplying $\mathbf{f}^T$ on both sides of the above equation, we get

$$(\mathbf{f}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{f}^T \mathbf{B} \mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) + \lambda \mathbf{f}^T \mathbf{f} + \frac{\mu}{2} \mathbf{f}^T \mathbf{c} = 0 \quad (25)$$

Using (23) and (22), we can simplify the above equation as

$$2 (\mathbf{f}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + \frac{\lambda}{X} = 0 \quad (26)$$

Finally, we can express λ in terms of $\mathbf{f}$ as

$$\lambda = -2X (\mathbf{f}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) = -2X \Delta_x \Delta_\omega \quad (27)$$

In order to evaluate μ , we multiply $\mathbf{c}^T$ on both sides of (24) to get

$$(\mathbf{c}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{c}^T \mathbf{B} \mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) + \lambda \mathbf{c}^T \mathbf{f} + \frac{\mu}{2} \mathbf{c}^T \mathbf{c} = 0$$

$$(\mathbf{c}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{c}^T \mathbf{B} \mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) + \frac{\mu}{2} \mathbf{c}^T \mathbf{c} = 0$$

Finally, we obtain an expression for μ in terms of $\mathbf{f}$ as

$$\mu = \frac{-2 [(\mathbf{c}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{c}^T \mathbf{B} \mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f})]}{\mathbf{c}^T \mathbf{c}}$$

$$= \frac{-2 [(\mathbf{c}^T \mathbf{S} \mathbf{f}) \Delta_\omega + (\mathbf{c}^T \mathbf{B} \mathbf{f}) \Delta_x]}{\mathbf{c}^T \mathbf{c}} \quad (28)$$

Using (24), (27), and (28), we arrive at the following **nonlinear** equation which the *optimal finite length discrete sequence* must satisfy:

IV. CONCLUSIONS

In this paper, we have proposed two solutions to the problem of obtaining a bandlimited function $f(x) \in \mathcal{B}$ which has the smallest SBP. The first solution is based on a modification of the Gaussian function; and the second, on the solution of a set of nonlinear algebraic equation obtained from the samples of the signal. It is found that the SBP of any $f(x) \in \mathcal{B}$ is strictly greater than the lower bound as in (4). Further, there are functions in $\mathcal{B}$ whose SBP is very close to the lower bound as in (4).

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$$(\mathbf{S}\mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{B}\mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f}) - 2X (\mathbf{f}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) \mathbf{f} - \frac{[(\mathbf{c}^T \mathbf{S} \mathbf{f}) (\mathbf{f}^T \mathbf{B} \mathbf{f}) + (\mathbf{c}^T \mathbf{B} \mathbf{f}) (\mathbf{f}^T \mathbf{S} \mathbf{f})]}{\mathbf{c}^T \mathbf{c}} \mathbf{c} = 0$$

We denote the above vector equation in terms of $\mathbf{f}$ as $\mathbf{L}(\mathbf{f})$ and rewrite it in a simpler form as,

$$\mathbf{L}(\mathbf{f}) = (\mathbf{S}\mathbf{f})\Delta_\omega + (\mathbf{B}\mathbf{f})\Delta_x - 2X\Delta_x \Delta_\omega \mathbf{f} - \frac{[(\mathbf{c}^T \mathbf{S} \mathbf{f}) \Delta_\omega + (\mathbf{c}^T \mathbf{B} \mathbf{f}) \Delta_x]}{\mathbf{c}^T \mathbf{c}} \mathbf{c} = 0 \quad (29)$$

The above equation, is solved by the Newton-Raphson technique by computing the Jacobian matrix of $\mathbf{L}$. For details, see [6]. The results are presented in Figs. 2 and 3. They represent the optimal final length discrete sequences.

Fig. 2. Optimal bandlimited signal of length 101

Fig. 3. Optimal bandlimited signal of length 201